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NEW GENERALIZED *h*-IMPLICATIONS

D. PEI✉[1] and Y. ZHU[1]

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Abstract

A new generalized class of fuzzy implications, called (h, f, g)-implications, is introduced and discussed in this paper. The results show that the new fuzzy implications possess some good properties, such as the left neutrality property and the exchange principle.

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1. Introduction

Similar to cases that generalize classical conjunction and disjunction to t-norms and t-conorms, fuzzy implications are generalizations of the classical implication of classical logic. As we know, fuzzy implications have a significant role in approximate reasoning and fuzzy control [2–5, 8, 11–13, 15]. Furthermore, these have been applied to many other fields including fuzzy decision making, image processing, expert systems, data mining, fuzzy relational equation, fuzzy mathematical morphology, fuzzy subsethood measures and so on.

In a survey paper on fuzzy implication functions, Mas et al. [8] showed the importance of having different fuzzy implications since they were used to represent imprecise knowledge. Recently, Fodor and Torrens [4] studied the historical development of logical connectives containing fuzzy implication functions for fuzzy sets and fuzzy logic.

In the past, many authors studied the ways or methods of generating fuzzy implications. Among these different methods, the most popular ones should be residuated implications, and so on, which are obtained from t-norms, t-conorms and negations (see the following book and articles [2, 3, 16]). Moreover, these methods have been extended to fuzzy implications derived from copulas, uninorms, quasi-copulas, overlap functions, aggregation operators and so on. At the same time, the

[1]Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China;
e-mail: peidw@163.com, kimi725@sina.com.

above implications for discrete fuzzy numbers and interval values are presented (see the following articles [4, 8]).

Other methods of generating fuzzy implications are obtained from given implications, such as the so-called threshold generation method, the convex combinations, ordinal sum method, ordering method and the method from a new binary operation on the set of all fuzzy implications. Most of the above mentioned methods preserve the most usual properties of initial implications.

Unlike the above methods derived from binary operators, a class of fuzzy implications are constructed, based on generating functions. Yager proposed f- and g-generated implications [15] (also see the following book and articles [1, 2, 9]). Recently, Xie and Liu [14] proposed a generalization of Yager's f-generated implications. Similarly, Hilnena et al. [6] generated a class of new implications through two fuzzy negations and a uninorm. Massanet and Torrens introduced h-implications by means of the additive generators of representable uninorms [10]. Liu proposed a new class of fuzzy implications through the so-called generalized h-generators [7].

In this paper, we propose a new class of fuzzy implications as new generalizations of h-implications, and investigate some basic properties of these new implications.

2. Preliminaries

To make this work self-contained, we recall some necessary concepts and known results used in the rest of the paper. We assume that the readers are familiar with the basic facts about t-norms and t-conorms [2].

Definition 2.1 [2]. A function $I : [0, 1]^2 \to [0, 1]$ is called a fuzzy implication, (shortly, implication), if, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions hold:

(I1) if $x_1 \le x_2$, then $I(x_1, y) \ge I(x_2, y)$, that is, $I(\cdot, y)$ is decreasing;
(I2) if $y_1 \le y_2$, then $I(x, y_1) \le I(x, y_2)$, that is, $I(x, \cdot)$ is increasing;
(I3) $I(0, 0) = 1$;
(I4) $I(1, 1) = 1$;
(I5) $I(1, 0) = 0$.

The set of all fuzzy implications is denoted by $\mathcal{FI}$.

Definition 2.2 [2]. A fuzzy implication I is said to satisfy:

(i) the left neutrality property (NP), if $I(1, y) = y$, $y \in [0, 1]$;
(ii) the exchange principle (EP), if $I(x, I(y, z)) = I(y, I(x, z))$, with $x, y, z \in [0, 1]$;
(iii) the identity principle (IP), if $I(x, x) = 1$, $x \in [0, 1]$;
(iv) the ordering property (OP), if $I(x, y) = 1 \iff x \le y$, with $x, y \in [0, 1]$;
(v) the law of contraposition (CP(N)) with respect to a negation N, if $I(x, y) = I(N(y), N(x))$, with $x, y \in [0, 1]$.

