Teaching Notes

Bicentric polygons, their incircles, circumcircles and the circles between

A bicentric polygon is one which possesses both an incircle and a circumcircle. To every bicentric polygon belong sets of circles which relate to them in the way described below (Figure 1). We take the triangle as our exemplar. We state results which students can verify using elementary methods. They may consult [1] for our own proofs but in several cases they may spot a neater approach. If so they should submit it to this journal as Feedback.

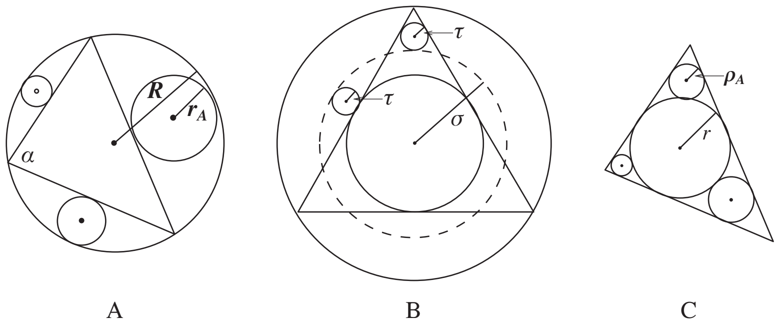

FIGURE 1

In all cases the small circles are of maximum size.

In Figure 1(A) the small circles are tangent *in*ternally to the *circum*circle, *ex*ternally to the *sides*.

We shall call these type *I* circles.

In Figure 1(C) the small circles are tangent *ex*ternally to the *in*circle, internally to the side-pairs defining each *vertex*.

We shall call these type *II* circles.

In Figure 1(B) the triangle is regular, the small circles equal. Between the incircle and the circumcircle is a circle we shall call the in-out circle. It is tangent internally to a set of type *II* circles, *ex*ternally to a set of type *I* circles.

Notation

A, C: triangle sides a opposite angle α, etc.

A: Circumradius R, small circle radii r_A, etc.

C: Inradius r, small circle radii ρ_A, etc.

B: In-out radius σ, small circle radii τ.

https://doi.org/10.1017/mag.2023.29 Published online by Cambridge University Press

Results

Students can take all the geometry they need from Figure 1.

Configuration A

1. $\frac{3R}{4} \leqslant r_A + r_B + r_C < R.$

2. $p_a q_b q_c + p_b q_c q_a + p_c q_a q_b = 4 p_a p_b p_c$,
where $p_a = \sqrt{r_A(R - r_A)}$, etc., $q_a = (R - 2r_A)$, etc.

Is it possible to derive an equation where R is the subject, enabling a comparison with 4?

Note in passing the standard identity $abc = 2(a + b + c)Rr$, which would enable us to rewrite equation 2 in terms of r.

Configuration C

3. $r \leqslant \rho_A + \rho_B + \rho_C < 2r$

Is a purely geometric proof possible here?

4. $r = \sqrt{\rho_A \rho_B} + \sqrt{\rho_B \rho_C} + \sqrt{\rho_C \rho_A}$. [2]

Configurations A and C

For regular n-gons,

2. becomes $R_e = \left(\frac{2}{1 - c}\right) r_e$,

4. becomes $r_e = \left(\frac{1 + c}{1 - c}\right) \rho_e$,

where $c = \cos \frac{\pi}{n}$, R_e, r_e are the respective circumradii and inradii, and r_e, ρ_e the respective small circle radii in the symmetrical case.

Configuration B

5. For the regular n-gon:
$\tau : r : \sigma : R :: c(1 - c) : c(1 + 3c) : c(3 + c) : (1 + 3c)$
where $c = \cos \frac{\pi}{n}$. **[5.1]**

For the equilateral triangle therefore: $\tau : r : \sigma : R :: 1 : 5 : 7 : 10$.

By Niven's theorem this ratio is integral for no other regular polygon.

6. $2(\tau + \sigma) > r + R$.

References

1. www.magicmathworks.org/follow-up/publications/6403
2. The Japanese collection ‘Seiyo Sampo’ (1781), **2**(84).

10.1017/mag.2023.29

PAUL STEPHENSON
Bohmerstrrasse 66, 45144,
Essen, Germany
e-mail: *pstephenson1@me.com*