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RANDOM MEASURABLE SETS AND COVARIOGRAM REALIZABILITY PROBLEMS

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Abstract

We provide a characterization of realisable set covariograms, bringing a rigorous yet abstract solution to the S_2 problem in materials science. Our method is based on the covariogram functional for random measurable sets (RAMS) and on a result about the representation of positive operators on a noncompact space. RAMS are an alternative to the classical random closed sets in stochastic geometry and geostatistics, and they provide a weaker framework that allows the manipulation of more irregular functionals, such as the perimeter. We therefore use the illustration provided by the S_2 problem to advocate the use of RAMS for solving theoretical problems of a geometric nature. Along the way, we extend the theory of random measurable sets, and in particular the local approximation of the perimeter by local covariograms.

Keywords: Random measurable set; realizability; S_2 problem; covariogram; perimeter; truncated moment problem

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1. Framework and main results

1.1. Introduction

An old and difficult problem in materials science is the S_2 problem, often posed in the following terms. Given a real function $S_2 \colon \mathbb{R}^d \to [0, 1]$, is there a stationary random set $X \subset \mathbb{R}^d$ whose standard two-point correlation function is S_2, i.e. such that

$$\mathbb{P}\{x, y \in X\} = S_2(x - y), \qquad x, y \in \mathbb{R}^d? \tag{1}$$

The S_2 problem is a *realizability problem* concerned with the existence of a (translation invariant) probability measure satisfying some prescribed marginal conditions.

This question is the stationary version of the problem of characterizing functions $S(x, y)$ satisfying

$$S(x, y) = \mathbb{P}\{x, y \in X\} = \mathbb{E}\, \mathbf{1}_X(x)\, \mathbf{1}_X(y).$$

The right-hand term is the second order moment of the random indicator field $x \mapsto \mathbf{1}_X(x)$, which justifies the term of *realizability problems*, concerned with the existence of a positive measure satisfying some prescribed moment conditions.

We can view the S_2 problem as a truncated version of the general *moment problem* that deals with the existence of a process for which all moments are prescribed. The main difficulty in

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considering only the moments up to some finite order is that this sequence of moments does not uniquely determine the possible solution. The appearance of second order realizability problems for random sets goes back to the 1950s; see, for example, [24] in the field of telecommunications. There are applications in materials science and geostatistics, and marginal problems in general are present under different occurrences in fields as various as quantum mechanics, computer science, or game theory; see [10] and the references therein.

Reconstruction of heterogeneous materials from a knowledge of limited microstructural information (a set of lower-order correlation functions) is a crucial issue in many applications. Finding a constructive solution to the realizability problem described above should allow us to test whether an estimated covariance indeed corresponds to a random structure, and propose an adapted reconstruction procedure. Studying this problem can serve many other purposes, especially in spatial modeling, where one needs to know the necessary admissibility conditions to propose new covariance models. A series of works by Torquato and his coauthors in the field of materials science gathers known necessary conditions and illustrates them for many 2D and 3D theoretical models, along with reconstruction procedures; see [15] and the survey [31, Section 2.2] and the references therein. This question was developed in parallel in the field of geostatistics, where some authors do not tackle this issue directly, but address the realizability problem within some particular classes of models, e.g. Gaussian, mosaic, or Boolean models; see [5], [8], [20], [21].

A related question concerns the *specific covariogram* of a stationary random set X, defined for all nonempty bounded open sets $U \subset \mathbb{R}^d$ by

$$\gamma_X^{\mathrm{s}}(y) = \frac{\mathbb{E}\mathcal{L}^d\{X \cap (y+X) \cap U\}}{\mathcal{L}^d(U)} = \mathbb{E}\mathcal{L}^d\{X \cap (y+X) \cap (0,1)^d\},$$

where $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$. The associated realizability problem, which consists of determining whether there exists a stationary random set X whose specific covariogram is a given function, is the *(specific) covariogram realizability problem*. Note that from a straightforward Fubini argument it follows that for any stationary random closed set (RACS) X,

$$\gamma_X^{\mathrm{s}}(y) = \int_{(0,1)^d} \mathbb{P}\{x \in X, x-y \in X\}\,\mathrm{d}x = S_2(-y) = S_2(y), \tag{2}$$

and, thus, the S_2 realizability problem and the specific covariogram problem are fundamentally the same.

Our main result provides an abstract and fully rigorous characterization of this problem for random measurable sets (RAMS) having locally finite mean perimeter. Furthermore, in the restrictive one-dimensional case ($d = 1$), results can be passed on to the classical framework of RACS. It will become clear in this paper why the covariogram approach in the framework of RAMS is more adapted to a rigorous mathematical study. RAMS are an alternative to the classical RACS in stochastic geometry and geostatistics as they provide a weaker framework allowing us to manipulate more irregular functionals, such as the perimeter. We therefore use the illustration provided by the S_2 problem to advocate the use of RAMS for solving theoretical problems of a geometric nature. Along the way, we extend the theory of RAMS, and, in particular, the local approximation of the perimeter by local covariograms. We remark that the framework of RAMS is related to that of 'random sets of finite perimeter' proposed recently by Rataj [27]. However, it is less restrictive since RAMS do not necessarily have a finite perimeter.

Our main result uses a fundamental relation between the Lipschitz property of the covariogram function of a random set, and the finiteness of its mean variational perimeter, unveiled in [11]. As in [19, Theorem 3.1] concerning point processes, we prove that the realizability of a given function $S_2 \colon \mathbb{R}^d \to \mathbb{R}$ can be characterized by two independent conditions: a positivity condition, and a regularity condition, namely the Lipschitz property of S_2. The positivity condition deals with the positivity of a linear operator extending S_2 on an appropriate space, and is of combinatorial nature. The proof of this main result relies on a theorem dealing with positive operators on a noncompact space recently derived in [19] to treat realizability problems for point processes. This general method therefore proves its versatility here by being applied to the framework of random sets in a very similar manner.

Checking whether S_2 satisfies the positivity condition is completely distinct from the concerns of this paper. It is a difficult problem that has a long history. It is more or less implicit in many articles, and was, to the best of the authors' knowledge, first addressed directly by Shepp [29], later on by Matheron [23], and more recently in [18] and [26]. It is equivalent to the study of the *correlation polytope* in the discrete geometry literature; see, for example, the works of Deza and Laurent [7]. Still, a deep mathematical understanding of the problem remains out of reach.

The plan of the paper is as follows. In the remainder of Section 1 we provide a quick overview of the mathematical objects involved here, namely RAMS, positivity, perimeter, and realizability problems, and we also state the main result of this paper, namely the specific covariogram realizability problem for stationary RAMS with finite specific perimeter. In Section 2, we develop the theory of RAMS, define different notions of perimeter, and explore the relations with RACS, while Section 3 is devoted to the local covariogram functional and its use for perimeter approximation. In Section 4, we provide the precise statement and the proof of the main result. We also show that our main result extends to the framework of one-dimensional stationary RACS.

1.2. RAMS and the variational perimeter

Details about RAMS are presented in Section 2, and here we provide the essential notation for stating the results. Call $\mathcal{M}$ the class of Lebesgue measurable sets of $\mathbb{R}^d$. A RAMS X is a random variable taking values in $\mathcal{M}$ endowed with the Borel σ-algebra induced by the local convergence in measure, which corresponds to the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-topology for the indicator functions; see Section 2.1 for details. We remark that under this topology, one is bound to identify two sets A and B lying within the same Lebesgue class (that is, such that their symmetric difference $A \Delta B$ is Lebesgue-negligible), and we indeed perform this identification on $\mathcal{M}$. Furthermore, say that a RAMS is *stationary* if its law is invariant under translations of $\mathbb{R}^d$.

One geometric notion that can be extended to RAMS is that of perimeter. For a deterministic measurable set A, the perimeter of A in an open set $U \subset \mathbb{R}^d$ is defined as the variation of the indicator function $\mathbf{1}_A$ in U, that is,

$$\mathrm{Per}(A; U) = \sup\left\{ \int_U \mathbf{1}_A(x) \operatorname{div} \varphi(x) \, \mathrm{d}x \colon \ \varphi \in \mathcal{C}^1_{\mathrm{c}}(U, \mathbb{R}^d), \ \|\varphi(x)\|_2 \le 1 \text{ for all } x \right\}, \tag{3}$$

where $\mathcal{C}^1_{\mathrm{c}}(U, \mathbb{R}^d)$ denotes the set of continuously differentiable functions $\varphi \colon U \to \mathbb{R}^d$ with compact support and $\|\cdot\|_2$ is the Euclidean norm [2]; see Section 2.2 for a discussion and some properties of variational perimeters. If X is a RAMS, then, for all open sets $U \subset \mathbb{R}^d$, $\mathrm{Per}(X; U)$ is a well-defined random variable because the map $A \mapsto \mathrm{Per}(A; U)$ is lower semi-continuous for the local convergence in measure in $\mathbb{R}^d$ [2, Proposition 3.38]. Besides, if X is stationary then $U \mapsto \mathbb{E}\{\mathrm{Per}(X; U)\}$ extends into a translation-invariant measure, and, thus, proportional to the Lebesgue measure. We call the *specific perimeter* or (*specific variation* [12]) of X the constant of proportionality that will be denoted by $\mathrm{Per}^{\mathrm{s}}(X)$ and is given by $\mathrm{Per}^{\mathrm{s}}(X) = \mathbb{E}\mathrm{Per}\{X; (0, 1)^d\}$. We refer to [12] for the computation of the specific perimeter of some classical random set models (Boolean models and Gaussian level sets).

1.3. Covariogram realizability problems

For a deterministic set A, we call the *local covariogram* of A the map

$$\delta_{y;W}(A) = \mathcal{L}^d(A \cap (y + A) \cap W), \qquad (y, W) \in \mathbb{R}^d \times \mathcal{W}, \tag{4}$$

where $\mathcal{W}$ denotes the set of *observation windows* defined by

$$\mathcal{W} = \{W \subset \mathbb{R}^d \text{ bounded open set such that } \mathcal{L}^d(\partial W) = 0\}.$$

Given a RAMS X, we denote by $\gamma_X(y; W) = \mathbb{E}\delta_{y;W}\{X\}$ the *(mean) local covariogram* of X. If X is stationary then the map $W \mapsto \gamma_X(y; W)$ is translation invariant and extends into a measure proportional to the Lebesgue measure. Hence, we call it the *specific covariogram* of X and denote it by the map $y \mapsto \gamma_X^{\mathrm{s}}(y)$, such that $\gamma_X(y; W) = \mathbb{E}\delta_{y;W}\{X\} = \gamma_X^{\mathrm{s}}(y)\mathcal{L}^d(W)$. Note that, we simply have $\gamma_X^{\mathrm{s}}(y) = \gamma_X(y, (0, 1)^d)$.

In this paper we are interested in the specific covariogram realizability problem. Given a function $S_2 : \mathbb{R}^d \to \mathbb{R}$, does there exist a stationary RAMS $X \in \mathcal{M}$ such that $S_2(y) = \gamma_X^{\mathrm{s}}(y)$ for all $y \in \mathbb{R}^d$?

The specific covariogram candidate S_2 has to verify some necessary structural condition to be realisable.

Definition 1. (*Covariogram admissible functions.*) A function $\gamma : \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}$ is said to be *$\mathcal{M}$-local covariogram admissible*, or just *admissible*, if for all 5-tuples $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$,

$$\left[\text{for all } A \in \mathcal{M},\ c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}(A) \geq 0\right] \Longrightarrow c + \sum_{i=1}^{q} a_i \gamma(y_i; W_i) \geq 0.$$

A function $S_2 : \mathbb{R}^d \to \mathbb{R}$ is said to be *$\mathcal{M}$-specific covariogram admissible*, or just *admissible*, if the function $(y; W) \mapsto S_2(y)\mathcal{L}^d(W)$ is *$\mathcal{M}$-local covariogram admissible*.

It is an immediate consequence of the positivity and linearity of the mathematical expectation that a realisable S_2 function is necessarily admissible. Checking whether a given S_2 is admissible, a problem of combinatorial nature, is difficult. It will not be addressed here, but as emphasized in (2), it is directly related to the positivity problem for two-point covering functions, which has been studied in numerous works; see [7], [18], [23], [26], [29], and the references therein. We remark that being admissible is a strong constraint on S_2 that conveys the usual properties of covariogram functions, and in particular $S_2(y) \geq 0$ for all $y \in \mathbb{R}^d$ (since for all $y \in \mathbb{R}^d$, $W \in \mathcal{W}$ and $A \in \mathcal{M}$, $\delta_{y;W}(A) \geq 0$).

In general, the admissibility of S_2 is not sufficient for S_2 to be realisable. Consider the linear operator Φ,

$$\Phi\left(c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}\right) = c + \sum_{i=1}^{q} a_i S_2(y_i)\mathcal{L}^d(W_i) \tag{5}$$

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on the subspace of functionals on $\mathcal{M}$ generated by the constant functions and the covariogram evaluations $A \mapsto \delta_{y;W}(A)$, $y \in \mathbb{R}^d$, $W \in \mathcal{W}$. The realizability of S_2 corresponds to the existence of a probability measure μ on $\mathcal{M}$ representing Φ, i.e. such that $\Phi(g) = \int_{\mathcal{M}} g \, \mathrm{d}\mu$ for g in the aforementioned subspace. In a noncompact space such as $\mathcal{M}$, the positivity of Φ, i.e. the admissibility of S_2, is not sufficient to represent it by a probability measure, as the σ-additivity is also needed.

It was shown in [19] that in such noncompact frameworks, the realizability problem would be better accompanied by an additional regularity condition formulated in terms of a function called a *regularity modulus*; see Section 4 for details. The perimeter function fulfills this role here, mostly because it can be approximated by linear combinations of covariograms, and has compact level sets. The well-posed realizability problem with regularity condition we consider here deals with the existence of a stationary RAMS $X \in \mathcal{M}$ such that

$$S_2(y) = \gamma_X^{\mathrm{s}}(y), \quad y \in \mathbb{R}^d, \qquad \mathrm{Per}^{\mathrm{s}}(X) = \mathbb{E}\mathrm{Per}\{X; (0, 1)^d\} < \infty.$$

The main result of this paper is the following theorem.

Theorem 1. *Let $S_2 : \mathbb{R}^d \mapsto \mathbb{R}$ be a function. Then S_2 is the specific covariogram of a stationary RAMS $X \in \mathcal{M}$ such that* $\mathrm{Per}^{\mathrm{s}}(X) < \infty$ *if and only if S_2 is admissible and Lipschitz at* 0 *along the d canonical directions.*

This result is analogous to the one obtained in [19] for point processes, since the realizability condition is shown to be a positivity condition plus a regularity condition, namely the Lipschitz property of S_2. As already discussed, a realisable function S_2 is necessarily admissible. Besides, extending results from [11], we show that a stationary RAMS X has a finite specific perimeter if and only if its specific covariogram γ_X^{s} is Lipschitz, and we obtain an explicit relation between the Lipschitz constant of S_2 and the specific perimeter; see Proposition 7. Hence, the direct implication of Theorem 1 is somewhat straightforward. The real difficulty consists in proving the converse implication. To do so we adapt the techniques of [19] to our context which involves several technicalities regarding the approximation of the perimeter by linear combinations of local covariogram functionals. We first establish the counterpart of Theorem 1 for the realizability of local covariogram function $\gamma : \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}$ (see Theorem 3) and we then extend this result to the case of the specific covariogram of stationary RAMS; see Theorem 5.