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REMARK 2.3. In fuzzy logic, there are three kinds of implications obtained from a t-norm, s-norm and negation which are very usual, that is, (S, N)-implication, R-implication (or residual implication), QL-implication. Given a t-norm T, a t-conorm S and a negation N,

(a) (S, N)-implication is defined by

$$I_{S,N}(x, y) = S(N(x), y), \quad x, y \in [0, 1];$$

(b) R-implication is defined by

$$I_T(x, y) = \sup\{t \in [0, 1] \mid T(x, t) \leq y\}, \quad x, y \in [0, 1];$$

(c) QL-operation (may not necessarily be an implication) is defined by

$$I_{T,S,N}(x, y) = S(N(x), T(x, y)), \quad x, y \in [0, 1].$$

More details including definitions, examples and properties of these implications may be found in the works of Baczyński and Jayaram [2] and Dubois and Prade [3]. Yager [15] introduced two new kinds of implications as follows.

DEFINITION 2.4 [15]. Let $f : [0, 1] \to [0, \infty]$ be a strictly decreasing continuous function with $f(1) = 0$. The function $I : [0, 1]^2 \to [0, 1]$, defined by

$$I(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1]$$

with the understanding that $0 \cdot \infty = \infty$, is called an *f-generated implication*. The function f itself is called an f-generator of I. In such a case, to emphasize the apparent relation we write I_f instead of I.

DEFINITION 2.5 [15]. Let $g : [0, 1] \to [0, \infty]$ be a strictly increasing continuous function with $g(0) = 0$. The function $I : [0, 1]^2 \to [0, 1]$, defined by

$$I(x, y) = g^{(-1)}\Big(\frac{1}{x} \cdot g(y)\Big), \quad x, y \in [0, 1]$$

with the understanding that $\infty \cdot 0 = \infty$, $1/0 = \infty$, is called a g-generated implication, where the function $g^{(-1)}$ is the pseudo-inverse of g given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \in [0, g(1)), \\ 1 & \text{if } x \in [g(1), \infty]. \end{cases}$$

The function g itself is called a g-generator of I. In such a case, we will write I^g instead of I.

Massanet and Torrens [10] introduced the so-called h-implications based on the additive generators of representable uninorms.

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DEFINITION 2.6 [10]. Fix an $e \in (0, 1)$ and let $h : [0, 1] \to [-\infty, \infty]$ be a strictly increasing continuous function with $h(1) = \infty$, $h(e) = 0$ and $h(0) = -\infty$. The function $I : [0, 1]^2 \to [0, 1]$ defined by,

$$I(x, y) = \begin{cases} 1 & \text{if } x = 0, \\ h^{-1}(x \cdot h(y)) & \text{if } x > 0,\ y \leq e, \\ h^{-1}\Big(\dfrac{1}{x} \cdot h(y)\Big) & \text{if } x > 0,\ y > e, \end{cases}$$

is called an h-implication. The function h itself is called a *h-generator* (with respect to e) of the function I. In such a case, we will write I^h instead of I.

DEFINITION 2.7 [2]. Let $I \in \mathcal{FI}$. The function $N_I : [0, 1] \to [0, 1]$ defined by $N_I = I(x, 0)$ is called the natural negation of I or the negation induced by I.

REMARK 2.8. From the book by Baczyński and Jayaram [2], we may show that if $I_{S,N}$ is an (S, N)-implication (see [16]), $I_{T,S,N}$ is a QL-operation and $I^{T,S,N}$ is a D-operation, then

$$N_{I_{S,N}} = N, \quad N_{I_{T,S,N}} = N, \quad N_{I^{T,S,N}} = N.$$

PROPOSITION 2.9 [2]. *If $I \in \mathcal{FI}$ satisfies the exchange principle* (EP) *and the left neutrality property* (NP) *then I satisfies the law of contraposition* CP*(N) if and only if $N = N_I$ and N_I is strong.*

3. New class of implications

In this section, we introduce a new class of fuzzy implications, discuss the intersection of the new class with other known classes of implications and study the law of importation for the new implications.

As stated above, Massanet and Torrens [10] introduced the h-implications by means of a h-generator (see Definition 2.6). In this section, similar to the case of the h-implications, we introduce a new class of implications.

