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EXTENDED ASYMPTOTIC IDENTIFIABILITY OF NONPARAMETRIC ITEM RESPONSE MODELS

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Nonparametric item response models provide a flexible framework in psychological and educational measurements. Douglas (Psychometrika 66(4):531–540, 2001) established asymptotic identifiability for a class of models with nonparametric response functions for long assessments. Nevertheless, the model class examined in Douglas (2001) excludes several popular parametric item response models. This limitation can hinder the applications in which nonparametric and parametric models are compared, such as evaluating model goodness-of-fit. To address this issue, We consider an extended nonparametric model class that encompasses most parametric models and establish asymptotic identifiability. The results bridge the parametric and nonparametric item response models and provide a solid theoretical foundation for the applications of nonparametric item response models for assessments with many items.

Key words: nonparametric item response theory, identifiability, asymptotic theory.

Item response theory (IRT) models play a crucial role in psychological measurements, educational testing, and political science (Van der Linden, 2018) . The parametric IRT models include a broad class of models that assume parametric forms of item characteristic curves (ICCs), such as the normal ogive models and the three-parameter logistic models. Nevertheless, it has been widely recognized that parametric families cannot always model ICCs well. This has spurred extensive research into the theory and applications of nonparametric IRT models; see reviews in Sijtsma (1998), Sijtsma and Molenaar (2002), and Chen et al. (2021). Nonparametric IRT models have been popularly used in assessing the goodness-of-fit of parametric IRT models and in providing robust measurements against model misfitting.

In the pursuit of nonparametric modeling of IRT models, one research line has focused on modeling nonparametric ICCs. These studies relax the assumptions of parametric ICCs via nonparametric functions such as splines or polynomials (Winsberg et al., 1984; Ramsay & Abrahamowicz, 1989; Ramsay & Winsberg, 1991; Ramsay, 1991; Douglas, 1997; Johnson, 2007; Peress, 2012; Falk & Cai, 2016).

When the true ICCs are assumed to belong to a very general function space, there may exist different sets of distinct ICCs that yield identical distributions of manifest variables. Understanding the identifiability of nonparametric IRT models is critical for relating the obtained estimates to the underlying true models. To address this issue, Douglas (2001) established the identifiability of nonparametric IRT models in an asymptotic sense with the number of items n going to infinity. The theoretical results provide foundations for various applications of nonparametric IRTs, including assessing the parametric model fit (Douglas & Cohen, 2001; Lee et al., 2009).

Nevertheless, as pointed out in Douglas (2001), their identifiability result relies on restrictive assumptions about the model class of ICCs, which are not met by some popular parametric item response models such as the normal ogive model. But in applications such as assessing the parametric model fit, it is often desired to consider a class of nonparametric IRTs that can encompass the widely used parametric IRTs. For example, Lee et al. (2009) proposed to evaluate

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the fit of a two-parameter logistic (2PL) model by comparing the estimated 2PL with another estimated nonparametric IRT. However, as the existing identifiability results in Douglas (2001) do not include 2PL in the model class, there may exist a nonparametric ICC that differs from the 2PL ICC but yields the same manifest distribution. Consequently, the discrepancy between the nonparametric and the 2PL ICCs might be a trivial result from non-identifiability rather than model misfit. This could lead to unreliability in goodness-of-fit measurements based on nonparametric IRTs.

To lay a solid theoretical foundation for related applications, it is imperative to establish identifiability for a model class that extends that in Douglas (2001). In this paper, we relax the assumptions made in Douglas (2001) so that the model class can encompass a broader range of parametric item response models, such as the normal ogive model and the four-parameter logistic model.