In addition, we study the links between RAMS and the more usual framework of RACS, which *in fine* enables us to obtain a result analogous to Theorem 1 for RACS of the real line (see Theorem 7); such a result was out of reach with previously developed methods.

2. RAMS

2.1. Definition of RAMS

RAMS are defined as random variables taking value in the set $\mathcal{M}$ of Lebesgue (classes of) sets of $\mathbb{R}^d$ endowed with the Borel σ-algebra $\mathcal{B}(\mathcal{M})$ induced by the natural topology, the so-called local convergence in measure. We recall that a sequence of measurable sets $(A_n)_{n\in\mathbb{N}}$ locally converges in measure to a measurable set A if for all bounded open sets $U \subset \mathbb{R}^d$, the sequence $\mathcal{L}^d((A_n \Delta A) \cap U)$ tends to 0, where Δ denotes the symmetric difference. The local convergence in measure simply corresponds to the convergence of the indicator functions $\mathbf{1}_{A_n}$ towards $\mathbf{1}_A$ in the space of locally integrable functions $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, and, consequently, $\mathcal{M}$ is a complete metrizable space. This is a consequence of the facts that $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a complete metrizable space and that the set of indicator functions is closed in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

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Definition 2. (*RAMS.*) A RAMS X is a measurable map $X : \omega \mapsto X(\omega)$ from $(\Omega, \mathcal{A})$ to $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, where $\mathcal{B}(\mathcal{M})$ denotes the Borel σ-algebra induced by the local convergence in measure.

Note that if X is a RAMS then $\omega \mapsto \mathbf{1}_{X(\omega)}$ is a random locally integrable function. This concept of random measurable (class of) set(s) is not standard, and, to the best of the authors' knowledge, it was first introduced in [30] for random subsets of the real interval [0, 1], as mentioned in [25].

In the remaining part of this section, we will discuss the link between RAMS and other classical random objects, namely random Radon measures, measurable subsets of $\Omega \times \mathbb{R}^d$, and RACS.

2.1.1. *Random Radon measures associated with RAMS.* Following the usual construction of random objects, a random Radon measure is defined as a measurable function from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to the space $\boldsymbol{M}^+$ of positive Radon measures on $\mathbb{R}^d$ equipped with the smallest σ-algebra for which the evaluation maps $\mu \mapsto \mu(B)$, $B \in \mathcal{B}(\mathbb{R}^d)$ relatively compact, are measurable; see, e.g. [6], [16], [28]. Any RAMS $X \subset \mathbb{R}^d$ canonically defines a random Radon measure that is the restriction to X of the Lebesgue measure, i.e. $B \mapsto \mathcal{L}^d(X \cap B)$ for Borel set $B \in \mathcal{B}(\mathbb{R}^d)$. The measurability of this restriction results from the observation that, for all $B \in \mathcal{B}(\mathbb{R}^d)$, the map $f \mapsto \int_B f(x)\,\mathrm{d}x$ is measurable for the L^1_{loc}-topology.

2.1.2. *Existence of a measurable graph representative.* For a RAMS $X : \Omega \to \mathcal{M}$, one can study the measurability properties of the graph $Y = \{(\omega, x) : x \in X(\omega)\} \subset \Omega \times \mathbb{R}^d$.

Definition 3. (*Measurable graph representatives.*) A subset $Y \subset \Omega \times \mathbb{R}^d$ is a *measurable graph representative* of a RAMS X if

(i) Y is a measurable subset of $\Omega \times \mathbb{R}^d$ (i.e. Y belongs to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$),

(ii) for almost every (a.e.) $\omega \in \Omega$, the ω-section $Y(\omega) = \{x \in \mathbb{R}^d : (\omega, x) \in Y\}$ is equivalent in measure to $X(\omega)$, i.e. $\mathcal{L}^d(Y(\omega) \Delta X(\omega)) = 0$.

Proposition 1. *Any measurable set $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$ canonically defines a RAMS by considering the Lebesgue class of its ω-sections:*

$$\omega \mapsto Y(\omega) = \{x \in \mathbb{R}^d : (\omega, x) \in Y\}.$$

Conversely, any RAMS X admits measurable graph representatives $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$.

Proof. The first point is trivial. Let us prove the second point. Consider the random Radon measure μ associated to X, i.e.

$$\mu(\omega, B) = \mathcal{L}^d(X(\omega) \cap B) = \int_B \mathbf{1}_{X(\omega)}(x)\,\mathrm{d}x.$$

By construction, this random Radon measure is absolutely continuous with respect to the Lebesgue measure. Then, according to the Radon–Nikodym theorem for random measures (see Theorem 8 in Appendix A), there exists a jointly measurable map $g : (\Omega \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)) \to \mathbb{R}$ such that for all $\omega \in \Omega$,

$$\mu(\omega, B) = \int_B g(\omega, x)\,\mathrm{d}x, \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

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Hence, for all $\omega \in \Omega$, $\mathbf{1}_{X(\omega)}(\cdot)$ and $g(\omega, \cdot)$ are both Radon–Nikodym derivatives of $\mu(\omega, \cdot)$ and, thus, are equal almost everywhere. In particular, for a.e. $x \in \mathbb{R}^d$, $g(\omega, x) \in \{0, 1\}$. Consequently, the function $(\omega, x) \mapsto \mathbf{1}_{\{g(\omega,x)=1\}}$ is also jointly measurable and is a Radon–Nikodym derivative of $\mu(\omega, \cdot)$ for all $\omega \in \Omega$, and, thus, the set

$$Y = \{(\omega, x) \in \Omega \times \mathbb{R}^d \colon g(\omega, x) = 1\}$$

is a measurable graph representative of X.

2.1.3. *RAMS and RACS.* Recall that $(\Omega, \mathcal{A}, \mathbb{P})$ denotes our probability space. Let $\mathcal{F} = \mathcal{F}(\mathbb{R}^d)$ be the set of all closed subsets of $\mathbb{R}^d$. Following [25, Definition 1.1] a RACS is defined as follows.

Definition 4. (*RACS.*) A map $Z \colon \Omega \to \mathcal{F}$ is called a RACS if for every compact set $K \subset \mathbb{R}^d$, $\{\omega \colon Z(\omega) \cap K \neq \varnothing\} \in \mathcal{A}$.

The framework of RACS is standard in stochastic geometry [22], [25]. Let us reproduce a result of Himmelberg that allows us to link the different notions of random sets; see [25, Theorem 2.3] or the original paper [13] for the complete theorem.

Theorem 2. (Himmelberg.) *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $Z \colon \Omega \to Z(\omega) \in \mathcal{F}$ be a map taking values into the set of closed subsets of $\mathbb{R}^d$. Consider the two following assertions:*

(i) *$\{\omega : \ Z \cap F \neq \varnothing\} \in \mathcal{A}$ for every closed set $F \subset \mathbb{R}^d$;*

(ii) *the graph of Z, i.e. the set $\{(\omega, x) \in \Omega \times \mathbb{R}^d \colon x \in Z(\omega)\}$, belongs to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$.*

Then the implication that (i) implies (ii) is always true, and if the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is complete, we have the equivalence (i) if and only if (ii).

In view of our definitions for random sets, Himmelberg's theorem can be rephrased in the following terms.

Proposition 2. (RACS and closed RAMS.) (i) *Any RACS Z has a measurable graph $Y = \{(\omega, x) \in \Omega \times \mathbb{R}^d \colon x \in Z(\omega)\}$, and, thus, also defines a unique RAMS.*

(ii) *Suppose that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is complete. Let $Y \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)$ be a measurable set such that for all $\omega \in \Omega$, its ω-section $Y(\omega) = \{x \in \mathbb{R}^d] \colon (\omega, x) \in Y\}$ is a closed subset of $\mathbb{R}^d$. Then, the map $\omega \mapsto Y(\omega)$ defines a RACS.*

2.2. RAMS of finite perimeter

For a closed set F, the perimeter is generally defined by the $(d-1)$-dimensional measure of the topological boundary, i.e. $\mathcal{H}^{d-1}(\partial F)$. This definition is not relevant for a measurable set $A \subset \mathbb{R}^d$, in the sense that the value $\mathcal{H}^{d-1}(\partial A)$ strongly depends on the representative of A within its Lebesgue class. The proper notion of perimeter for measurable sets is the variational perimeter that defines the perimeter as the variation of the indicator function of the set. An important feature of the variational perimeter is that it is lower semi-continuous for the convergence in measure, while the functional $F \mapsto \mathcal{H}^{d-1}(\partial F)$ is not lower semi-continuous on the set of closed sets $\mathcal{F}$ endowed with the hit-or-miss topology. This is a key aspect for this paper since it allows us to consider the variational perimeter as a regularity modulus for realisability problems in following the framework of [19].

2.2.1. *Variational perimeters.* Let U be an open subset of $\mathbb{R}^d$. Recall that the *(variational) perimeter* $\mathrm{Per}(A; U)$ of a measurable set $A \in \mathcal{M}$ in the open set U is defined by (3). Denote by S^{d-1} the unit sphere of $\mathbb{R}^d$. Closely related to the perimeter, we also define the *directional variation* in the direction $u \in S^{d-1}$ of A in U by (see [2, Section 3.11])

$$V_u(A; U) = \sup\left\{\int_U \mathbf{1}_A(x)\langle \nabla\varphi(x), u\rangle \,\mathrm{d}x : \varphi \in \mathcal{C}^1_{\mathrm{c}}(U, \mathbb{R}),\ |\varphi(x)| \le 1 \text{ for all } x\right\}.$$

For technical reasons, we also consider the *anisotropic perimeter*

$$A \mapsto \mathrm{Per}_{\boldsymbol{B}}(A; U) = \sum_{j=1}^{d} V_{e_j}(A; U),$$

which adds up the directional variations along the d directions of the canonical basis $\boldsymbol{B} = \{e_1, \dots, e_d\}$. In geometric measure theory, the functional $A \mapsto \mathrm{Per}_{\boldsymbol{B}}(A; U)$ is described as the anisotropic perimeter associated with the anisotropy function $x \mapsto \|x\|_\infty$; see, e.g. [4] and the references therein. Indeed, we can easily see that

$$\mathrm{Per}_{\boldsymbol{B}}(A; U) = \sup\left\{\int_U \mathbf{1}_A(x) \operatorname{div}\varphi(x) \,\mathrm{d}x : \varphi \in \mathcal{C}^1_{\mathrm{c}}(U, \mathbb{R}^d),\ \|\varphi(x)\|_\infty \le 1 \text{ for all } x\right\}.$$

Hence, the only difference between the variational definition of the isotropic perimeter $\mathrm{Per}(A; U)$ and the one of the anisotropic perimeter $\mathrm{Per}_{\boldsymbol{B}}(A; U)$ is that the test functions φ take values in the ℓ_2-unit ball B_d for the former whereas they take values in the ℓ_∞-unit ball $[-1, 1]^d$ for the latter. The set inclusions $B_d \subset [-1, 1]^d \subset \sqrt{d}B_d$ lead to the tight inequalities

$$\mathrm{Per}(A; U) \le \mathrm{Per}_{\boldsymbol{B}}(A; U) \le \sqrt{d}\,\mathrm{Per}(A; U).$$

Consequently, a set A has a finite perimeter $\mathrm{Per}(A; U)$ in U if and only if it has a finite anisotropic perimeter $\mathrm{Per}_{\boldsymbol{B}}(A; U)$. Let us mention that this equivalence is not true when considering only one directional variation $V_u(A; U)$. We say that a measurable set $A \subset \mathbb{R}^d$ has *locally finite perimeter* if A has a finite perimeter $\mathrm{Per}(A; U)$ in all bounded open sets $U \subset \mathbb{R}^d$.

To conclude, let us mention that if X is a RAMS then $\mathrm{Per}(X; U)$, $\mathrm{Per}_{\boldsymbol{B}}(X; U)$, and $V_u(X; U)$, $u \in S^{d-1}$ are well-defined random variables since the maps $A \mapsto \mathrm{Per}(A; U)$, $A \mapsto \mathrm{Per}_{\boldsymbol{B}}(A; U)$, and $A \mapsto V_u(A; U)$ are lower semi-continuous for the convergence in measure [2]. Consequently, we say that a RAMS X has almost sure (a.s.) finite (respectively locally finite) perimeter in U if the random variable $\mathrm{Per}(X; U)$ is a.s. finite (respectively if, for all bounded open sets $V \subset U$, $\mathrm{Per}(X; V)$ is a.s. finite).

Remark 1. Rataj [27] recently proposed a framework for 'random sets of finite perimeter' that models random sets as random variables in the space of indicator functions of sets of finite perimeter endowed with the Borel σ-algebra induced by the strict convergence in the space of functions of bounded variation [2, Section 3.1]. Since this convergence induced the L^1-convergence of indicator functions, any 'random set of finite perimeter' X uniquely defines a RAMS X having a.s. finite perimeter. One advantage of the RAMS framework is that it is more general in the sense that it enables us to consider random sets that do not have finite perimeter; see, e.g. Corollary 1.

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2.2.2. *Closed representative of one-dimensional sets of finite perimeter.* Although the general geometric structure of sets of finite perimeter is well known (see [2, Section 3.5]), it necessitates involved notions from geometric measure theory (rectifiable sets, reduced and essential boundaries, etc.). However, when restricted to the case of one-dimensional sets of finite perimeter, all the complexity vanishes since subsets of $\mathbb{R}$ having finite perimeter all correspond to finite unions of nonempty and disjoint closed intervals.

More precisely, according to [2, Proposition 3.52], if a nonnegligible measurable set $A \subset \mathbb{R}$ has finite perimeter in an interval $(a, b) \subset \bar{\mathbb{R}}$, there exists an integer p and p pairwise disjoint nonempty and closed intervals $J_i = [a_{2i-1}, a_{2i}] \subset \bar{\mathbb{R}}$, with $a_1 < a_2 < \cdots < a_{2p}$, such that

- $A \cap (a, b)$ is equivalent in measure to the union $\bigcup_i J_i$,
- the perimeter of A in (a, b) is the number of interval endpoints belonging to (a, b),

$$\mathrm{Per}(A; (a, b)) = \#\{a_1, a_2, \ldots, a_{2p}\} \cap (a, b).$$

We remark that a set of the form $A = \bigcup_i [a_{2i-1}, a_{2i}]$ is closed, and that such a set satisfies the identity $\mathrm{Per}(A; (a, b)) = \mathcal{H}^0(\partial A \cap (a, b))$, where ∂A denotes the topological boundary of A and $\mathcal{H}^0$ is the Hausdorff measure of dimension 0 on $\mathbb{R}$ (i.e. the counting measure), while in the general case one only has $\mathrm{Per}(A; (a, b)) \leq \mathcal{H}^0(\partial A \cap (a, b))$ since A may contain isolated points.