DEFINITION 3.1. Let $c \in (0, 1)$ and $h : [0, 1] \to [-\infty, \infty]$ be a h-generator. A function $I : [0, 1]^2 \to [0, 1]$ defined by

$$I(x, y) = \begin{cases} 1 & \text{if } x = 0, \\ h^{-1}(f(x) \cdot h(y)) & \text{if } x > 0,\ y \leq c, \\ h^{-1}(g(x) \cdot h(y)) & \text{if } x > 0,\ y > c, \end{cases}$$

with the understanding that $\infty \cdot 0 = 0$, is called an (h, f, g)-operation, where $f : [0, 1] \to [0, 1]$ is an increasing function satisfying $f(0) = 0$, $f(1) = 1$; $g : [0, 1] \to [1, \infty]$ is a decreasing function satisfying $g(0) = \infty$ and $g(1) = 1$.

In such case, to emphasize the apparent relation, we write $I_{h,f,g}$ instead of I.

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EXAMPLE 3.2. (i) Let $f(x) = x^2$, $g(x) = 1/x^2$ and $h(x) = \ln(x/(1-x))$ (with respect to $c = 1/2$). Then for $x, y \in [0, 1]$ we have

$$I_{h,f,g}(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{y^{x^2}}{(1-y)^{x^2} + y^{x^2}} & \text{if } x > 0,\ y \le \dfrac{1}{2}, \\ \dfrac{y^{1/x^2}}{(1-y)^{1/x^2} + y^{1/x^2}} & \text{if } x > 0,\ y > \dfrac{1}{2}. \end{cases}$$

(ii) Let $f(x) = x^3$, $g(x) = 1/x^3$ and $h(x) = \ln(-(1/\beta)\cdot\ln(1-x))$ (with respect to $c = 1 - \exp(-\beta)$ and $\beta > 0$). Then for $x, y \in [0, 1]$ we have

$$I_{h,f,g}(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ 1 - e^{[-\beta\cdot\{(-1/\beta)\cdot\ln(1-y)\}]^{x^2}} & \text{if } x > 0,\ y \le 1 - e^{-\beta}, \\ 1 - e^{[-\beta\cdot\{(-1/\beta)\cdot\ln(1-y)\}]^{1/x^2}} & \text{if } x > 0,\ y > 1 - e^{-\beta}. \end{cases}$$

The next proposition shows that an (h, f, g)-operation is indeed a fuzzy implication.

PROPOSITION 3.3. *If $I = I_{h,f,g}$ is an (h, f, g)-operation, then I is a fuzzy implication, that is, $I \in \mathcal{FI}$.*

REMARK 3.4. (i) Based on the above results, we can say "(h, f, g)-implications" instead of "(h, f, g)-operations".

(ii) If $f(x) = x$ and $g(x) = 1/x$, then the (h, f, g)-implication is the same as the h-implication. This fact shows that the new implication $I_{h,f,g}$ is a natural generalization of the h-implications.

In the next theorem, we investigate a necessary and sufficient condition under which two (h, f, g)-implications are equal, when functions f and g are fixed.

THEOREM 3.5. *Let $f : [0, 1] \to [0, 1]$ be a strictly increasing continuous function satisfying $f(0) = 0$, $f(1) = 1$, and let $g : [0, 1] \to [1, \infty]$ be a strictly decreasing continuous function satisfying $g(0) = \infty$ and $g(1) = 1$. Also let $h_1(x)$ and $h_2(x)$ be two h-generators with respect to a fixed $c \in (0, 1)$. Then the following statements are equivalent:*

(i) $I_{h_1,f,g} = I_{h_2,f,g}$;

(ii) *there exists constants $a, b \in (0, \infty)$ such that*

$$h_2(x) = \begin{cases} a \cdot h_1(x) & \text{if } x \in [0, c], \\ b \cdot h_1(x) & \text{if } x \in (c, 1]. \end{cases}$$

PROOF. First we prove that (i) implies (ii). Assume that $I_1 = I_{h_1,f,g}$ and $I_2 = I_{h_2,f,g}$ are two (h, f, g)-implications with $I_1 = I_2$.