Following Douglas (2001), we focus on monotone, unidimensional, and locally independent item response models. Specifically, let $\boldsymbol{Y}_n = (Y_1, \ldots, Y_n)$ represent n observed dichotomous variables, and denote the i-th ICC as $P_i(\theta) = P(Y_i = 1 \mid \Theta = \theta) \in [0, 1]$, which is unidimensional and strictly increasing for $i = 1, \ldots, n$. Here, $P_i(\theta)$ represents the probability that a person with a given ability level θ will answer the i-th item correctly. Then for the manifest distribution of $\boldsymbol{Y}_n$, a locally independent IRT model consists of a collection of functions $\{P_1, P_2, \ldots, P_n\}$ and a probability density function f of θ that satisfy

$$P\left[Y_{i_1} = 1, Y_{i_2} = 1, \ldots, Y_{i_k} = 1\right] = \int \prod_{j=1}^{k} P_{i_j}(\theta) f(\theta) \mathrm{d}\theta \tag{1}$$

for any nonempty subsets $\{i_1, i_2, \ldots, i_k\}$ of the integers $\{1, 2, \ldots, n\}$. Given a fixed probability density function $f(\theta)$, we say that the ICCs are identifiable, if for any other collection of ICCs $\left\{P_1^*, P_2^*, \ldots, P_n^*\right\}$ satisfying (1), we have $P_i = P_i^*$ for all $i \in \{1, 2, \ldots, n\}$.

It is worth noting that the analyses in this paper and Douglas (2001) restrict f to be a fixed density function. Allowing the transformation of f can introduce non-identifiability issues that may not contribute meaningfully to the analysis. To illustrate, consider a smooth and invertible function $F(\cdot)$, and define a transformed latent trait $\lambda = F^{-1}(\theta)$, where $F^{-1}(\cdot)$ represents the inverse function of $F(\cdot)$. Then the manifest distribution (1) is equivalent to

$$P\left[Y_{i_1} = 1, Y_{i_2} = 1, \ldots, Y_{i_k} = 1\right] = \int \prod_{j=1}^{k} P_{i_j}[F(\lambda)] f[F(\lambda)] F'(\lambda) d\lambda,$$

which gives rise to another IRT model with a different set of ICCs $\{P_1[F(\lambda)], \ldots, P_n[F(\lambda)]\}$ coupled with the latent trait density $f[F(\lambda)]F'(\lambda)$. As the choice of f can be arbitrary, we consider a fixed f for theoretical convenience. In Sect. 1.2, we will show that our results encompass the commonly used normal distribution of the latent trait with a proper transformation.

In Sect. 1, we introduce an analysis framework of triangular sequences similarly to Douglas (2001). Under this framework, we introduce conditions on the nonparametric item response models and present examples to demonstrate that the new conditions can significantly relax assumptions in Douglas (2001). Section 2 presents the asymptotic identifiability results under relaxed assumptions and proofs. Section 3 discusses the practical implications of the results. Then appendix provides technical lemmas and the proofs of lemmas and propositions.

1. Set-Up

We consider the triangular sequence of item response variables that can be expressed as

$$\begin{aligned}
&\boldsymbol{Y}_k = \left(Y_{k,1}, Y_{k,2}, \ldots, Y_{k,k}\right)\\
&\boldsymbol{Y}_{k+1} = \left(Y_{k+1,1}, Y_{k+1,2}, \ldots, Y_{k+1,k}, Y_{k+1,k+1}\right)\\
&\boldsymbol{Y}_{k+2} = \left(Y_{k+2,1}, Y_{k+2}, \ldots, 2, Y_{k+2,k+1}, Y_{k+2,k+2}\right)\\
&\ldots\ldots
\end{aligned}$$

Item response vectors in the sequence are not required to share any items with one another. That is to say, the vectors in the sequence are allowed to be disjoint, or they may overlap to any extent. Moreover, we let $\mathcal{F}_k$ denote the probability distribution of $\boldsymbol{Y}_k$. Then $\{\mathcal{F}_k, \mathcal{F}_{k+1}, \ldots\}$ form a triangle sequence of distributions of item response vectors. In addition, we define a triangular sequence of ICCs as

$$\begin{aligned}
&\mathcal{P}_k = \left\{P_{k,1}, P_{k,2}, \ldots, P_{k,k}\right\}\\
&\mathcal{P}_{k+1} = \left\{P_{k+1,1}, P_{k+1,2}, \ldots, P_{k+1,k}, P_{k+1,k+1}\right\}\\
&\mathcal{P}_{k+2} = \left\{P_{k+2,1}, P_{k+2,2}, \ldots, P_{k+2,k+1}, P_{k+2,k+2}\right\}\\
&\ldots\ldots
\end{aligned}$$