In the general case, since $A \subset \mathbb{R}$ may have locally finite perimeter, then there exists a unique countable or finite family of closed and disjoint intervals $J_i = [a_{2i-1}, a_{2i}]$, $i \in I \subset \mathbb{Z}$, such that A is equivalent in measure to $\bigcup_{i \in I} J_i$, and for all bounded open intervals (a, b), $\mathrm{Per}(A; (a, b))$ is the number of interval endpoints belonging to (a, b).

Using both this observation and Proposition 2, we obtain the following proposition.

Proposition 3. *Suppose that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is complete. Let X be a RAMS of $\mathbb{R}$ that has a.s. locally finite perimeter. Then, there exists a RACS $Z \subset \mathbb{R}$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for all $a < b \in \mathbb{R}$,*

$$\mathcal{L}^1(X(\omega) \Delta Z(\omega)) = 0, \qquad \mathrm{Per}(X(\omega); (a, b)) = \mathcal{H}^0(\partial Z(\omega) \cap (a, b)).$$

Proof. First, we remark that a measurable set of finite perimeter $A \subset \mathbb{R}$ equivalent in measure to $\bigcup_{i \in I} [a_{2i-1}, a_{2i}]$ for some finite or countable index set $I \subset \mathbb{Z}$ has the Lebesgue density

$$D(x, A) = \lim_{r \to 0+} \frac{\mathcal{L}^1(A \cap (x - r, x + r))}{2r} = \begin{cases} 1 & \text{if } x \text{ is in some open interval } (a_{2i-1}, a_{2i}), \\ \frac{1}{2} & \text{if } x \text{ is an interval endpoint } a_{2i-1} \text{ or } a_{2i} \text{ for some } i \in I, \\ 0 & \text{if } x \notin \bigcup_{i \in I} [a_{2i-1}, a_{2i}]. \end{cases}$$

Let X be a RAMS of $\mathbb{R}$ that has a.s. locally finite perimeter. Let $\Omega' \in \mathcal{A}$ be a subset of Ω of probability 1 such that for all $\omega \in \Omega'$, X has locally finite perimeter. For all $\omega \in \Omega'$, the Lebesgue class $X(\omega)$ admits a representative that is the union of an at most countable family of nonempty and disjoint closed intervals. According to the above observation, for a fixed $\omega \in \Omega'$, the density $D(x, X(\omega))$ exists for all $x \in \mathbb{R}$, and the good representative of X is given

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by $\{x \in \mathbb{R}: D(x, X(\omega)) > 0\}$. Let $(r_n)_{n\in\mathbb{N}}$ be a positive sequence decreasing to 0, and define for all $\omega \in \Omega$,

$$g(\omega, x) = \limsup_{n\to+\infty} \frac{\mathcal{L}^1(X(\omega) \cap (x - r_n, x + r_n))}{2r_n}.$$

According to the proof of Theorem 8 and Proposition 1,

$$Y = \{(\omega, x) \in \Omega \times \mathbb{R}^d : g(\omega, x) > 0\}$$

is a measurable set that is a measurable graph representative of X. Besides, for a given $\omega \in \Omega'$, since $D(x, X(\omega))$ exists for all $x \in \mathbb{R}$,

$$D(x, X(\omega)) = g(\omega, x), \qquad x \in \mathbb{R}.$$

Hence, for all $\omega \in \Omega'$, the ω-section $Y(\omega)$ of Y is the union of an at most countable and locally finite family of nonempty and disjoint closed intervals, and in particular a closed set. Thus, by Proposition 2, the map $\omega \mapsto Y(\omega)$ defines a RACS.

2.2.3. *Non-closed RAMS in dimension $d > 1$.* In contrast to the one-dimensional case, in dimension $d > 1$ there exist measurable sets of finite perimeters that do not have closed representative in their Lebesgue class. A set A obtained as the union of an infinite family of open balls with small radii and with centers forming a dense subset of $[0, 1]^d$ is considered in [2, Example 3.53]. It has finite perimeter, finite measure $\mathcal{L}^d(A) < 1$, and is such that $\mathcal{L}^d(A \cap U) > 0$ for any open subset U of $[0, 1]^d$.

Such a set clearly has no closed representative, because if it had one, say F, then F would charge every open subset of $[0, 1]^d$, and therefore it would be dense in $[0, 1]^d$. Since F is closed, we would have $F = [0, 1]^d$, which contradicts $\mathcal{L}^d(F) = \mathcal{L}^d(A) < 1$.

3. Local covariogram and perimeter approximation

In this section we establish general properties of the local covariogram of a measurable set, as well as the mean local covariogram of a RAMS. A particular emphasis is given to the relation between the local perimeter of a set and the Lipschitz constant of its local covariogram in order to adapt the results of [11] to the local covariogram functional.

3.1. Definition and continuity

The local covariogram of a measurable set $A \in \mathcal{M}$ is defined in (4). We remark that for all $y \in \mathbb{R}^d$, and $W \in \mathcal{W}$,

$$\delta_{y;W}(A) = \delta_{y;W}(A \cap (W \cup (-y + W))), \quad A \in \mathcal{M}, \tag{6}$$

so that only the part of A included in the domain $W \cup (-y + W)$ has an influence on the value of $\delta_{y;W}(A)$, hence, local covariograms are indeed local. Before enunciating specific results of interest for our realisability problem, we prove that local covariograms are continuous for the local convergence in measure.

Proposition 4. (Continuity of local covariograms.) (i) *For all $A \in \mathcal{M}$ and $W \in \mathcal{W}$, the map $y \mapsto \delta_{y;W}(A)$ is uniformly continuous over $\mathbb{R}^d$.*

(ii) *Let $A \in \mathcal{M}$ and $y \in \mathbb{R}^d$. Then, for all $U, W \in \mathcal{W}$,*

$$|\delta_{y;U}(A) - \delta_{y;W}(A)| \le \mathcal{L}^d(U \Delta W).$$

In particular, the map $W \mapsto \delta_{y;W}(A)$ is continuous for the convergence in measure.

(iii) *Let $A, B \in \mathcal{M}$, and let $W \in \mathcal{W}$. Then, for all $y \in \mathbb{R}^d$,*

$$|\delta_{y;W}(A) - \delta_{y;W}(B)| \leq 2\mathcal{L}^d((A \Delta B) \cap (W \cup (-y + W))).$$

In particular, the map $A \mapsto \delta_{y;W}(A)$ is continuous for the local convergence in measure.

Proof. (i) The convolution interpretation for local covariograms yields

$$\delta_{y;W}(A) = \int_{\mathbb{R}^d} \mathbf{1}_{A\cap W}(x)\, \mathbf{1}_{-A}(y - x)\, \mathrm{d}x = \mathbf{1}_{A\cap W} * \mathbf{1}_{-A}(y).$$

Since $\mathbf{1}_{A\cap W} \in L^1(\mathbb{R}^d)$ and $\mathbf{1}_{-A} \in L^\infty(\mathbb{R}^d)$, the uniform continuity is ensured by the L^p-$L^{p'}$-convolution theorem; see, e.g. [14, Proposition 3.2].

(ii) Using the general inequality $|\mathcal{L}^d(A_1) - \mathcal{L}^d(A_2)| \leq \mathcal{L}^d(A_1 \Delta A_2)$, we obtain

$$|\delta_{y;U}(A) - \delta_{y;W}(A)| \leq \mathcal{L}^d((A \cap (A + y) \cap U)\Delta(A \cap (A + y) \cap W)) \leq \mathcal{L}^d(U \Delta W).$$

(iii) If A and B have finite Lebesgue measure,

$$\begin{aligned}
&|\delta_{y;W}(A) - \delta_{y;W}(B)| \\
&\quad = |\, \mathbf{1}_{A\cap W} * \mathbf{1}_{-A}(y) - \mathbf{1}_{B\cap W} * \mathbf{1}_{-B}(y)| \\
&\quad \leq |\, \mathbf{1}_{A\cap W} * \mathbf{1}_{-A}(y) - \mathbf{1}_{A\cap W} * \mathbf{1}_{-B}(y) + \mathbf{1}_{A\cap W} * \mathbf{1}_{-B}(y) - \mathbf{1}_{B\cap W} * \mathbf{1}_{-B}(y)| \\
&\quad \leq |\, \mathbf{1}_{A\cap W} *(\mathbf{1}_{-A} - \mathbf{1}_{-B})(y)| + |(\mathbf{1}_{A\cap W} - \mathbf{1}_{B\cap W}) * \mathbf{1}_{-B}(y)| \\
&\quad \leq \|\, \mathbf{1}_{A\cap W} \|_\infty \|\, \mathbf{1}_{-A} - \mathbf{1}_{-B} \|_1 + \|\, \mathbf{1}_{A\cap W} - \mathbf{1}_{B\cap W} \|_1 \|\, \mathbf{1}_{-B} \|_\infty \\
&\quad \leq \mathcal{L}^d(A \Delta B) + \mathcal{L}^d((A \cap W)\Delta(B \cap W)) \\
&\quad \leq 2\mathcal{L}^d(A \Delta B).
\end{aligned}$$

The announced general inequality is obtained from (6) which ensures that we can replace A and B by $A \cap (W \cup (-y + W))$ and $B \cap (W \cup (-y + W))$ without changing the values of $\delta_{y;W}(A)$ and $\delta_{y;W}(B)$.

3.2. Local covariogram and anisotropic perimeter

As for the case of covariogram [11], difference quotients at 0 of local covariograms are related to the directional variations of the set A. This is clarified by the following results, where $V_u(f; U)$ denotes the directional variation of $f \in L^1(U)$ in U in the direction $u \in S^{d-1}$, i.e.

$$V_u(f; U) = \sup\left\{\int_U f(x)\langle\nabla\varphi(x), u\rangle\, \mathrm{d}x : \varphi \in \mathcal{C}^1_{\mathrm{c}}(U, \mathbb{R}),\ |\varphi(x)| \leq 1 \text{ for all } x\right\}.$$

Recall that by definition, for a set $A \in \mathcal{M}$, $V_u(A; U)$ stands for $V_u(\mathbf{1}_A; U))$, and that $A \ominus B$ denotes the Minkowski difference of two measurable sets A and B. The following proposition states a well-known result from the theory of functions of bounded variation that is fully proved in [12].

Proposition 5. *Let U be an open subset of $\mathbb{R}^d$ and $u \in S^{d-1}$. Then, for all functions $f \in L^1(U)$ and $\varepsilon \in \mathbb{R}$,*

$$\int_{U\ominus[0,\varepsilon u]} \frac{|f(x + \varepsilon u) - f(x)|}{|\varepsilon|}\, \mathrm{d}x \leq V_u(f; U),$$

where $[0, \varepsilon u]$ *denotes the segment* $\{t\varepsilon u \colon t \in [0,1]\}$*, and*

$$\lim_{\varepsilon\to 0}\int_{U\ominus[0,\varepsilon u]}\frac{|f(x+\varepsilon u)-f(x)|}{|\varepsilon|}\,\mathrm{d}x = V_u(f;U).$$

The next two propositions show that when f is the indicator function of a set A, the integral

$$\int_{U\ominus[0,\varepsilon u]}\frac{|f(x+\varepsilon u)-f(x)|}{|\varepsilon|}\,\mathrm{d}x$$

can be expressed as a linear combination of local covariograms $\delta_{y;W}(A)$. Since this linear combination will be central in the following results, we introduce the notation

$$\sigma_{u;W}(A) = \frac{1}{\|u\|}(\delta_{0;W\ominus[-u,0]}(A) - \delta_{u;W\ominus[-u,0]}(A) + \delta_{0;W\ominus[0,u]}(A) - \delta_{-u;W\ominus[0,u]}(A))$$

for any $A \in \mathcal{M}$, $u \neq 0$, and $W \in \mathcal{W}$. We remark that for $W \in \mathcal{W}$, $y \in \mathbb{R}^d$, $A \in \mathcal{M}(\mathbb{R}^d)$,

$$\delta_{0;W}(A) - \delta_{y;W}(A) = \mathcal{L}^d(A\cap W) - \mathcal{L}^d(A\cap(y+A)\cap W) = \mathcal{L}^d((A\setminus(y+A))\cap W). \quad (7)$$

Proposition 6. (Local covariogram and anisotropic perimeter.) *For all* $A \in \mathcal{M}$*,* $W \in \mathcal{W}$*,* $\varepsilon \in \mathbb{R}$*, and* $u \in S^{d-1}$*,*

$$0 \le \sigma_{\varepsilon u;W}(A) \le V_u(A;W), \qquad \lim_{\varepsilon\to 0}\sigma_{\varepsilon u;W}(A) = V_u(A;W). \quad (8)$$

When summing along the d directions of the canonical basis $\boldsymbol{B} = \{e_1, e_2, \dots, e_d\}$*, we obtain similar results for the anisotropic perimeter, i.e. for all* $A \in \mathcal{M}$ *and* $\varepsilon \in \mathbb{R}$*,*

$$0 \le \sum_{j=1}^{d}\sigma_{\varepsilon e_j;W}(A) \le \mathrm{Per}_{\boldsymbol{B}}(A;W), \qquad \lim_{\varepsilon\to 0}\sum_{j=1}^{d}\sigma_{\varepsilon e_j;W}(A) = \mathrm{Per}_{\boldsymbol{B}}(A;W).$$

Proof. These inequalities are immediate from Proposition 5, and the equality

$$\int_{W\ominus[0,\varepsilon u]}|\mathbf{1}_A(x+\varepsilon u) - \mathbf{1}_A(x)|\,\mathrm{d}x = |\varepsilon|\sigma_{\varepsilon u;W}(A)$$

holds for all $A \in \mathcal{M}$, $W \in \mathcal{W}$, $u \in S^{d-1}$, and $\varepsilon \in \mathbb{R}$. Indeed,

$$\begin{aligned}\int_{W\ominus[0,\varepsilon u]}|\mathbf{1}_A(x+\varepsilon u) - \mathbf{1}_A(x)|\,\mathrm{d}x &= \int_{W\ominus[0,\varepsilon u]}|\mathbf{1}_{-\varepsilon u+A}(x) - \mathbf{1}_A(x)|\,\mathrm{d}x\\ &= \mathcal{L}^d(((-\varepsilon u + A)\Delta A)\cap(W\ominus[0,\varepsilon u]))\\ &= \mathcal{L}^d(((-\varepsilon u + A)\setminus A)\cap(W\ominus[0,\varepsilon u]))\\ &\quad + \mathcal{L}^d((A\setminus(-\varepsilon u + A))\cap(W\ominus[0,\varepsilon u])).\end{aligned}$$

Applying the translation by vector, εu yields

$$\mathcal{L}^d(((-\varepsilon u + A)\setminus A)\cap(W\ominus[0,\varepsilon u])) = \mathcal{L}^d((A\setminus(\varepsilon u + A))\cap(\varepsilon u + (W\ominus[0,\varepsilon u]))).$$

We remark that $\varepsilon u + (W\ominus[0,\varepsilon u]) = W\ominus[-\varepsilon u, 0]$ and, thus, using (7),

$$\begin{aligned}&\int_{W\ominus[0,\varepsilon u]}|\mathbf{1}_A(x+\varepsilon u) - \mathbf{1}_A(x)|\,\mathrm{d}x\\ &\quad = \mathcal{L}^d((A\setminus(\varepsilon u + A))\cap(W\ominus[-\varepsilon u,0])) + \mathcal{L}^d((A\setminus(-\varepsilon u + A))\cap(W\ominus[0,\varepsilon u]))\\ &\quad = \delta_{0;W\ominus[-\varepsilon u,0]}(A) - \delta_{\varepsilon u;W\ominus[-\varepsilon u,0]}(A) + \delta_{0;W\ominus[0,\varepsilon u]}(A) - \delta_{-\varepsilon u;W\ominus[0,\varepsilon u]}(A).\end{aligned}$$

We turn to the counterpart of Proposition 6 for mean local covariograms of RAMS. For a RAMS X, γ_X denotes the *(mean) local covariogram* of the RAMS X defined by $\gamma_X(y; W) = \mathbb{E}\delta_{y;W}\{X\}$, $y \in \mathbb{R}^d$, $W \in \mathcal{W}$, and define similarly $\sigma_X(u; W) = \mathbb{E}\sigma_{u;W}\{X\}$.