From the definition of (h, f, g)-implications,

$$h_1^{-1}(f(x)\cdot h_1(y)) = h_2^{-1}(f(x)\cdot h_2(y)), \quad x > 0,\ y \le c. \tag{3.1}$$

$$h_1^{-1}(g(x)\cdot h_1(y)) = h_2^{-1}(g(x)\cdot h_2(y)), \quad x > 0,\ y > c. \tag{3.2}$$

When $x > 0$ and $y \le c$,

$$\begin{aligned} \text{equation (3.1)} \Leftrightarrow \quad & h_2 \circ h_1^{-1}(f(x) \cdot h_1(y)) = f(x) \cdot h_2(y) \\ \Leftrightarrow \quad & h_2 \circ h_1^{-1}(f(x) \cdot h_1(y)) = f(x) \cdot (h_2 \circ h_1^{-1})(h_1(y)), \end{aligned}$$

and when $x > 0$ and $y > c$,

$$\begin{aligned} \text{equation (3.2)} \Leftrightarrow \quad & h_2 \circ h_1^{-1}(g(x) \cdot h_1(y)) = g(x) \cdot h_2(y) \\ \Leftrightarrow \quad & h_2 \circ h_1^{-1}(g(x) \cdot h_1(y)) = g(x) \cdot (h_2 \circ h_1^{-1})(h_1(y)). \end{aligned}$$

Substituting $k = h_2 \circ h_1^{-1}$ and $z = h_1(y)$, we obtain the following equations:

$$k(f(x) \cdot z) = f(x) \cdot k(z), \quad x \in (0, 1],\ z \in [-\infty, 0], \tag{3.3}$$

and

$$k(g(x) \cdot z) = g(x) \cdot k(z), \quad x \in (0, 1],\ z \in (0, \infty], \tag{3.4}$$

where $k : [-\infty, \infty] \to [-\infty, \infty]$ is a continuous strictly increasing bijection with $k(0) = 0$.

In equation (3.3), letting $z = -1$ yields

$$k(-f(x)) = f(x) \cdot k(-1), |\ x \in (0, 1]. \tag{3.5}$$

Fix arbitrarily $z \in (-\infty, 0)$. Obviously, there exists $x_1 \in (0, 1]$ such that $f(x_1) \cdot z \in [-1, 0)$. From equations (3.3) and (3.5), we get

$$k(z) = \frac{1}{f(x_1)} \cdot k(f(x_1) \cdot z) = \frac{1}{f(x_1)} \cdot (-f(x_1)) \cdot z \cdot k(-1) = -z \cdot k(-1).$$

Now, by the definition of k, we have $h_2 \circ h_1^{-1}(z) = -z \cdot (h_2 \circ h_1^{-1})(-1)$, and hence $h_2(y) = -h_1(y) \cdot (h_2 \circ h_1^{-1})(-1) = h_1(y) \cdot (-h_2 \circ h_1^{-1}(-1)), y \in (0, c)$. Thus, letting $a = -h_2 \circ h_1^{-1}(-1) \in (0, \infty)$, yields $h_2(y) = a \cdot h_1(y)$, $y \in (0, c)$. When $y = 0, c$, we also have $h_2(x) = a \cdot h_1(x)$, since $h_1(0) = h_2(0) = -\infty$ and $h_1(c) = h_2(c) = 0$. Hence, $h_2(y) = a \cdot h_1(y)$, $y \in [0, c]$.

In equation (3.4), letting $z = 1$ yields

$$k(g(x)) = g(x) \cdot k(1), \quad x \in (0, 1]. \tag{3.6}$$

Fix arbitrarily $z \in (0, \infty)$. Then, there exists $x_2 \in (0, 1]$ such that $g(x_2) \cdot z \in [1, \infty)$. From equations (3.4) and (3.6),

$$k(z) = \frac{1}{g(x_2)} \cdot k(g(x_2) \cdot z) = \frac{1}{g(x_2)} \cdot g(x_2) \cdot z \cdot k(1) = z \cdot k(1).$$

Now, by the definition of k, we have $h_2 \circ h_1^{-1}(z) = z \cdot (h_2 \circ h_1^{-1})(1)$, and hence $h_2(y) = h_1(y) \cdot (h_2 \circ h_1^{-1})(1) = h_1(y) \cdot (h_2 \circ h_1^{-1}(1))$, $y \in (c, 1)$. Thus, letting $b = h_2 \circ h_1^{-1}(1) \in (0, \infty)$, yields $h_2(y) = b \cdot h_1(y)$, $y \in (c, 1)$. When $y = 1$, we also have $h_2(x) = b \cdot h_1(x)$ since $h_1(1) = h_2(1) = \infty$. Hence, $h_2(y) = b \cdot h_1(y)$, $y \in (c, 1]$.