Let $f(\theta)$ be a fixed probability density function of the latent trait Θ. We say that $\{\mathcal{P}_k, \mathcal{P}_{k+1}, \ldots\}$, coupled with f, is a model for the sequence of manifest distributions $\{\mathcal{F}_k, \mathcal{F}_{k+1}, \ldots\}$, if for each k, all equations between the manifest distributions and the integrals, specified as in (1), are satisfied. We establish the asymptotic identifiability in the sense that if the two sequences of models, $\{\mathcal{P}_k, \mathcal{P}_{k+1}, \ldots\}$ and $\{\mathcal{P}^*_k, \mathcal{P}^*_{k+1}, \ldots\}$, are for the same sequence of manifest distributions, their pointwise difference converges to as the number of items n increase to infinity.

We next state conditions that specify the class of item response models to consider. Then we provide examples showing that the specified model class can include a wide range of popular item response models and greatly extends the model class in Douglas (2001).

1.1. Conditions

Condition 1. *Unidimensionality and local independence: The latent variable Θ is a scalar-valued random variable and item responses are mutually independent conditioning on Θ.*

Condition 2. *Θ follows $U(0, 1)$, a uniform distribution on the interval $(0, 1)$.*

Given the unidimensionality in Condition 1, Condition 2 can be viewed as the choice of a specific parameterization for Θ. In particular, when Λ is a random variable with a continuous cumulative distribution function $F : \mathbb{R} \to (0, 1)$, we define $\Theta = F(\Lambda)$, resulting in $\Theta \sim U(0, 1)$. This transformation suggests that any latent trait following a continuous distribution can be equivalently transformed to $U(0, 1)$.

Condition 3. *Given each pair (n, i), the first-order derivative $P'_{n,i}(\theta)$ exists and is continuous in the open interval $(0, 1)$. For $0 < \alpha < \beta < 1$, there exist constants $m_{\alpha\beta}$ and $M_{\alpha\beta}$ that do not depend on (n, i) such that for $\theta \in [\alpha, \beta]$, $0 < m_{\alpha\beta} < P'_{n,i}(\theta) < M_{\alpha\beta} < \infty$.*

Condition 3 requires that the derivatives $P'_{n,i}(\theta)$'s are uniformly bounded from below and above on a compact interval $[\alpha, \beta] \subseteq (0, 1)$. This is a notable relaxation of Assumption 4 in Douglas (2001), which requires that $P'_{n,i}(\theta)$'s are uniformly bounded over the entire interval $(0, 1)$. With $m_{\alpha\beta} > 0$ in Condition 3, we ensure that $P_{n,i}(\theta)$'s are strictly increasing with respect to θ, which is a commonly accepted assumption in the literature. Nevertheless, we point out that all the analyses can be readily extended to cases where $P_{n,i}(\theta)$'s are strictly decreasing with respect to θ by consider the transformation $\tilde{\Theta} = 1 - \Theta \sim U(0, 1)$ and $\tilde{P}_{n,i}(\theta) = P_{n,i}(1-\theta)$ so that $\tilde{P}'_{n,i}(\theta) = -P'_{n,i}(1-\theta)$.

Condition 4. *For each $i \in \{1, \ldots, n\}$, there exist constants $\kappa_{n,i} < \gamma_{n,i} \in [0, 1]$ such that $P_{n,i}(\theta) \in [\kappa_{n,i}, \gamma_{n,i}]$. Moreover, for any $\epsilon > 0$, there exist constants l_ϵ and $u_\epsilon \in (0, 1)$ such that for all (n, i),*

$$\sup_{\theta \in [0, l_\epsilon]} [P_{n,i}(\theta) - \kappa_{n,i}] \leqslant \epsilon, \quad \text{and} \quad \sup_{\theta \in [u_\epsilon, 1]} [\gamma_{n,i} - P_{n,i}(\theta)] \leqslant \epsilon.$$

Condition 4 implies that for each (n, i), $\lim_{\theta \downarrow 0} P_{n,i}(\theta) = \kappa_{n,i}$ and $\lim_{\theta \uparrow 1} P_{n,i}(\theta) = \gamma_{n,i}$, where $\theta \downarrow 0$ and $\theta \uparrow 0$ represent the one-sided limits "from above" and "from below", respectively. When choosing $\kappa_{n,i} = 0$ and $\gamma_{n,i} = 1$, Condition 4 implies that Assumption 5 in Douglas (2001) holds, i.e., the ICCs converge to 0 and 1 on the two end points of $(0, 1)$, respectively. In contrast, Condition 4 allows more flexible limiting values. This would enlarge the model class to include models with guessing and missing parameters; please see more detailed discussions in Example 2 below.