Corollary 1. *Let X be a RAMS. Then, for all $W \in \mathcal{W}$ and $u \in S^{d-1}$,*

$$\sigma_X(\varepsilon u; W) \le \mathbb{E}V_u\{X; W\}, \qquad \mathbb{E}V_u\{X; W\} = \lim_{\varepsilon \to 0} \sigma_X(\varepsilon u; W).$$

Proof. This is straightforward from (8). If $\mathbb{E}V_u\{X; W\} < \infty$, apply the Lebesgue theorem with the a.s. convergence and domination given by (8), while if $\mathbb{E}V_u\{X; W\} = \infty$, apply Fatou's lemma.

We turn to similar results for stationary RAMS. First, recall that if X is a stationary RAMS, the specific covariogram of X is defined by $\gamma_X^{\mathrm{s}}(y) = \gamma_X(y; (0, 1)^d) \in [0, 1]$. By analogy with the specific perimeter $\mathrm{Per}^{\mathrm{s}}(X) = \mathbb{E}\mathrm{Per}\{X; (0, 1)^d\}$, the *specific anisotropic perimeter* of X is defined by $\mathrm{Per}_{\boldsymbol{B}}^{\mathrm{s}}(X) = \mathbb{E}\mathrm{Per}_{\boldsymbol{B}}\{X; (0, 1)^d\} \in [0, +\infty]$ and for all $u \in S^{d-1}$, the *specific variation* of X in direction u is given by $V_u^{\mathrm{s}}(X) = \mathbb{E}V_u\{X; (0, 1)^d\}$.

For a function $F\colon \mathbb{R}^d \to \mathbb{R}$, define the Lipschitz constant in the jth direction at $y \in \mathbb{R}^d$ by

$$\mathrm{Lip}_j(F, y) = \sup_{t \in \mathbb{R}} \frac{|F(y + te_j) - F(y)|}{|t|},$$

and denote $\mathrm{Lip}_j(F) = \sup_{y \in \mathbb{R}^d} \mathrm{Lip}_j(F, y)$. Note that a function F is Lipschitz in the usual sense if and only each constant $\mathrm{Lip}_j(F)$ is finite for $j = 1, \dots, d$.

Proposition 7. *Let X be a stationary RAMS and let γ_X^{s} be its specific covariogram. Then γ_X^{s} is even and, for all $y, z \in \mathbb{R}^d$,*

$$|\gamma_X^{\mathrm{s}}(y) - \gamma_X^{\mathrm{s}}(z)| \le \gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(y - z).$$

In particular, γ_X^{s} is Lipschitz over $\mathbb{R}^d$ if and only if γ_X^{s} is Lipschitz at 0. Besides, for all $j \in \{1, \dots, d\}$,

$$\frac{\gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(\varepsilon e_j)}{|\varepsilon|} \le \frac{1}{2} V_{e_j}^{\mathrm{s}}(X), \qquad \varepsilon \ne 0,$$

and

$$\mathrm{Lip}_j(\gamma_X^{\mathrm{s}}) = \mathrm{Lip}_j(\gamma_X^{\mathrm{s}}, 0) = \lim_{\varepsilon \to 0} \frac{\gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(\varepsilon e_j)}{|\varepsilon|} = \frac{1}{2} V_{e_j}^{\mathrm{s}}(X).$$

The proof of the proposition is an adaptation of similar results for covariogram functions [11]. We first state a lemma regarding local covariogram of deterministic sets.

Lemma 1. *For all $y, z \in \mathbb{R}^d$, $W \in \mathcal{W}$, and $A \in \mathcal{M}$,*

$$\delta_{y;W}(A) - \delta_{z;W}(A) \le \delta_{0;-y+W}(A) - \delta_{z-y;-y+W}(A). \tag{9}$$

Proof. We have

$$\begin{aligned}
\delta_{y;W}(A) - \delta_{z;W}(A) &= \mathcal{L}^d(A \cap (y + A) \cap W) - \mathcal{L}^d(A \cap (z + A) \cap W) \\
&\le \mathcal{L}^d((A \cap (y + A) \cap W) \setminus (A \cap (z + A) \cap W)) \\
&\le \mathcal{L}^d(((y + A) \cap W) \setminus ((z + A) \cap W)) \\
&\le \mathcal{L}^d((y + A) \cap W) - \mathcal{L}^d((y + A) \cap (z + A) \cap W) \\
&\le \mathcal{L}^d(A \cap (-y + W)) - \mathcal{L}^d(A \cap (z - y + A) \cap (-y + W)) \\
&\le \delta_{0;-y+W}(A) - \delta_{z-y;-y+W}(A).
\end{aligned}$$

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Proof of Proposition 7. Let us first check that γ_X^{s} is even. For all $y \in \mathbb{R}^d$,

$$\begin{aligned}\gamma_X^{\mathrm{s}}(-y) &= \mathbb{E}\{\mathcal{L}^d(X \cap (-y + X) \cap (0,1)^d\} \\ &= \mathbb{E}\{\mathcal{L}^d((y+X) \cap X \cap (y + (0,1)^d)\} \\ &= \gamma_X^{\mathrm{s}}(y).\end{aligned}$$

Let us turn to the inequality. As a direct consequence of (9),

$$\gamma_X(y; W) - \gamma_X(z; W) \le \gamma_X(0; -y + W) - \gamma_X(z - y; -y + W).$$

But, since $\gamma_X(y; W) = \gamma_X^{\mathrm{s}}(y)\mathcal{L}^d(W)$, $\gamma_X^{\mathrm{s}}(y) - \gamma_X^{\mathrm{s}}(z) \le \gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(z-y)$, and interchanging y and z yields $|\gamma_X^{\mathrm{s}}(y) - \gamma_X^{\mathrm{s}}(z)| \le \gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(y - z)$. This inequality yields

$$\mathrm{Lip}_j(\gamma_X^{\mathrm{s}}) = \sup_{y \in \mathbb{R}^d,\ \varepsilon \in \mathbb{R}} \frac{|\gamma_X^{\mathrm{s}}(y + \varepsilon e_j) - \gamma_X^{\mathrm{s}}(y)|}{|\varepsilon|} = \sup_{\varepsilon \in \mathbb{R}} \frac{\gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(\varepsilon e_j)}{|\varepsilon|} = \mathrm{Lip}_j(\gamma_X^{\mathrm{s}}, 0).$$

Since γ_X^{s} is even, for all $\varepsilon \neq 0$,

$$\begin{aligned}\frac{\varphi_x^{\mathrm{s}}}{|\varepsilon|} &= \frac{1}{2}\frac{\varphi_x^{\mathrm{s}} + \psi_x^{\mathrm{s}}}{|\varepsilon|} \\ &= \frac{1}{2}\sup_{c>0} \frac{\varphi_x^{\mathrm{s}} + \psi_x^{\mathrm{s}}}{|\varepsilon|}\frac{c^d - |\varepsilon| c}{c^d} \\ &= \frac{1}{2}\sup_{c>0} \frac{\varphi_x^{\mathrm{s}}\mathcal{L}^d((0,c)^d \ominus [-\varepsilon e_j, 0]) + \psi_x^{\mathrm{s}}\mathcal{L}^d((0,c)^d \ominus [0, \varepsilon e_j])}{|\varepsilon|\mathcal{L}^d((0,c)^d)} \\ &= \frac{1}{2}\sup_{c>0} \sigma_X(\varepsilon e_j; (0,c)^d)\frac{1}{\mathcal{L}^d((0,c)^d)} \\ &\le \frac{1}{2}\sup_{c>0} \mathbb{E}V_{e_j}\{X; (0,c)^d\}\frac{1}{\mathcal{L}^d((0,c)^d)} \\ &= \frac{1}{2}V_{e_j}^{\mathrm{s}}(X),\end{aligned}$$

where the inequality follows from (8). In this equation we make the substitution $\varphi_x^{\mathrm{s}} = \gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(\varepsilon e_j)$ and $\psi_x^{\mathrm{s}} = \gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(-\varepsilon e_j)$. It follows that

$$\mathrm{Lip}_j(\gamma_X^{\mathrm{s}}, 0) = \sup_{\varepsilon \in \mathbb{R}} \frac{\gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(\varepsilon e_j)}{|\varepsilon|} \le \frac{1}{2}V_{e_j}^{\mathrm{s}}(X).$$

Besides, for all $\varepsilon \neq 0$ and $c > 0$,

$$\frac{1}{2}\sigma_X(\varepsilon e_j; (0,c)^d)\frac{1}{\mathcal{L}^d((0,c)^d)} \le \frac{\gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(\varepsilon e_j)}{|\varepsilon|} \le \frac{1}{2}V_{e_j}^{\mathrm{s}}(X),$$

and according to Corollary 1, the left-hand term tends to $\frac{1}{2}V_{e_j}^{\mathrm{s}}(X)$ when $\varepsilon \to 0$. Hence,

$$\lim_{\varepsilon \to 0} \frac{\gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(\varepsilon e_j)}{|\varepsilon|} = \frac{1}{2}V_{e_j}^{\mathrm{s}}(X) = \sup_{\varepsilon \in \mathbb{R}} \frac{\gamma_X^{\mathrm{s}}(0) - \gamma_X^{\mathrm{s}}(\varepsilon e_j)}{|\varepsilon|}.$$

3.3. Anisotropic perimeter approximation for pixelized sets

The proofs of our main results rely on several approximations involving pixelized sets and discretized covariograms. In the previous section we proved that the directional variations as well as the anisotropic perimeter can be computed from limits of difference quotients at 0 of the local covariogram. Here, we show that for pixelized sets, the anisotropic perimeter $\mathrm{Per}_{\boldsymbol{B}}(A; W)$ can be expressed as a finite difference at 0 of the local covariogram functionals.

For $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, we consider the pixels $C_k^n = (1/n)k + [0, n^{-1}]^d$, $k \in \mathbb{Z}^d$, that are the cells of the lattice $n^{-1}\mathbb{Z}^d$. Denote by $\mathcal{M}_n$ the algebra of $\mathbb{R}^d$ induced by the sets C_k^n, $k \in \mathbb{Z}^d$, and denote $\mathcal{W}_n = \mathcal{W} \cap \mathcal{M}_n$. For any $W \in \mathcal{W}_n$, also denote $\mathcal{M}_n(W) = \{A \in \mathcal{M}_n : A \subset W\}$ the sets of pixelized sets contained inside W. We remark that for any set $A \in \mathcal{M}_n$, there is a unique subset I_A of $\mathbb{Z}^d$ such that A is equivalent in measure to $\bigcup_{k \in I_A} C_k^n$.

Proposition 8. *Let $\boldsymbol{B} = (e_1, e_2, \ldots, e_d)$ be the canonical basis of $\mathbb{R}^d$ and let $n \in \mathbb{N}^*$. For all $A \in \mathcal{M}_n$, $W \in \mathcal{W}_n$, and $j \in \{1, \ldots, d\}$,*

$$V_{e_j}(A; W) = \sigma_{n^{-1}e_j; W}(A).$$

Hence, for all $A \in \mathcal{M}_n$ and $W \in \mathcal{W}_n$, $\mathrm{Per}_{\boldsymbol{B}}(A; W) = \sum_{j=1}^d \sigma_{n^{-1}e_j; W}(A)$.

Proof. Let $0 < \varepsilon \leq n^{-1}$. Consider the quantity

$$\begin{aligned}\delta_{\varepsilon e_j; W \ominus [-\varepsilon e_j, 0]}(A) &= \mathcal{L}^d(A \cap (\varepsilon e_j + A) \cap (W \ominus [-\varepsilon e_j, 0])) \\ &= \mathcal{L}^d\left(\left(\bigcup_{k \in I_A} C_k^n\right) \cap \left(\bigcup_{l \in I_A} (\varepsilon e_j + C_l^n)\right) \cap (W \ominus [-\varepsilon e_j, 0])\right).\end{aligned}$$

The two unions are over sets with pairwise negligible intersection, whence

$$\delta_{\varepsilon e_j; W \ominus [-\varepsilon e_j, 0]}(A) = \sum_{k,l \in I_A} \mathcal{L}^d(C_k^n \cap (C_l^n + \varepsilon e_j) \cap (W \ominus [-\varepsilon e_j, 0])).$$

Since $0 < \varepsilon \leq n^{-1}$, for $k, l \in I_A$,

$$\begin{aligned}&\mathcal{L}^d(C_k^n \cap (C_l^n + \varepsilon e_j) \cap (W \ominus [-\varepsilon e_j, 0])) \\ &\quad = \begin{cases} n^{-(d-1)}(n^{-1} - \varepsilon) & \text{if } l = k \in I_W, \\ \varepsilon n^{-(d-1)} & \text{if } l = k - e_j \text{ and } k, l \in I_W, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

These assertions are straightforward, we simply have to be cautious in the case $l = k - e_j$, $k \in I_W$, $l \notin I_W$, contribution of which is 0. Summing up those contributions and doing similar computations for the quantities $\delta_{0; W \ominus [-\varepsilon e_j, 0]}(A)$, $\delta_{-\varepsilon e_j; W \ominus [0, \varepsilon e_j]}(A)$, and $\delta_{0; W \ominus [0, \varepsilon e_j]}(A)$ yields, for some real numbers α, β independent of ε, for all $\varepsilon \in (0, n^{-1}]$,

$$\sigma_{\varepsilon e_j; W}(A) = \frac{\alpha}{\varepsilon} + \beta.$$

Proposition 6 then implies that $\alpha = 0$, $\beta = V_{e_j}(A; W)$, which yields the desired conclusion with $\varepsilon = n^{-1}$.

4. Realisability result

In this section we make explicit some considerations related to our realisability result and provide its proof.