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Next, we prove that (ii) implies (i). Let h_1 be a h-generator and $a, b \in (0, \infty)$. Define

$$h_2(x) = \begin{cases} a \cdot h_1(x) & \text{if } x \in [0, c], \\ b \cdot h_1(x) & \text{if } x \in (c, 1]. \end{cases}$$

Evidently, h_2 is a well-defined h-generator. Moreover, for any $z \in [-\infty, \infty]$,

$$h_2{}^{-1}(z) = \begin{cases} h_1^{-1}(z/a) & \text{if } z \in [-\infty, 0], \\ h_1^{-1}(z/b) & \text{if } z \in (0, \infty]. \end{cases}$$

Next, we prove that $I_1 = I_{h_1,f,g}$ and $I_2 = I_{h_2,f,g}$ are equal. Note that when $x = 0$, the equality $I_1(0, y) = I_2(0, y)$ holds for all $y \in [0, 1]$. When $x > 0$ and $y \le c$,

$$\begin{aligned} I_2(x, y) &= h_2^{-1}(f(x) \cdot h_2(y)) = h_2^{-1}(f(x) \cdot a \cdot h_1(y)) \\ &= h_1^{-1}\Big(\frac{f(x) \cdot a \cdot h_1(y)}{a}\Big) \\ &= I_1(x, y). \end{aligned}$$

Also, when $x > 0$ and $y > c$, we obtain

$$\begin{aligned} I_2(x, y) &= h_2^{-1}(g(x) \cdot h_2(y)) = h_2^{-1}(g(x) \cdot b \cdot h_1(y)) \\ &= h_1^{-1}\Big(\frac{g(x) \cdot b \cdot h_1(y)}{b}\Big) = I_1(x, y), \end{aligned}$$

which proves that $I_1 = I_2$. □

Corollary 3.6. *Let $f : [0, 1] \to [0, 1]$ be a strictly increasing continuous function satisfying $f(0) = 0$, $f(1) = 1$ and $g : [0, 1] \to [1, \infty]$ be a strictly decreasing continuous function satisfying $g(0) = \infty$ and $g(1) = 1$. Also, let $h_1(x)$ and $h_2(x)$ be two h-generators with respect to a fixed $c \in (0, 1)$ such that $-h_2 \circ h_1^{-1}(-1) = h_2 \circ h_1(1)$. Then the following statements are equivalent:*

(i) $I_{h_1,f,g} = I_{h_2,f,g}$;
(ii) *there exists a constant $a \in (0, \infty)$ such that $h_2(x) = a \cdot h_2(x)$ for all $x \in [0, 1]$.*

Proposition 3.7. *If $I = I_{h,f,g}$ is an (h, f, g)-implication where $g(x) < \infty$ and $f(x) > 0$, $x \in (0, 1]$, then we have the following statements:*

(i) *I satisfies the left neutrality property* (NP)*;*
(ii) $I(x, y) \le c \Longleftrightarrow (x > 0$ *and* $y \le c)$, $I(x, y) > c \Longleftrightarrow (x = 0$ *or* $y > c)$*;*
(iii) *I satisfies the exchange principle* (EP)*;*
(iv) $I(x, x) = 1 \Longleftrightarrow (x = 0$ *or* $x = 1)$*. This shows that I does not satisfy the identity principle* (IP)*;*
(v) $I(x, y) = 1 \Longleftrightarrow (x = 0$ *or* $y = 1)$*. This shows that I does not satisfy the ordering property* (OP)*.*

The following proposition discusses the natural negations of the (h, f, g)-implications.

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Proposition 3.8. *If $I = I_{h,f,g}$ is an (h, f, g)-implication, then*

$$N_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ c & \text{if } f(x) = 0,\ x \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

In particular, if $f(x) > 0$ for all $x \in (0, 1]$, then $N_I = N_1$ is defined by

$$N_1(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Corollary 3.9. *If $I = I_{h,f,g}$ is an (h, f, g)-implication, then I does not satisfy the law of contraposition* CP(N) *with any fuzzy negation N.*

Following this, we explore the continuity of (h, f, g)-implications.