In summary, Conditions 1–2 are the same as Assumptions 1–3 in Douglas (2001), whereas Conditions 3 and 4 considerably relax Assumptions 4 and 5 in Douglas (2001), respectively. To demonstrate this, we next provide examples with rigorous theoretical justifications.

1.2. Examples

Before presenting specific examples, we point out that the model of manifest distributions consists of both the ICCs and the distribution of the latent trait. In practice, a non-uniform distribution of the latent trait, e.g., the standard normal distribution, may be more commonly used. As discussed as after Condition 2, $\Theta \sim U(0, 1)$ represents just one specific parameterization of the latent trait. Given a continuous distribution of the latent trait that is not $U(0, 1)$, we can reparametrize the latent trait and the ICCs to obtain an equivalent model.

For instance, consider an item response model with an ICC denote as $Q(\cdot)$, and it is coupled with the latent trait Λ following a distribution with a smooth cumulative distribution function denoted as F. Suppose Λ has a density function $f(\lambda)$, and F has an inverse function denoted as F^{-1}. Then $\Theta = F(\Lambda) \sim U(0, 1)$, and $F^{-1}(\Theta)$ is a random variable with the cumulative distribution F. Then we can construct an equivalent item response model with the ICC

$$P(\theta) = Q[F^{-1}(\theta)]. \tag{2}$$

By the chain rule and the inverse function theorem in calculus,

$$P'(\theta) = \frac{Q'[F^{-1}(\theta)]}{f[F^{-1}(\theta)]}.$$

In the classical IRT models, it is common to assume that the latent trait $\Lambda \sim N(0, 1)$, i.e., the standard normal distribution. Then we can plug in $F(\cdot) = \Phi(\cdot)$, where $\Phi(\cdot)$ represents the cumulative distribution function of $N(0, 1)$.

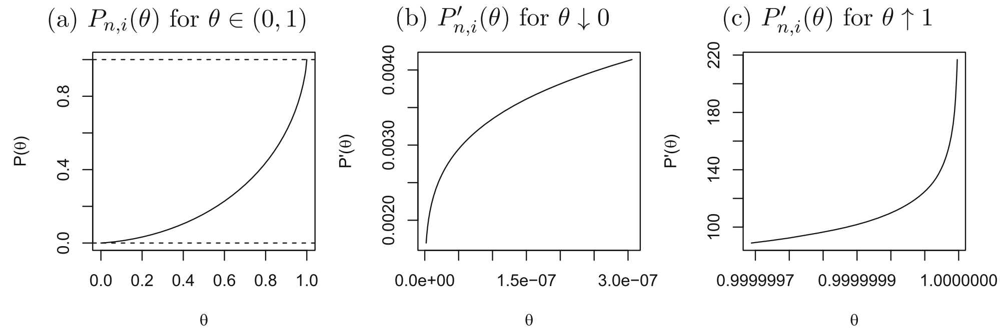

FIGURE 1.
$P_{n,i}(\theta)$ and $P'_{n,i}(\theta)$ for the normal ogive model in (3) with $a_{n,i} = b_{n,i} = 1$.

Example 1. Two-Parameter Normal Ogive Model. Consider the normal ogive model where the latent trait $\Lambda \sim N(0, 1)$, and each ICC $Q_{n,i}(\lambda) = \Phi(a_{n,i}(\lambda - b_{n,i}))$ is determined by two parameters $(a_{n,i}, b_{n,i})$. By (2), we have an equivalent model with $\Theta \sim U(0, 1)$ and the ICC

$$P_{n,i}(\theta) = \Phi(a_{n,i}[\Phi^{-1}(\theta) - b_{n,i}]), \quad \text{and} \quad P'_{n,i}(\theta) = \frac{a_{n,i}\phi(a_{n,i}[\Phi^{-1}(\theta) - b_{n,i}])}{\phi[\Phi^{-1}(\theta)]}, \tag{3}$$

where $\phi(x)$ denotes the density function of $N(0, 1)$. Douglas (2001) has pointed out that this model would violate their Assumption 4. In Proposition 1 below, we formally demonstrate that $P'_{n,i}(\theta)$ cannot be uniformly bounded over the entire interval (0, 1), Furthermore, we prove that Conditions 3 and 4 are satisfied as long as the parameters $(a_{n,i}, b_{n,i})$ are uniformly bounded.