4.1. Realisability problem and regularity modulus

Recall that the local covariogram of a RAMS X is $\gamma_X(y; W) = \mathbb{E}\delta_{y;W}\{X\}$. We introduce a regularized realisability problem for a local covariogram. Put $U_n = (-n, n)^d$. Define the *weighted anisotropic perimeter* by

$$\mathrm{Per}_{\boldsymbol{B}}^{\beta}(A) = \sum_{n\geq 1} \beta_n \mathrm{Per}_{\boldsymbol{B}}(A; U_n),$$

where the sequence (β_n) is set to $\beta_n = 2^{-n}(2n)^{-d}$ so that $\sum_{n\geq 1} \beta_n \mathcal{L}^d(U_n) = 1$. For a given function $\gamma\colon \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}$, define

$$\begin{aligned}\sigma_\gamma(u; W) = \frac{1}{\|u\|}[&\gamma(0; W \ominus [-u, 0]) - \gamma(u; W \ominus [-u, 0]) \\ &+ \gamma(0; W \ominus [0, u]) - \gamma(-u; W \ominus [0, u])].\end{aligned}$$

Define for all windows $W \in \mathcal{W}$ the constant $L_j(\gamma, W) \in [0, +\infty]$ by

$$L_j(\gamma, W) = \sup_{\varepsilon\in\mathbb{R}} \sigma_\gamma(\varepsilon e_j; W), \qquad j \in \{1, \ldots, d\}. \tag{10}$$

The constant $L_j(\gamma, W)$ is related to the Lipschitz property of γ in its spatial variable. The motivation for considering this particular constant comes from Corollary 1, which shows that if γ_X is the local covariogram of a RAMS X, then

$$\mathbb{E}V_{e_j}\{X; W\} = \sup_{\varepsilon\in\mathbb{R}} \sigma_{\gamma_X}(\varepsilon e_j; W).$$

Theorem 3. *Let $\gamma\colon \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}$ be a function and $r \geq 0$. Then γ is realisable by a RAMS X such that*

$$\mathbb{E}\mathrm{Per}_{\boldsymbol{B}}^{\beta}\{X\} \leq r$$

if and only if γ is admissible (see Definition 1) and

$$\sum_{n\geq 1} \beta_n \left(\sum_{j=1}^{d} L_j(\gamma, U_n)\right) \leq r, \tag{11}$$

where for all $j \in \{1, \ldots, d\}$ and $n \geq 1$, the constant $L_j(\gamma, U_n)$ is defined by (10).

The stationary counterpart of the above theorem is stated and proved in Section 4.2. Let us specialize a general definition from [19, Definition 2.5] to our framework.

Definition 5. (*Regularity moduli.*) Let $\boldsymbol{G}$ be a vector space of measurable real functions on $\mathcal{M}$. A *$\boldsymbol{G}$-regularity modulus* on $\mathcal{M}$ is a lower semi-continuous function $\chi : \mathcal{M} \mapsto [0, +\infty]$ such that, for all $g \in \boldsymbol{G}$, the level set

$$H_g = \{A \in \mathcal{M} \colon \chi(A) \leq g(A)\} \subset \mathcal{M}$$

is relatively compact for the convergence in measure.

We provide the following result, which is a straightforward consequence of [19, Proposition 2.2 and Theorem 2.6] for bounded continuous functions; see in particular the discussion after the proof of Theorem 2.6.

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Theorem 4. (Lachièze-Rey–Molchanov [19].) *Let $\boldsymbol{G}$ be a vector space of real continuous bounded functions on $\mathcal{M}$ that comprises constant functions. Let χ be a $\boldsymbol{G}$-regularity modulus, and Φ be a linear function on $\boldsymbol{G}$ such that $\Phi(1) = 1$. Then, for any given $r \geq 0$, there exists a RAMS $X \in \mathcal{M}$ such that*

$$\mathbb{E}g\{X\} = \Phi(g), \qquad g \in \boldsymbol{G}, \qquad \mathbb{E}\chi\{X\} \leq r \tag{12}$$

if and only if

$$\sup_{g \in \boldsymbol{G}} \inf_{A \in \mathcal{M}} \chi(A) - g(A) + \Phi(g) \leq r. \tag{13}$$

In our setting, call $\boldsymbol{G}$ the vector space generated by the constant functionals and the local covariogram functionals $A \mapsto \delta_{y;W}(A)$, $y \in \mathbb{R}^d$, $W \in \mathcal{W}$.

Proposition 9. *It holds that $\mathrm{Per}^{\beta}_{\boldsymbol{B}}$ is a $\boldsymbol{G}$-regularity modulus (and, therefore, a $\boldsymbol{G}^*$-regularity modulus for any subspace $\boldsymbol{G}^* \subset \boldsymbol{G}$).*

Proof. By definition of a regularity modulus, we have to show that the $\mathrm{Per}^{\beta}_{\boldsymbol{B}}$-level sets are relatively compact. Consider a sequence (A_n) such that $\mathrm{Per}^{\beta}_{\boldsymbol{B}}(A_n) \leq c$ for all $n \in \mathbb{N}$. Then, for all $n, m \in \mathbb{N}$, $\mathrm{Per}_{\boldsymbol{B}}(A_n; U_m) \leq c/\beta_m < \infty$, and, thus, (A_n) is a sequence of sets of locally finite perimeter whose perimeter in any open bounded set $U \subset \mathbb{R}^d$ is uniformly bounded. According to [2, Theorem 3.39], there exists a subsequence of (A_n) that locally converges in measure in $\mathbb{R}^d$.

For $g \in \boldsymbol{G}$, denote by $\mathrm{dom}(g)$ the smallest open set such that for every measurable set A, $g(A) = g(A \cap \mathrm{dom}(g))$. If g has the form

$$g = \sum_{i=1}^{q} a_i \delta_{y_i;W_i},$$

we have $\mathrm{dom}(g) \subset \cup_i (W_i \cup (-y_i + W_i))$, there is not equality because such a decomposition is not unique.

We turn to the proof of Theorem 3. It involves several technical lemmas that are stated within the proof when needed. Their demonstrations are delayed to the end of the section.

Proof of Theorem 3. (i) *Necessity.* If X is a RAMS then the admissibility of γ_X is the consequence of the positivity of the mathematical expectation; see the discussion below Definition 1. According to Proposition 6, for all $n \geq 1$, with probability 1,

$$\sigma_{\varepsilon e_j;U_n}(X) \leq V_{e_j}(X; U_n).$$

After taking the expectation, the supremum of the left-hand member over $\varepsilon > 0$ is $L_j(\gamma, W)$. Summing over j yields $\sum_{j=1}^{d} L_j(\gamma, U_n) \leq \mathrm{Per}_{\boldsymbol{B}}(X; U_n)$, and multiplying by β_n and summing over n yields (11).

(ii) *Sufficiency.* Call $\boldsymbol{G}_n \subset \boldsymbol{G}$ the set of functionals $g = c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}$ such that for all i, $y_i \in n^{-1}\mathbb{Z}^d$, and $W_i \in \mathcal{W}_n$, i.e. the closures of the W_i are pixelized sets. Denote by $\boldsymbol{G}^* = \bigcup_{n \geq 1} \boldsymbol{G}_n$. We remark that each $\boldsymbol{G}_n$ is a vector space and that $\boldsymbol{G}^* \subset \boldsymbol{G}$ is a vector space as well: indeed, if $g_1 \in \boldsymbol{G}_n$ and $g_2 \in \boldsymbol{G}_m$ then $g_1 + g_2 \in \boldsymbol{G}_{mn}$. To apply Theorem 4 to $\boldsymbol{G}^*$, we need to show that

$$\sup_{n \geq 1} \sup_{g \in \boldsymbol{G}_n} \inf_{A \in \mathcal{M}} \mathrm{Per}^{\beta}_{\boldsymbol{B}}(A) - g(A) + \Phi(g) \leq r,$$

where Φ is defined by

$$\Phi\Big(c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}\Big) = c + \sum_{i=1}^{q} \gamma(y_i, W_i).$$

We first remark that Φ is a positive operator because γ is admissible; see Definition 1. Let $g \in \boldsymbol{G}_n$. Define $m_g = \inf_{A\in\mathcal{M}} \operatorname{Per}^{\beta}_{\boldsymbol{B}}(A) - g(A) + \Phi(g)$. Let $p \in \mathbb{N}$ large enough such that $\operatorname{dom}(g) \subset (-p, p)^d$. For all $c > 0$ denote $\mathcal{M}^c_n = \mathcal{M}_n((-c, c)^d)$. We have

$$m_g \le \inf_{A\in\mathcal{M}^p_n} \operatorname{Per}^{\beta}_{\boldsymbol{B}}(A) - g(A) + \Phi(g)$$

because $\mathcal{M}^p_n \subset \mathcal{M}$. The proof is based on an approximation of the perimeter by a discretized functional with compact domain, summarized by the following lemma.

Lemma 2. *For $n, p \ge 1$, put $U^p_n = (-p - 1/n, p + 1/n)^d$. There exists $g_{n,p} \in \boldsymbol{G}_n$ with $\operatorname{dom}(g_{n,p}) \subset U^p_n$ such that*

$$g_{n,p}(A) = \operatorname{Per}^{\beta}_{\boldsymbol{B}}(A) \quad \textit{for all } A \in \mathcal{M}^p_n.$$

Its explicit expression is

$$g_{n,p}(A) = \sum_{m=1}^{p} \beta_m \sum_{j=1}^{d} \sigma_{n^{-1}e_j;U_n}(A) + \Big(\sum_{m=p+1}^{+\infty} \beta_m\Big) \sum_{j=1}^{d} \sigma_{n^{-1}e_j;U_p}(A).$$

Furthermore, for all $A \in \mathcal{M}_n$,

$$|g_{n,p}(A) - g_{n,p}(A \cap (-p, p)^d)| \le E_{n,p},$$

where $E_{n,p} = 8dn2^{-p}(p + 1)^{-d}((p + 1/n)^d - p^d)$.

Therefore, $g_{n,p} = \operatorname{Per}^{\beta}_{\boldsymbol{B}}$ on $\mathcal{M}^p_n$, and

$$m_g \le \inf_{A\in\mathcal{M}^p_n} g_{n,p}(A) - g(A) + \Phi(g) \le \inf_{A\in\mathcal{M}^{p+1/n}_n} g_{n,p}(A) - g(A) + \Phi(g) + E_{n,p},$$

because $g(A) = g(A \cap \operatorname{dom}(g)) = g(A \cap (-p, p)^d) = g(A \cap (-p - 1/n, p + 1/n)^d)$, and $|g_{n,p}(A) - g_{n,p}(A \cap (-p, p)^d)| \le E_{n,p}$, where the error term $E_{n,p}$ is computed in Lemma 2. We need the following lemma, also proved afterwards.

Lemma 3. *Any functional $g \in \boldsymbol{G}_n$ reaches its infimum on an element of $\mathcal{M}_n(\operatorname{dom}(g))$.*

We have $\operatorname{dom}(g_{n,p} - g) \subset U^p_n$, whence by Lemma 3, $g_{n,p} - g$ reaches its infimum over $\mathcal{M}$ on $\mathcal{M}^{p+1/n}_n$, and

$$\inf_{A\in\mathcal{M}^{p+1/n}_n} g_{n,p}(A) - g(A) = \inf_{A\in\mathcal{M}} g_{n,p}(A) - g(A) = \inf_{A\in\mathcal{M}} (g_{n,p} - g)(A) \le \Phi(g_{n,p} - g),$$

where the last inequality is a consequence of the positivity of Φ. Therefore,

$$m_g \le \Phi(g_{n,p} - g) + \Phi(g) + E_{n,p} = \Phi(g_{n,p}) + E_{n,p}. \tag{14}$$

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Let us bound $\Phi(g_{n,p})$. Recall that by definition $\Phi(\delta_{y;W}) = \gamma(y;W)$. The definition of the constants $L_j(\gamma, W)$ and the expression of $g_{n,p}$ yield

$$\Phi(g_{n,p}) \leq \sum_{m=1}^{p} \beta_m \left(\sum_{j=1}^{d} L_j(\gamma, U_m) \right) + \left(\sum_{m=p+1}^{+\infty} \beta_m \right) \left(\sum_{j=1}^{d} L_j(\gamma, U_n^p) \right).$$

Lemma 4. *For all admissible functions γ, $L_j(\gamma; W) \leq L_j(\gamma, W')$ for any $j \in \{1, \dots, d\}$ and $W, W' \in \mathcal{W}$ such that $W \subset W'$.*

By Lemma 4, since γ is admissible, and for all $m > p$, $U_n^p \subset U_m$,

$$\begin{aligned} \left(\sum_{m=p+1}^{+\infty} \beta_m \right) \left(\sum_{j=1}^{d} L_j(\gamma, U_n^p) \right) &= \sum_{m=p+1}^{+\infty} \beta_m \left(\sum_{j=1}^{d} L_j(\gamma, U_n^p) \right) \\ &\leq \sum_{m=p+1}^{+\infty} \beta_m \left(\sum_{j=1}^{d} L_j(\gamma, U_m) \right). \end{aligned}$$

Hence,

$$\Phi(g_{n,p}) \leq \sum_{m=1}^{+\infty} \beta_m \left(\sum_{j=1}^{d} L_j(\gamma, U_m) \right) \leq r.$$

Returning to (14) yields

$$m_g \leq r + E_{n,p}.$$

Since for all $n \geq 1$, $E_{n,p}$ tends to 0 as p tends to $+\infty$, we have $m_g \leq r$. Since $n \geq 1$ and $g \in \boldsymbol{G}_n$ were arbitrarily chosen, we conclude that

$$\sup_{g \in \boldsymbol{G}^*} \inf_{A \in \mathcal{M}} \mathrm{Per}_{\boldsymbol{B}}^{\beta}(A) - g(A) + \Phi(g) \leq r.$$

Hence, we can apply Theorem 4 to ensure that there exists a RAMS X solution of the problem (12). This RAMS X satisfies $\mathbb{E}\mathrm{Per}_{\boldsymbol{B}}^{\beta}\{X\} \leq r$ and, for y in the set $\mathbb{Q}^d$ of vectors with rational coordinates,

$$W \in \mathcal{W} \cap \bigcup_{n \in \mathbb{N}^*} \mathcal{M}_n, \qquad \gamma_X(y; W) = \gamma(y; W).$$

It only remains to show that this equality between γ_X and γ extends to all couples $(y; W) \in \mathbb{R}^d \times \mathcal{W}$ using the continuity of both γ_X and γ.

First, regarding the W-variable, since γ_X and γ are both admissible, and by Proposition 4 $|\delta_{y;U}(A) - \delta_{y;W}(A)| \leq \mathcal{L}^d(U \Delta W)$ for all $U, W \in \mathcal{W}$, both γ_X and γ are continuous with respect to the convergence in measure. Besides, the set of pixelized sets $\mathcal{W} \cap \bigcup_{n \in \mathbb{N}^*} \mathcal{M}_n$ is dense in $\mathcal{W}$ for the convergence in measure. Indeed, given $W \in \mathcal{W}$, it is easily shown by dominated convergence that the sequence $W_n = \bigcup_{k \in \mathbb{Z}^d} \{C_k^n : C_k^n \subset W\}$ converges in measure towards W, since due to the hypothesis $\mathcal{L}^d(\partial W) = 0$, for almost all $x \in \mathbb{R}^d$ either $x \in W$ or $x \in \mathbb{R}^d \setminus \bar{W}$, where $\bar{W}$ denotes the closure of W.

Regarding the y-variable, $y \mapsto \gamma_X(y; W) = \mathbb{E}\delta_{y;W}\{X\}$ is continuous since $y \mapsto \delta_{y;W}(X)$ is a.s. continuous and bounded by $\mathcal{L}^d(W)$. To conclude the proof, we show that $y \mapsto \gamma(y; W)$ is also continuous over $\mathbb{R}^d$, which is the purpose of the following lemma.