Proposition 3.10. *Let $I = I_{h,f,g}$ be an (h, f, g)-implication where f and g are continuous functions. Then I is continuous except at the points $(0, y)$ with $y \leq c$.*

Next several theorems show the fact that the intersections of (h, f, g)-implications, (S, N)-implications, QL-implications, D-implications and R-implications generated from left-continuous t-norms are almost empty.

Theorem 3.11. *Let $I = I_{h,f,g}$ be an (h, f, g)-implication where for f there exists $x_1 \in (0, 1)$ such that $f(x_1) > 0$. Then I is not an (S, N)-implication.*

Proof. Assume that I is a (S, N)-implication obtained form a s-norm S and a fuzzy negation N. From Remark 2.8, we have $N_I = N$. However, by Proposition 3.8,

$$N_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ c & \text{if } f(x) = 0,\ x \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, we know that the (S, N)-implication obtained from N_I is

$$I(x, y) = \begin{cases} 1 & \text{if } x = 0, \\ S(c, y) & \text{if } f(x) = 0,\ x \neq 0, \\ y & \text{otherwise}. \end{cases}$$

Then when $y \in (c, 1)$, we have

$$I(x_1, y) = h^{-1}(f(x_1) \cdot h(y)) = y.$$

We get $f(x_1) \cdot h(y) = h(y)$, a contradiction. This shows that I is not a (S, N)-implication. □

Similarly, for QL-implications we can obtain the following similar conclusion.

Theorem 3.12. *Let $I = I_{h,f,g}$ be an (h, f, g)-implication. Then I is not a QL-implication.*

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THEOREM 3.13. *Let $I = I_{h,f,g}$ be an (h, f, g)-implication. Then I is not a D-implication.*

PROPOSITION 3.14 [2]. *For a function $I : [0, 1]^2 \to [0, 1]$, the following statements are equivalent:*

(i) *I is an R-implication generated from a left-continuous t-norm;*
(ii) *I satisfies* IP, EP, OP *and it is right-continuous with respect to the second variable.*

THEOREM 3.15. *Let $I = I_{h,f,g}$ be an (h, f, g)-implication. Then I is not an R-implication obtained from a left-continuous t-norm.*

Now we investigate the intersections between (h, f, g)-implications and f- and g-generated implications.

PROPOSITION 3.16 [2]. *Let f be an f-generator, I_f an f-generated implication. Then:*

(i) *the natural negation N_{I_f} is a strict negation if and only if $f(0) < \infty$;*
(ii) *I_f is continuous except at the point (0,0) if and only if $f(0) = \infty$.*

THEOREM 3.17. *Let $I = I_{h,f,g}$ be an (h, f, g)-implication. Then I is not an f-generated implication.*

PROOF. If $f(0) < \infty$, then from Proposition 3.16, N_I is a strict negation, which contradicts Proposition 3.8.

If $f(0) = \infty$, then from Proposition 3.16, I is continuous except at the point (0,0), which contradicts Proposition 3.10. Hence, I is not a f-generated implication. □

PROPOSITION 3.18 [2]. *Let g be a g-generator, I^g a g-generated implication. Then I^g is continuous except at the point (0,0).*

From this proposition, we can easily prove the following theorem.

THEOREM 3.19. *Let $I = I_{h,f,g}$ be a (h, f, g)-implication. Then I is not a g-generated implication.*

In classical logic, $(p \wedge q) \to r \equiv (p \to (q \to r))$ is a tautology which is called the *law of importation* (LI). The general form of the above equivalence is given as

$$I(T(x, y), z) = I(x, I(y, z)), \quad x, y, z \in [0, 1],$$

where $I \in \mathcal{FI}$, T is a t-norm. In this case, we say that the implication I satisfies the LI with respect to T.

The following conclusion shows the relationship between the $I_{h,f,g}$- implications and the LI.