Proposition 1. *Suppose ICCs in the sequence* $\{Q_{n,i}(\lambda)\}$ *follow the normal ogive models with parameters* $(a_{n,i}, b_{n,i})$ *and the latent trait* $\Lambda \sim N(0, 1)$. *Let* $\{P_{n,i}(\theta)\}$ *denote the corresponding transformed ICCs following* (3) *with the latent trait* $\Theta = \Phi(\Lambda) \sim U(0, 1)$.

(i) *When* $|a_{n,i}| \neq 1$ *or* $b_{n,i} \neq 0$, $\lim_{\theta\downarrow 0} P'_{n,i}(\theta)$ *and* $\lim_{\theta\uparrow 1} P'_{n,i}(\theta)$ *are either 0 or* $+\infty$.
(ii) *Assume there exist constants* $C_a, C_b > 0$ *independent with* (n, i) *such that* $a_{n,i} \in [1/C_a, C_a]$ *and* $|b_{n,i}| \in [1/C_b, C_b]$ *for all* (n, i). *Then the transformed ICCs* $\{P_{n,i}(\theta)\}$ *satisfy Conditions* 3 *and* 4 *with* $\kappa_{n,i} = 0$ *and* $\gamma_{n,i} = 1$.

We visually illustrate Proposition 1 by plotting $P_{n,i}(\theta)$ over (0, 1) and $P'_{n,i}(\theta)$ when θ is close to 0 and 1, respectively. Figure 1 suggests that $P'_{n,i}(\theta)$ approaches infinitesimal proximity to 0 and ∞ as θ converges to 0 and 1, respectively, which is consistent with Proposition 1 (a). Nevertheless, when θ is bounded away from 0 and 1, $P'(\theta)$ is finite and strictly positive. The above discussions focus on two-parameter normal ogive model, chosen for simplicity and to align with the discussions in Douglas (2001). Similar conclusions can also be established beyond this model class, even with additional parameters in the model. As an example, we next examine the four-parameter logistic model.

Example 2. Four-Parameter Logistic Model. Typically, the four-parameter logistic (4PL) model assumes that the latent trait $\Lambda \sim N(0, 1)$ and the ICC $Q_{n,i}(\lambda) = c_{n,i} + (d_{n,i} - c_{n,i})g[a_{n,i}(\lambda - b_{n,i})]$ depends on four parameters $(a_{n,i}, b_{n,i}, c_{n,i}, d_{n,i})$ and $g(x) = e^x/(1 + e^x)$. When we consider the reparametrized latent trait $\Theta = \Phi^{-1}(\Lambda) \sim U(0, 1)$, by (2), the equivalently transformed ICC is

$$P_{n,i}(\theta) = c_{n,i} + \left(d_{n,i} - c_{n,i}\right) g\left[a_{n,i}\left(\Phi^{-1}(\theta) - b_{n,i}\right)\right]. \tag{4}$$

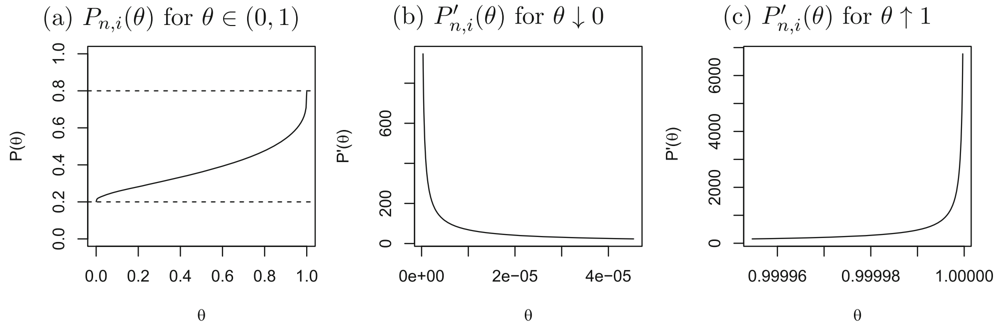

FIGURE 2.
$P_{n,i}(\theta)$ and $P'_{n,i}(\theta)$ for the 4PL in (4) with $a_{n,i} = b_{n,i} = 1$, $c_{n,i} = 0.2$, and $d_{n,i} = 0.8$.