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Lemma 5. *Let γ be an admissible function. Let $y \in \mathbb{R}^d$, $W \in \mathcal{W}$, and $r > 0$. For all $z \in \mathbb{R}^d$ and $\rho > 0$, denote by $C(z, \rho)$ the hypercube of center z and half size length ρ, i.e. $C(z, \rho) = \{z' \in \mathbb{R}^d : \|z' - z\|_\infty \leq \rho\}$.*

Then, for all $z, z' \in C(y, r)$,

$$|\gamma(z; W) - \gamma(z'; W)| \leq \sum_{j=1}^{d} L_j(\gamma, W \oplus C(-y, 3r))|z'_j - z_j|.$$

In particular, if γ satisfies (11), then for all $W \in \mathcal{W}$, the map $y \mapsto \gamma(y; W)$ is locally Lipschitz.

We turn to the proofs of the lemmas.

Proof of Lemma 2. We first remark that for all sets A of finite perimeter such that $A \subset (-p, p)^d$ for some integer $p \geq 1$ and for all $m > p$, since $A \cap (U_m \setminus (-p, p)^d) = \varnothing$,

$$\mathrm{Per}_{\boldsymbol{B}}(A; U_m) = \mathrm{Per}_{\boldsymbol{B}}\left(A; \left(-p - \frac{1}{n}, p + \frac{1}{n}\right)^d\right) = \mathrm{Per}_{\boldsymbol{B}}(A; U_n^p).$$

Consequently,

$$\mathrm{Per}_{\boldsymbol{B}}^{\beta}(A) = \sum_{m\geq 1} \beta_m \mathrm{Per}_{\boldsymbol{B}}(A; U_m) = \sum_{m=1}^{p} \beta_m \mathrm{Per}_{\boldsymbol{B}}(A; U_m) + \left(\sum_{m=p+1}^{+\infty} \beta_m\right) \mathrm{Per}_{\boldsymbol{B}}(A; U_n^p).$$

According to Proposition 8, for all pixelized sets $A \in \mathcal{M}_n^p$, all the perimeters $\mathrm{Per}_{\boldsymbol{B}}(A; U_m)$, $1 \leq m \leq p$, and $\mathrm{Per}_{\boldsymbol{B}}(A; U_n^p)$, can be expressed as some linear combination of local covariograms, hence,

$$\mathrm{Per}_{\boldsymbol{B}}^{\beta}(A) = \sum_{m=1}^{p} \beta_m \sum_{j=1}^{d} \sigma_{n^{-1}e_j; U_m}(A) + \left(\sum_{m=p+1}^{+\infty} \beta_m\right) \sum_{j=1}^{d} \sigma_{n^{-1}e_j; U_n^p}(A).$$

The linear combination on the right-hand side is an element of $\boldsymbol{G}$ that will be denoted by $g_{n,p}$ in what follows. Note that $\mathrm{dom}(g_{n,p}) \subset U_n^p = (-p - 1/n, p + 1/n)^d$. It remains to show that the inequality $|g_{n,p}(A) - g_{n,p}(A \cap (-p, p)^d)| \leq E_{n,p}$. Using Proposition 4, for all $A, B \in \mathcal{M}$, $j \in \{1, \ldots, d\}$, and $W \in \{U_1, \ldots, U_p, U_n^p\}$,

$$|\sigma_{n^{-1}e_j; W}(A) - \sigma_{n^{-1}e_j; W}(B)| \leq 8n\mathcal{L}^d((A \Delta B) \cap W).$$

Hence, for all $A \in \mathcal{M}_n$,

$$\begin{aligned} |g_{n,p}(A) - g_{n,p}(A \cap (-p, p)^d)| &= |g_{n,p}(A \cap U_n^p) - g_{n,p}(A \cap (-p, p)^d)| \\ &\leq \left(\sum_{m=p+1}^{+\infty} \beta_m\right) d8n\mathcal{L}^d((A \cap U_n^p) \Delta (A \cap (-p, p)^d)), \end{aligned}$$

since for all $m \in \{1, \ldots, p\}$, $((A \cap U_n^p) \Delta (A \cap (-p, p)^d)) \cap U_m = \varnothing$. For all $m \geq p + 1$, $\beta_m = 2^{-m}(2m)^{-d} \leq 2^{-m}(2(p + 1))^{-d}$. Hence,

$$\sum_{m=p+1}^{+\infty} \beta_m \leq 2^{-d}(p + 1)^{-d} \sum_{m=p+1}^{+\infty} 2^{-m} = 2^{-d-p}(p + 1)^{-d}.$$

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Besides, $\mathcal{L}^d((A \cap U_n^p)\Delta(A \cap (-p, p)^d)) \le \mathcal{L}^d(U_n^p \setminus (-p, p)^d) = 2^d((p + 1/n)^d - p^d)$. Finally,

$$|g_{n,p}(A \cap U_n^p) - g_{n,p}(A \cap (-p, p)^d)| \le 8dn2^{-p}(p+1)^{-d}\left(\left(p + \frac{1}{n}\right)^d - p^d\right).$$

Proof of Lemma 3. Put $W = \mathrm{dom}(g) \in \mathcal{W}_n$. Then $g(A) = g(A \cap W)$ for any $A \in \mathcal{M}$. Now that the problem is restricted to the bounded pixelized domain W, it remains to show that the extrema of g on $\mathcal{M}(W)$ are reached by sets of $\mathcal{M}_n(W)$. Let us turn to the details.

Without loss of generality, assume that g has the form

$$g = \sum_{i=1}^{q} a_i \delta_{y_i;W_i}$$

for some $y_i \in n^{-1}\mathbb{Z}^d$, $W_i \in \mathcal{W}_n$, and $a_i \in \mathbb{R}$. Denote by $I_n(W)$ the set of all indexes $k \in \mathbb{Z}^d$ such that the hypercube C_k^n is included in $\bar{W}$, we then also have $\bar{W} = \bigcup_{k \in I_n(W)} C_k^n$. For $A \in \mathcal{M}(W)$, $n \ge 1$, denote by $A_k^n = A \cap C_k^n, k \in I_n(W)$, the intersection of A with the hypercube C_k^n, and by $\tilde{A}_k^n = -k + nA_k^n$ its rescaled translated version comprised in $[0, 1]^d$. Consider the probability space $(\Omega = [0, 1)^d, \mathcal{A} = \mathcal{B}([0, 1)^d), \mathbb{P} = \mathcal{L}^d)$, on which we define the $\{0, 1\}^{I_n(W)}$-valued random vector $\boldsymbol{Y}^A(w) = (\mathbf{1}_{\tilde{A}_k^n}(\omega))_{k \in I_n(W)}, \omega \in \Omega$. The measures of the pairwise intersections $\mathcal{L}^d(\tilde{A}_k^n \cap \tilde{A}_l^n)$ can thus be seen as the components of the covariance matrix $\boldsymbol{C}(A) = (C_{k,l}(A))_{k,l \in I_n(W)}$ of the random vector $\boldsymbol{Y}^A$, i.e.

$$C_{k,l}(A) = \mathbb{E}\{Y_k^A Y_l^A\} = \mathbb{E}\{\mathbf{1}_{\{\omega \in \tilde{A}_k^n\}} \mathbf{1}_{\{\omega \in \tilde{A}_l^n\}}\} = \mathcal{L}^d(\tilde{A}_k^n \cap \tilde{A}_l^n), \qquad k, l \in I_n(W).$$

Let us prove that $g(A)$ can be written as

$$g(A) = \sum_{k,l \in I_n(W)} \beta_{k,l} \mathcal{L}^d(\tilde{A}_k^n \cap \tilde{A}_l^n)$$

for some coefficients $\beta = (\beta_{k,l})_{k,l \in I_n(W)}$ depending solely on g. Putting $k_i = ny_i \in \mathbb{Z}^d$, we have

$$\begin{aligned}
g(A) &= \sum_{i=1}^{q} a_i \delta_{n^{-1}k_i, W_i}(A) \\
&= \sum_{i=1}^{q} a_i \mathcal{L}^d(A \cap (n^{-1}k_i + A) \cap W_i) \\
&= \sum_{i=1}^{q} a_i \sum_{k \in I_n(W)} \mathbf{1}_{\{C_k^n \subset \bar{W}_i\}} \int_{C_k^n} \mathbf{1}_{\{x \in A, x \in n^{-1}k_i + A\}} \, \mathrm{d}x \\
&= \sum_{i=1}^{q} a_i \sum_{k,l \in I_n(W)} \mathbf{1}_{\{l = k - k_i\}} \mathbf{1}_{\{C_k^n \subset \bar{W}_i\}} \int_{C_k^n} \mathbf{1}_{\{x \in A, x \in n^{-1}(k-l) + A\}} \, \mathrm{d}x \\
&= \sum_{k,l \in I_n(W)} \sum_{i=1}^{q} a_i \mathbf{1}_{\{l = k - k_i\}} \mathbf{1}_{\{C_k^n \subset \bar{W}_i\}} \int_{C_0^n} \mathbf{1}_{\{x \in -n^{-1}k + A, x \in -n^{-1}l + A\}} \, \mathrm{d}x \\
&= \sum_{k,l \in I_n(W)} \underbrace{n^{-d}\left(\sum_{i=1}^{q} a_i \mathbf{1}_{\{l = k - k_i\}} \mathbf{1}_{\{C_k^n \subset \bar{W}_i\}}\right)}_{=\beta_{k,l}} \mathcal{L}^d(\tilde{A}_k^n \cap \tilde{A}_j^n).
\end{aligned}$$

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Then, we can write

$$g(A) = \sum_{k,l \in I_n(W)} \beta_{k,l} C_{k,l}(A) = \langle \beta, \boldsymbol{C}(A) \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the classical scalar product between matrices. Denote by Γ_n the set of covariance matrices of all random vectors having values in $\{0, 1\}^{I_n(W)}$. Since for every set A we can associate some covariance matrix $\boldsymbol{C}(A)$ such that $g(A) = \langle \beta, \boldsymbol{C}(A) \rangle$, one can write

$$\inf_{A \in \mathcal{M}(W)} g(A) \geq \inf_{C \in \Gamma_n} \langle \beta, C \rangle.$$

The optimization problem on the right-hand side of this inequality is a linear programming problem on the bounded convex set Γ_n. Hence, we are ensured that there exists an optimal solution C^* of this problem which is an extreme point of Γ_n. As shown in [18, Theorem 2.5], the extreme points of Γ_n are covariance matrices associated with deterministic random vectors; see also [7], where Γ_n is called the *correlation polytope* and studied more deeply. Therefore, there exists a fixed vector $\boldsymbol{z}^* \in \{0, 1\}^{I_n(W)}$ such that $C^*_{k,l} = z^*_k z^*_l$ minimizes $\langle \beta, C \rangle$. Given this vector $\boldsymbol{z}^* \in \{0, 1\}^{I_n(W)}$, define the set A^* as the union of the hypercubes

$$A^* = \bigcup_{\{k:\, z^*_k = 1\}} C^n_k \cap W.$$

Then we see that the covariance matrix $\boldsymbol{C}(A^*)$ associated with the deterministic set A^* is equal to C^*. Furthermore, it is clear that A^* is measurable with respect to the σ-algebra generated by the C^n_k, meaning exactly $A^* \in \mathcal{M}_n(W)$. Hence, we have shown that

$$\inf_{A \in \mathcal{M}(K)} g(A) \geq \inf_{C \in \Gamma_n} \langle \beta, C \rangle = \min_{A \in \mathcal{M}_n(W)} g(A).$$

Since $\mathcal{M}_n(W) \subset \mathcal{M}(W)$ the reverse inequality is immediate, and, thus,

$$\inf_{A \in \mathcal{M}(W)} g(A) = \min_{A \in \mathcal{M}_n(W)} g(A).$$

Proof of Lemma 4. First, we remark that if W and W' are two observation windows such that $W \subset W'$, then, for all $y \in \mathbb{R}^d$ and $A \in \mathcal{M}$,

$$0 \leq \delta_{0;W}(A) - \delta_{y;W}(A) \leq \delta_{0;W'}(A) - \delta_{y;W'}(A).$$

Indeed, $W \subset W'$ yields $(A \setminus (y + A)) \cap W \subset (A \setminus (y + A)) \cap W'$, and, thus, taking the Lebesgue measure and using (7) it follows that $0 \leq \delta_{0;W}(A) - \delta_{y;W}(A) \leq \delta_{0;W'}(A) - \delta_{y;W'}(A)$.

Let W and $W' \in \mathcal{W}$ such that $W \subset W'$, $j \in \{1, \ldots, d\}$, and we show that $L_j(\gamma; W) \leq L_j(\gamma, W')$. Suppose that $L_j(\gamma, W')$ is finite, otherwise there is nothing to show. Since $W \subset W'$ we also have $W \ominus [-\varepsilon e_j, 0] \subset W' \ominus [-\varepsilon e_j, 0]$ and $W \ominus [0, \varepsilon e_j] \subset W' \ominus [0, \varepsilon e_j]$. Hence, according to the preliminary remark, for all $A \in \mathcal{M}$, $\varepsilon \in \mathbb{R}$, $\sigma_{\varepsilon e_j; W}(A) \leq \sigma_{\varepsilon e_j; W'}(A)$. Since γ is admissible, this implies that for all $\varepsilon \in \mathbb{R}$, $\sigma_\gamma(\varepsilon e_j; W) \leq \sigma_\gamma(\varepsilon e_j; W')$. Hence, by definition of $L_j(\gamma, W')$, $\sigma_\gamma(\varepsilon e_j; W) \leq L_j(\gamma, W')$, and, thus, $L_j(\gamma; W) \leq L_j(\gamma, W')$.