PROPOSITION 3.20. *Suppose that $I = I_{h,f,g}$ is an (h, f, g)-implication. If I satisfies the* LI *with a t-norm T, then T is positive, that is, $T(x, y) = 0$ if and only if $x = 0$ or $y = 0$.*

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THEOREM 3.21. *Suppose that T is a t-norm and $I = I_{h,f,g}$ is an (h, f, g)-implication, where f is a strictly increasing continuous function and g is a strictly decreasing continuous function. Then the following statements are equivalent:*

(i) *I satisfies the* LI *with respect to T;*
(ii) $T(x,y) = f^{-1}(f(x) \cdot f(y)) = g^{-1}(g(x) \cdot g(y))$, $x, y \in [0, 1]$.

4. Conclusions

In this paper, we proposed a class of new fuzzy implications, called (h, f, g)-implications, which were generalizations of the h-implications proposed by Massanet and Torrens [10]. We investigated some basic properties of these new implications. In the future, we plan to explore the law of importation for the new implications, and establish the relationship of the new implications with the R-implications, (S, N)-implications, D-implications, QL-implications, f-generated and g-generated implications. This may lead to possible applications of these new implications in approximate reasoning and fuzzy control.

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References

[1] M. Baczyński and B. Jayaram, "Yager's classes of fuzzy implications: some properties and intersections", *Kybernetika* **43** (2007) 157–182; http://eudml.org/doc/33849.
[2] M. Baczyński and B. Jayaram, *Fuzzy implications* (Springer, Berlin, 2008); doi:10.1007/978-3-540-69082-5.
[3] D. Dubois and H. Prade, "Fuzzy sets in approximate reasoning (I)", *Fuzzy Sets Syst.* **40** (1991) 143–202; doi:10.1016/0165-0114(91)90050-Z.
[4] J. Fodor and J. Torrens, "An overview of fuzzy logic connectives on the unit interval", *Fuzzy Sets Syst.* **281** (2015) 183–187; doi:10.1016/j.fss.2015.05.016.
[5] S. Gottwald, *A treatise on many-valued logics*, Volume 9 of *Studies in Logic and Computation* (Research Studies Press Ltd, Baldock, 2001).
[6] D. Hilnena, M. Kalina and P. Kral, "A class of implications related to Yager's f-implications", *Inform. Sci.* **260** (2014) 171–184; doi:10.1016/j.ins.2013.09.045.
[7] H. W. Liu, "A new class of fuzzy implications derived from generalized h-generators", *Fuzzy Sets Syst.* **24** (2013) 63–92; doi:10.1016/j.fss.2012.11.022.
[8] M. Mas, M. Monserrat, J. Torrens and E. Trillas, "A survey on fuzzy implications functions", *IEEE Trans. Fuzzy Syst.* **15** (2007) 1107–1121; doi:10.1109/TFUZZ.2007.896304.
[9] S. Massanet and J. Torrens, "On the characterization of Yager's implications", *Inform. Sci.* **201** (2012) 1–18; doi:10.1016/j.ins.2012.03.008.
[10] S. Massanet and J. Torrens, "On a new class of fuzzy implications: h-implications and generalizations", *Inform. Sci.* **181** (2014) 2111–2127; doi:10.1016/j.ins.2011.01.030.
[11] D. Pei, "R_0 implication: characteristics and applications", *Fuzzy Sets Syst.* **131** (2002) 297–302; doi:10.1016/S0165-0114(02)00053-2.

[12] D. Pei, "The unified algorithms of triple I methods for fuzzy reasoning", *Inform. Sci.* **178** (2008) 520–530; doi:10.1016/j.ins.2007.09.003.
[13] G. J. Wang, "On the logic foundation of fuzzy reasoning", *Inform. Sci.* **117** (1999) 47–88; doi:10.1016/S0020-0255(98)10103-2.
[14] A. F. Xie and H. W. Liu, "A generalization of Yager's f-generated implications", *Internat. J. Approx. Reason.* **54** (2013) 35–46; doi:10.1016/j.ijar.2012.08.005.
[15] R. R. Yager, "On some new classes of implication operators and their role in approximate reasoning", *Inform. Sci.* **167** (2004) 193–216; doi:10.1016/j.ins.2003.04.001.
[16] Y. Zhu and D. Pei, "On the characterizations of D-implications", *Fuzzy Syst. Math.* **29** (2015) 88–98.