Proof of Lemma 5. Recall that it has been shown in the proof of Proposition 7 (see (9)) that for all $y, z \in \mathbb{R}^d$, $W \in \mathcal{W}$, and $A \in \mathcal{M}$,

$$\delta_{y;W}(A) - \delta_{z,W}(A) \leq \delta_{0,-y+W}(A) - \delta_{z-y,-y+W}(A).$$

Let γ be an admissible function and let $y \in \mathbb{R}^d$, $W \in \mathcal{W}$ and $r > 0$ be fixed. Let $z, z' \in C(y, r)$ be such that $z' = z + te_j$ for some $t \in \mathbb{R}$ and $j \in \{1, \ldots, d\}$. Since γ is admissible, the above inequality ensures that

$$\gamma(z; W) - \gamma(z'; W) \leq \gamma(0; -z + W) - \gamma(te_j; -z + W).$$

As a consequence of (7), since γ is admissible, the difference at 0 means that the map $U \mapsto \gamma(0; U) - \gamma(te_j; U)$ is an increasing function of U. Hence, since $W \subset (W \oplus [-te_j, 0]) \ominus [-te_j, 0]$,

$$\begin{aligned}
&\gamma(z; W) - \gamma(z'; W) \\
&\quad \leq \gamma(0; -z + (W \oplus [-te_j, 0]) \ominus [-te_j, 0]) - \gamma(te_j; -z + (W \oplus [-te_j, 0]) \ominus [-te_j, 0]) \\
&\quad \leq |t| \frac{\gamma(0; -z + (W \oplus [-te_j, 0]) \ominus [-te_j, 0]) - \gamma(te_j; -z + (W \oplus [-te_j, 0]) \ominus [-te_j, 0])}{|t|} \\
&\quad \leq |t| \sup_{\varepsilon \in \mathbb{R}} \frac{\gamma(0; -z + (W \oplus [-te_j, 0]) \ominus [-\varepsilon e_j, 0]) - \gamma(\varepsilon e_j; -z + (W \oplus [-te_j, 0]) \ominus [-\varepsilon e_j, 0])}{|\varepsilon|} \\
&\quad \leq |t| L_j(\gamma, -z + W \oplus [-te_j, 0]).
\end{aligned}$$

According to Lemma 4, $W \mapsto L_j(\gamma; W)$ is increasing. Since $z \in C(y, r)$ and $|t| = \|z - z'\|_\infty \leq \|z - y\|_\infty + \|y - z'\|_\infty \leq 2r$, we have $-z + W \oplus [-te_j, 0] \subset W \oplus C(-y, 3r)$. Hence, for all $z, z' \in C(y, r)$ be such that $z' = z + te_j$,

$$\gamma(z; W) - \gamma(z'; W) \leq L_j(\gamma, W \oplus C(-y, 3r))|t|.$$

Exchanging z and z', we obtain $|\gamma(z; W) - \gamma(z'; W)| \leq L_j(\gamma, W \oplus C(-y, 3r))|t|$. To finish, consider a couple of points $z, z' \in C(y, r)$ that are not necessarily aligned along an axis. Consider the finite sequence of vector $\boldsymbol{u}_0 = z, u_1, \ldots, u_d = z'$ defined such that the j first coordinates of u_j are the ones of z' while its $d - j$ last coordinates are the ones of z, so that $\boldsymbol{u}_0 = z$, $u_d = z'$ and $u_j - u_{j-1} = (z'_j - z_j)e_j$. Clearly, each u_j belongs to the hypercube $C(y, r)$, and, thus, applying the d inequalities obtained above, we obtain

$$\begin{aligned}
|\gamma(z; W) - \gamma(z'; W)| &= \left| \sum_{j=1}^{d} \gamma(u_j; W) - \gamma(u_{j-1}; W) \right| \\
&\leq \sum_{j=1}^{d} \left| \gamma(u_j; W) - \gamma(u_{j-1}; W) \right| \\
&\leq \sum_{j=1}^{d} L_j(\gamma, W \oplus C(-y, 3r))|z'_j - z_j|.
\end{aligned}$$

If γ satisfies (11) then, for all $n \in \mathbb{N}^*$ and $j \in \{1, \ldots, d\}$, the constants $L_j(\gamma, U_n)$ are all finite. According to Lemma 4, this implies that the d constants $L_j(\gamma, W \oplus C(-y, 3r))$, $j \in \{1, \ldots, d\}$, are all finite for any fixed $y \in \mathbb{R}^d$, $W \in \mathcal{W}$, and $r > 0$, and, thus, the map $y \mapsto \gamma(y; W)$ is locally Lipschitz.

4.2. Stationary case

The following theorem is the main result of this paper. It is a refined version of Theorem 1 given in the introduction.

Theorem 5. *Let* $S_2 \colon \mathbb{R}^d \to \mathbb{R}$ *be a function and* $r \geq 0$*. Then there is a stationary RAMS* X *such that*

$$S_2(y) = \gamma_X^{\mathrm{s}}(y), \quad y \in \mathbb{R}^d, \qquad \mathrm{Per}_{\boldsymbol{B}}^{\mathrm{s}}(X) \leq r$$

if and only if S_2 *is admissible and*

$$\sum_{j=1}^{d} \mathrm{Lip}_j(S_2, 0) \leq \frac{r}{2}.$$

We shall use a variant of [19, Theorem 2.10(ii)], where the monotonicity assumption is replaced by a domination.

Theorem 6. *Let* $\boldsymbol{G}$*,* χ*,* Φ *be as in Theorem 4, and assume that* $\boldsymbol{G}$ *is stable under the action of a group of transformations* Θ *of* $\mathbb{R}^d$*: for all* $\theta \in \Theta$*,* $g \in \boldsymbol{G}$*,* $\theta g \colon A \mapsto g(\theta A)$ *is a function of* $\boldsymbol{G}$*. Furthermore, assume that there is a sequence* $(g_n)_{n \geq 1}$ *of functions of* $\boldsymbol{G}$ *such that* $0 \leq g_n \leq \chi$ *and*

$$g_n(A) \longrightarrow \chi(A) \quad \text{as } n \to +\infty, \; A \in \mathcal{M},$$

and that χ *is sub-invariant: for every* $\theta \in \Theta$*, there is a constant* $C_\theta > 0$ *such that*

$$\chi(\theta A) \leq C_\theta \chi(A), \qquad A \in \mathcal{M}. \tag{15}$$

Then, if Φ *is invariant under the action of* Θ*, i.e.*

$$\Phi(\theta g) = \Phi(g), \qquad g \in \boldsymbol{G}, \; \theta \in \Theta,$$

for any given $r \geq 0$*, there exists a* Θ*-invariant RAMS* X *such that*

$$\mathbb{E} g\{X\} = \Phi(g), \quad g \in \boldsymbol{G}, \qquad \mathbb{E}\chi\{X\} \leq r$$

if and only if (13) holds.

Proof. The proof is the same as [19, Theorem 2.10(ii)], which itself is based on [17, Proposition 4.1]. The proof consists of checking the hypotheses of the Markov–Kakutani fixed point theorem. Let M be the family of random elements X that realise Φ on $\boldsymbol{G}$, and satisfy $\mathbb{E}\chi\{\theta X\} \leq r$ for every $\theta \in \mathbb{R}^d$. The family M is easily seen to be convex with respect to the addition of measures, it is compact by [19, Theorem 2.8], and invariant under the action of Θ thanks to the Θ-invariance of Φ. It remains to prove that M is not empty. Since (13) is in order, Theorem 4 yields the existence of a RAMS X realising Φ and such that $\mathbb{E}\chi\{X\} \leq r$. For $\theta \in \Theta$, by the Lebesgue theorem, we obtain

$$\mathbb{E}\chi\{\theta X\} = \mathbb{E} \lim_n g_n\{\theta X\} = \lim_n \mathbb{E} g_n\{\theta X\} = \lim_n \mathbb{E} g_n\{X\} = \mathbb{E} \lim_n g_n\{X\} = \mathbb{E}\chi\{X\} \leq r,$$

where we have used the fact that $\mathbb{E}\sup_n g_n\{X\} < \infty$ and

$$\mathbb{E} \sup_n g_n\{\theta X\} \leq \mathbb{E}\chi\{\theta X\} \leq C_\theta \mathbb{E}\chi\{X\} < \infty.$$

It follows that $M \ni X$ is nonempty, whence by the Markov–Kakutani theorem the mappings $X \mapsto \theta X$, $\theta \in \Theta$, admit a common fixed point X (considered here as a probability measure), which is therefore invariant under Θ.

Proof of Theorem 5. (i) *Necessity.* Assume that S_2 is the specific covariogram of a stationary RAMS X with $\mathrm{Per}^{\mathrm{s}}_{\boldsymbol{B}}(X) \le r$. Then S_2 is admissible, and by Proposition 7,

$$\sum_{j=1}^{d} \mathrm{Lip}_j(S_2, 0) = \sum_{j=1}^{d} \frac{1}{2} V^{\mathrm{s}}_{e_j}(X) = \frac{1}{2}\mathrm{Per}^{\mathrm{s}}_{\boldsymbol{B}}(X) \le \frac{r}{2}.$$

(ii) *Sufficiency.* Define $\gamma(y; W) = \mathcal{L}^d(W)S_2(y)$. Then, for all $W \in \mathcal{W}$ and $j \in \{1, \dots, d\}$,

$$\begin{aligned} L_j(\gamma, W) &= \sup_{\varepsilon\in\mathbb{R}} \frac{1}{|\varepsilon|}[\mathcal{L}^d(W \ominus [-\varepsilon e_j, 0])(S_2(0) - S_2(\varepsilon e_j)) \\ &\qquad + \mathcal{L}^d(W \ominus [0, \varepsilon e_j])(S_2(0) - S_2(-\varepsilon e_j))] \\ &\le 2\mathcal{L}^d(W)\,\mathrm{Lip}_j(S_2, 0). \end{aligned}$$

Hence, since $\sum_{j=1}^{d} \mathrm{Lip}_j(S_2, 0) \le r/2$,

$$\sum_{n\ge1} \beta_n \left(\sum_{j=1}^{d} L_j(\gamma, U_n)\right) \le \sum_{n\ge1} \beta_n 2\mathcal{L}^d(U_n) \sum_{j=1}^{d} \mathrm{Lip}_j(S_2) \le r.$$

Hence, according to Theorem 3, γ is the local covariogram of a RAMS X, and consequently, according to Theorem 4, γ satisfies (13) with $\chi = \mathrm{Per}^{\beta}_{\boldsymbol{B}}$ and $\boldsymbol{G}$ the space of all functionals of the form $g = \sum_{i=1}^{q} a_i \delta_{y_i;W_i}$. For $t \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, $W \in \mathcal{W}$, and $A \in \mathcal{M}$, we have

$$\begin{aligned} \theta_t \delta_{y;W}(A) &= \delta_{y;W}(\theta_t A) \\ &= \mathcal{L}^d((t + A) \cap (t + y + A) \cap W) \\ &= \mathcal{L}^d(A \cap (A + y) \cap (-t + W)) \\ &= \delta_{y;-t+W}(A), \end{aligned}$$

whence the space $\boldsymbol{G}$ generated by constant functions and functions $\delta_{y;W}$, $y \in \mathbb{R}^d$, $W \in \mathcal{W}$ is invariant under the action of the group $\Theta = \{\theta_t : t \in \mathbb{R}^d\}$ of translations. The linear functional defined by

$$\Phi(\delta_{y;W}) = \mathcal{L}^d(W)\gamma^{\mathrm{s}}(y)$$

is invariant under the action of translations θ_t, $t \in \mathbb{R}^d$. For $t \in \mathbb{R}^d$, let $\lceil t \rceil_\infty = \lceil \|t\|_\infty \rceil$ be the smallest integer larger than $\|t\|_\infty$. Then, recalling that U_n denotes the hypercube $(-n, n)^d$, $-t + U_n \subset (-n - \|t\|_\infty, n + \|t\|_\infty)^d \subset U_{n+\lceil t\rceil_\infty}$. Hence, for $A \in \mathcal{M}$,

$$\mathrm{Per}^{\beta}_{\boldsymbol{B}}(t+A) = \sum_{n=1}^{+\infty} \beta_n \mathrm{Per}_{\boldsymbol{B}}(t+A; U_n) = \sum_{n=1}^{+\infty} \beta_n \mathrm{Per}_{\boldsymbol{B}}(A; -t+U_n) \le \sum_{n=1}^{+\infty} \beta_n \mathrm{Per}_{\boldsymbol{B}}(A; U_{n+\lceil t\rceil_\infty}).$$

Since $\beta_n = 2^{-n}(2n)^{-d} = 2^{\lceil t\rceil_\infty}((n + \lceil t\rceil_\infty)/n)^d \beta_{n+\lceil t\rceil_\infty} \le 2^{\lceil t\rceil_\infty} \lceil t\rceil_\infty^d \beta_{n+\lceil t\rceil_\infty}$, we have

$$\mathrm{Per}^{\beta}_{\boldsymbol{B}}(t+A) \le 2^{\lceil t\rceil_\infty} \lceil t\rceil_\infty^d \sum_{n=1}^{+\infty} \beta_{n+\lceil t\rceil_\infty} \mathrm{Per}_{\boldsymbol{B}}(A; U_{n+\lceil t\rceil_\infty}) \le 2^{\lceil t\rceil_\infty} \lceil t\rceil_\infty^d \mathrm{Per}^{\beta}_{\boldsymbol{B}}(A), \qquad A \in \mathcal{M},$$

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whence (15) is in order for $\chi = \mathrm{Per}^{\beta}_{\boldsymbol{B}}$. To apply Theorem 6 it only remains to check that $\mathrm{Per}^{\beta}_{\boldsymbol{B}}$ can be pointwise approximated from below by functions from $\boldsymbol{G}$. According to Proposition 6, for $A \in \mathcal{M}$, and U a bounded open set of $\mathbb{R}^d$,

$$\mathrm{Per}_{\boldsymbol{B}}(A; U) = \lim_n g_n^U(A)$$

for some function $g_n^U \in \boldsymbol{G}$ mentioned in Proposition 6 that satisfy

$$0 \le g_n^U \le \mathrm{Per}_{\boldsymbol{B}}(\cdot; U).$$

Define

$$g_n(A) = \sum_{m=1}^{n} \beta_m g_n^{U_m}(A).$$

Let $A \in \mathcal{M}$ with $\mathrm{Per}^{\beta}_{\boldsymbol{B}}(A) < \infty$. Since for every $m \ge 1$, $g_n^{U_m}(A) \to \mathrm{Per}(A; U_m)$ as $n \to \infty$, the Lebesgue theorem with $0 \le g_n(A) \le \mathrm{Per}^{\beta}_{\boldsymbol{B}}(A) < \infty$ ensures that $g_n(A) \to \mathrm{Per}^{\beta}_{\boldsymbol{B}}(A)$ as $n \to \infty$.

If $A \in \mathcal{M}$ is such that $\mathrm{Per}^{\beta}_{\boldsymbol{B}}(A) = \infty$, let $M > 0$, and n_0 be such that $\sum_{m=1}^{n_0} \beta_m \mathrm{Per}_{\boldsymbol{B}}(A; U_m) \ge M + 1$. Let $n_1 \ge n_0$ be such that for $n \ge n_1$, $g_n^{U_m}(A) \ge \mathrm{Per}_{\boldsymbol{B}}(A; U_m) - 1$ for $1 \le m \le n_0$. Then, for $n \ge n_1$,

$$g_n(A) \ge \sum_{m=1}^{n_0} \beta_m \mathrm{Per}_{\boldsymbol{B}}(A; U_m) - \sum_{m \ge 1} \beta_m \ge M + 1 - 1 \ge M.$$

It follows that $g_n(A) \to \infty = \mathrm{Per}^{\beta}_{\boldsymbol{B}}(A)$.

Hence, according to Theorem 6, there exists a stationary RAMS X realising γ, which implies that $\gamma_X^{\mathrm{s}} = S_2$. Then, according to Proposition 7,

$$\mathrm{Per}^{\mathrm{s}}_{\boldsymbol{B}}(X) = \sum_{j=1}^{d} V^{\mathrm{s}}_{e_j}(X) = 2 \sum_{j=1}^{d} \mathrm{Lip}_j(S_2, 0) \le r.$$

4.3. Covariogram realisability problem for RACS of $\mathbb{R}$

The goal of this section is to establish a result similar to Theorem 5 for the specific covariogram of one-dimensional stationary RACS.

First let us discuss the definition of local covariogram admissibility of functions in arbitrary dimension $d \ge 1$. By analogy with the definition of $\mathcal{M}$-local covariogram admissible functions (see Definition 1), when considering RACS of $\mathbb{R}^d$, we say that a function $\gamma : \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}$ is *$\mathcal{F}$-local covariogram admissible* if for all 5-tuples $(q \ge 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$,

$$\left[\text{for all } F \in \mathcal{F},\ c + \sum_{i=1}^{q} a_i \delta_{y_i; W_i}(F) \ge 0\right] \Longrightarrow c + \sum_{i=1}^{q} a_i \gamma(y_i; W_i) \ge 0.$$

Besides, we say that $S_2 : \mathbb{R} \to \mathbb{R}$ is *$\mathcal{F}$-specific covariogram admissible* if $(y, W) \mapsto S_2(y)\mathcal{L}^d(W)$ is $\mathcal{F}$-local covariogram admissible. However, this distinction is superfluous since these two notions of admissibility are strictly equivalent.

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Proposition 10. *A function* $\gamma : \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}$ *is* $\mathcal{F}$*-local covariogram admissible if and only if it is* $\mathcal{M}$*-local covariogram admissible.*

Proof. The proof of this equivalence relies on the continuity of local covariograms for the convergence in measure and the density of compact sets due to the Lusin theorem. It consists of showing that for all 5-tuples $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$,

$$\left[\text{for all } F \in \mathcal{F},\ c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}(F) \geq 0\right] \Longleftrightarrow \left[\text{for all } A \in \mathcal{M},\ c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}(A) \geq 0\right].$$

Since $\mathcal{F} \subset \mathcal{M}$, the implication $\Leftarrow$ is clear. To show the converse, let $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathcal{W}^q, c \in \mathbb{R})$ be such that for all $F \in \mathcal{F}$, $c + \sum_{i=1}^{q} a_i \delta_{y_i}(F) \geq 0$, and we show that this inequality is valid for any $A \in \mathcal{M}$. One can suppose that A is bounded since according to (6), one can replace A by $A \cap \bigcup_{i=1}^{q} W_i \cup (-y_i + W_i)$ without changing the value of $c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}(A)$. Then, by the Lusin theorem (see, e.g. [9]), there exists a sequence of compact sets $K_n \subset A$ that converges in measure towards A, i.e. $\mathcal{L}^d(A \Delta K_n) \to 0$. Since each K_n is closed, for all n, $c + \sum_{i=1}^{q} a_i \delta_{y_i}(K_n) \geq 0$. Since the sequence $(K_n)_{n\in\mathbb{N}}$ converges in measure towards A, thanks to the fact that the local covariogram $B \mapsto \delta_{y;W}(B)$ is continuous for the local convergence in measure (see Proposition 4) for all $(y_i; W_i)$, $\delta_{y_i;W_i}(K_n)$ tends to $\delta_{y_i;W_i}(A)$, and, thus, letting n tend to $+\infty$ the inequality $c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}(A) \geq 0$ follows.

Now that this technical point has been clarified we are in position to formulate our result for the realisability of specific covariogram of stationary RACS of $\mathbb{R}$.

Theorem 7. *Suppose that the probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *is complete. Let* $S_2 : \mathbb{R} \to \mathbb{R}$ *be a given function and let* $r > 0$*. Then* S_2 *is the covariogram of a stationary RACS* $Z \subset \mathbb{R}$ *such that*

$$\mathbb{E}\{\mathcal{H}^0(\partial Z) \cap (0, 1)\} \leq r$$

if and only if S_2 *is* $\mathcal{F}$*-specific covariogram admissible and Lipschitz with Lipschitz constant* $L \leq r/2$.

Proof. (i) *Necessity.* If there exists a stationary RACS $Z \subset \mathbb{R}$ such that $\mathbb{E}\{\mathcal{H}^0(\partial Z) \cap (0, 1)\} \leq r$ then S_2 is necessarily $\mathcal{F}$-specific covariogram admissible and, according to Proposition 7, S_2 is Lipschitz with Lipschitz constant $L = \frac{1}{2}\mathbb{E}\{\text{Per}(Z); (0, 1)\}$. But $\text{Per}(Z; (0, 1)) \leq \mathcal{H}^0(\partial Z \cap (0, 1))$ yields $L \leq \frac{1}{2}\mathbb{E}\{\mathcal{H}^0(\partial Z \cap (0, 1))\} \leq r/2$.

(ii) *Sufficiency.* Suppose that S_2 is $\mathcal{F}$-specific covariogram admissible and Lipschitz with Lipschitz constant $L \leq r/2$. Then, by Proposition 10, γ is $\mathcal{M}$-specific covariogram admissible, and, thus, by Theorem 5 there exists a RAMS $X \subset \mathbb{R}$ such that S_2 is the specific covariogram of X and $\mathbb{E}\{\text{Per}(X); (0, 1)\} \leq r$. By Proposition 3, there exists a RACS $Z \subset \mathbb{R}$, equivalent in measure to X, such that $\text{Per}(X; (0, 1)) = \mathcal{H}^0(\partial Z \cap (0, 1))$ a.s. Then the specific covariogram of Z is also equal to S_2 and $\mathbb{E}\{\mathcal{H}^0(\partial Z \cap (0, 1))\} = \mathbb{E}\{\text{Per}(X; (0, 1))\} \leq r$.

Note that although the geometry of sets with finite perimeter on the line seems quite simplistic, a direct proof of the realisability result above is far from trivial.

Appendix A. The Radon–Nikodym theorem for random measures

Theorem 8. *Let* U *be an open subset of* $\mathbb{R}^d$*. Let* $\mu : \Omega \mapsto \boldsymbol{M}(U)$ *be a random signed Radon measure on* U *such that for all* $\omega \in \Omega$*, the measure* $\mu(\omega, \cdot)$ *is absolutely continuous with*

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respect to the Lebesgue measure $\mathcal{L}^d$. *Then, there exists a jointly measurable map* f *on* $(\Omega \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(U))$ *such that for all* $\omega \in \Omega$, $f(\omega, \cdot)$ *is a Radon–Nikodym derivative of* $\mu(\omega, \cdot)$ *with respect to the Lebesgue measure* $\mathcal{L}^d$.

Proof. This proof follows the outline of [3, Exercise 6.10.72]. It is enough to consider the case $U = \mathbb{R}^d$, since for $U \subset \mathbb{R}^d$ we can always extend the random measure by 0 over $\mathbb{R}^d$ and take the restriction of f to U afterwards. Denote by $B(x, r)$ the open ball of center x and radius r, and by κ_d the Lebesgue measure of the unit ball of $\mathbb{R}^d$, so that for all $x \in \mathbb{R}^d$ and $r > 0$, $\mathcal{L}^d(B(x, r)) = \kappa_d r^d$. For any $\omega \in \Omega$, according to the Besicovitch derivation theorem (see, e.g. [2, Theorem 2.22]), the derivative of the measure $\mu(\omega, \cdot)$ with respect to $\mathcal{L}^d$, i.e.

$$\lim_{r \to 0+} \frac{\mu(\omega, B(x, r))}{\kappa_d r^d}, \qquad x \in \mathbb{R}^d,$$

exists for $\mathcal{L}^d$-almost all $x \in \mathbb{R}^d$, is in $L^1(\mathbb{R}^d)$, and is a Radon–Nikodym derivative of the measure $\mu(\omega, \cdot)$. Let $(r_n)_{n \in \mathbb{N}}$ be a positive sequence decreasing to 0. For all $\omega \in \Omega$, $x \in \mathbb{R}^d$, and $n \in \mathbb{N}$, define

$$f_n(\omega, x) = \frac{\mu(\omega, B(x, r_n))}{\kappa_d r_n^d}.$$

As a consequence of the Besicovitch derivation theorem, for all $\omega \in \Omega$, the function

$$f(x, \omega) = \limsup_{n \to +\infty} f_n(\omega, x)\, \mathbf{1}_{\{\limsup_{n \to +\infty} f_n(\omega, x) = \liminf_{n \to +\infty} f_n(\omega, x)\}}$$

is a Radon–Nikodym derivative of $\mu(\omega, \cdot)$ with respect to the Lebesgue measure $\mathcal{L}^d$. Let us show that this function f is jointly measurable, i.e. $\mathcal{A} \otimes \mathcal{B}(U)$-measurable. Given the definition of f, and since the lim sup and lim inf of a countable sequence of measurable functions is a measurable function, it is enough to show that the functions f_n are jointly measurable. Let $n \in \mathbb{N}$. For all $x \in \mathbb{R}^d$, by definition of a random Radon measure, the map

$$\omega \mapsto f_n(\omega, x) = \frac{\mu(\omega, B(x, r_n))}{\kappa_d r_n^d}$$

is $\mathcal{A}$-measurable. Let us show that for all $\omega \in \Omega$, the map

$$x \mapsto f_n(\omega, x) = \frac{\mu(\omega, B(x, r_n))}{\kappa_d r_n^d}$$

is continuous over $\mathbb{R}^d$. Indeed, let $x \in \mathbb{R}^d$ and $(x_k)_{k \in \mathbb{N}}$ a sequence of points that tends to x. Then, for all $y \in \mathbb{R}^d \setminus \partial B(x, r_n)$, $\mathbf{1}_{\{y \in B(x_k, r_n)\}}$ tends to $\mathbf{1}_{\{y \in B(x, r_n)\}}$. Since the sphere $\partial B(x, r_n)$ is Lebesgue negligible, by absolute continuity, $\partial B(x, r_n)$ is also $\mu(\omega, \cdot)$-negligible. Hence, $\mathbf{1}_{\{y \in B(x_k, r_n)\}}$ tends to $\mathbf{1}_{\{y \in B(x, r_n)\}}$ for $\mu(\omega, \cdot)$-almost all $y \in \mathbb{R}^d$. Besides, since the sequence (x_k) tends to x, it is bounded, and, thus, there exists $R > 0$ such that $x_k \in B(0, R)$ for all $k \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$,

$$\left| \frac{\mathbf{1}_{\{y \in B(x_k, r_n)\}}}{\kappa_d r_n^d} \right| \leq \frac{\mathbf{1}_{\{y \in B(0, R + r_n)\}}}{\kappa_d r_n^d} \in L^1(\mu(\omega, \cdot)).$$

Hence, by dominated convergence,

$$\lim_{k \to +\infty} \frac{\mu(\omega, B(x_k, r_n))}{\kappa_d r_n^d} = \frac{\mu(\omega, B(x, r_n))}{\kappa_d r_n^d},$$

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that is $f_n(\omega, \cdot)$ is continuous at x. In conclusion, $\omega \mapsto f_n(\omega, x)$ is measurable and $x \mapsto f_n(\omega, x)$ is continuous, i.e. f_n is a Carathéodory function. Since $\mathbb{R}^d$ is a separable metric space, we can conclude that f_n is jointly measurable [1, Section 4.10].

References

[1] Aliprantis, C. D. and Border, K. C. (2006). *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd edn. Springer, Berlin.

[2] Ambrosio, L., Fusco, N. and Pallara, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press.

[3] Bogachev, V. I. (2007). *Measure Theory*, Vol. II. Springer, Berlin.

[4] Caselles, V., Chambolle, A., Moll, S. and Novaga, M. (2008). A characterization of convex calibrable sets in $\mathbb{R}^N$ with respect to anisotropic norms. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25,** 803–832.

[5] Chilès, J.-P. and Delfiner, P. (1999). *Geostatistics: Modeling Spatial Uncertainty*. John Wiley, New York.

[6] Daley, D. J. and Vere-Jones, D. (2008). *An Introduction to the Theory of Point Processes*, Vol. II: *General Theory and Structure*, 2nd edn. Springer, New York.

[7] Deza, M. M. and Laurent, M. (1997). *Geometry of Cuts and Metrics*. Springer, Berlin.

[8] Emery, X. (2010). On the existence of mosaic and indicator random fields with spherical, circular, and triangular variograms. *Math. Geosci.* **42,** 969–984.

[9] Evans, L. C. and Gariepy, R. F. (1992). *Measure Theory and Fine Properties of Functions*. CRC, Boca Raton, FL.

[10] Fritz, T. and Chaves, R. (2013). Entropic inequalities and marginal problems. *IEEE Trans. Inf. Theory* **59,** 803–817.

[11] Galerne, B. (2011). Computation of the perimeter of measurable sets via their covariogram. Applications to random sets. *Image Anal. Stereol.* **30,** 39–51.

[12] Galerne, B. (2014). Random fields of bounded variation and computation of their variation intensity. *Tech. Rep.* 2014-25, Laboratoire MAP5, Université Paris Descartes.

[13] Himmelberg, C. J. (1975). Measurable relations. *Fund. Math* **87,** 53–72.

[14] Hirsch, F. and Lacombe, G. (1999). *Elements of Functional Analysis* (Graduate Texts Math. **192**). Springer, New York.

[15] Jiao, Y., Stillinger, F. H. and Torquato, S. (2007). Modeling heterogeneous materials via two-point correlation functions: basic principles. *Phys. Rev. E* **76,** 031110.

[16] Kallenberg, O. (1986). *Random Measures*, 4th edn. Akademie-Verlag, Berlin.

[17] Kuna, T., Lebowitz, J. L. and Speer, E. R. (2011). Necessary and sufficient conditions for realizability of point processes. *Ann. Appl. Prob.* **21,** 1253–1281.

[18] Lachièze-Rey, R. (2013). Realisability conditions for second-order marginals of biphased media. *Random Structures Algorithms* 10.1002/rsa.20546.

[19] Lachièze-Rey, R. and Molchanov, I. (2015). Regularity conditions in the realisability problem with applications to point processes and random closed sets. *Ann. Appl. Prob.* **25,** 116–149.

[20] Lantuéjoul, C. (2002). *Geostatistical Simulation: Models and Algorithms*. Springer, Berlin.

[21] Masry, E. (1972). On covariance functions of unit processes. *SIAM J. Appl. Math.* **23,** 28–33.

[22] Matheron, G. (1975). *Random Sets and Integral Geometry*. John Wiley, New York.

[23] Matheron, G. (1993). Une conjecture sur la covariance d'un ensemble aléatoire. In *Cahiers de Géostatistique, Fascicule 3, Compte-Rendu des Journées de Géostatistique* (Fontainebleau, 1993), pp. 107–113.

[24] McMillan, B. (1955). History of a problem. *J. Soc. Ind. Appl. Math.* **3,** 119–128.

[25] Molchanov, I. (2005). *Theory of Random Sets*. Springer, London.

[26] Quintanilla, J. A. (2008). Necessary and sufficient conditions for the two-point phase probability function of two-phase random media. *Proc. R. Soc. London A* **464,** 1761–1779.

[27] Rataj, J. (2014). Random sets of finite perimeter. *Math. Nachr.* **288,** 1047–1056.

[28] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin.

[29] Shepp, L. A. (1963). On positive-definite functions associated with certain stochastic processes. Tech. Rep. 63-1213-11, Bell Laboratories.

[30] Straka, F. and Štěpán, J. (1988). Random sets in [0,1]. In *Transactions of the Tenth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes*, Vol. B, Reidel, Dordrecht, pp. 349–356.

[31] Torquato, S. (2002). *Random Heterogeneous Materials: Microstructure and Macroscopic Properties*. Springer, New York.

https://doi.org/10.1239/aap/1444308874 Published online by Cambridge University Press