This formulation can cover the Rasch model, 2PL, and 3PL models as special cases by setting some parameters to 0. In (4), $\lim_{\theta\downarrow 0} P_{n,i}(\theta) = c_{n,i}$, $\lim_{\theta\uparrow 1} P_{n,i}(\theta) = d_{n,i}$, and

$$P'_{n,i}(\theta) = (d_{n,i} - c_{n,i})a_{n,i} \frac{g'\left[a_{n,i}\left(\Phi^{-1}(\theta) - b_{n,i}\right)\right]}{\phi[\Phi^{-1}(\theta)]}.$$

In Proposition 2 below, we formally show that $P'_{n,i}(\theta)$ is unbounded over the entire interval (0, 1) and prove that Conditions 3 and 4 can be satisfied.

Proposition 2. *Suppose ICCs in the sequence $\{Q_{n,i}(\lambda)\}$ follow the 4PL models with the latent trait $\Lambda \sim N(0, 1)$ and the parameters $(a_{n,i}, b_{n,i}, c_{n,i}, d_{n,i})$ satisfying $c_{n,i} < d_{n,i} \in [0, 1]$. Let $\{P_{n,i}(\theta)\}$ denote the corresponding transformed ICCs following (4) with the latent trait $\Theta = \Phi(\Lambda) \sim U(0, 1)$.*

(i) *When $a_{n,i} \neq 0$, $\lim_{\theta\downarrow 0} P'_{n,i}(\theta) = \lim_{\theta\uparrow 1} P'_{n,i}(\theta) = +\infty$.*
(ii) *Assume there exist constants $C_a, C_b, C_{c,d} > 0$ independent with (n, i) such that $a_{n,i} \in [1/C_a, C_a]$, $|b_{n,i}| \in [1/C_b, C_b]$, and $d_{n,i} - c_{n,i} \in [1/C_{c,d}, C_{c,d}]$ for all (n, i). Then the transformed ICCs $\{P_{n,i}(\theta)\}$ satisfy Conditions 3 and 4 with $\kappa_{n,i} = c_{n,i}$ and $\gamma_{n,i} = d_{n,i}$.*

We visually illustrate Proposition 2 by plotting $P_{n,i}(\theta)$ over (0, 1) and $P'_{n,i}(\theta)$ when θ is close to 0 and 1, respectively. Figure 2a shows that $P_{n,i}(\theta)$ converges to 0.2 and 0.8 at the two ends points of (0,1), respectively. Therefore, Assumption 5 in Douglas (2001) is violated in this case. Moreover, Fig. 2b, c suggests that $P'_{n,i}(\theta)$ diverges to ∞ as θ converges to 0 and 1, which aligns with Proposition 2a and shows that Assumption 4 in Douglas (1997) is violated.

Remark 1. We point out that Douglas (2001) proposed another two Assumptions $4'$ and $5'$ and established Proposition 1 showing that an ICC satisfying their Assumptions 1–5 can be approximated by another ICC satisfying Assumptions 1–3, $4'$, and $5'$. In this paper, Condition 3 is similar to Assumption $4'$, and Condition 4 further relaxes Assumption $5'$ by allowing more general limits at the two end points 0 and 1. For instance, Assumptions $4'$ and $5'$ are satisfied by Example 1 above, but Assumption $5'$ is violated by Example 2. Furthermore, we note that Douglas (2001) didn't directly establish asymptotic identifiability under the extended model class in this paper.

2. Results

We next present Theorem 1, the main result for the asymptotic equivalence between two sequences of ICCs under relaxed assumptions.

Theorem 1. *For any two sequences of ICCs* $\{\mathcal{P}_k, \mathcal{P}_{k+1}, \ldots\}$ *and* $\{\mathcal{P}^*_k, \mathcal{P}^*_{k+1}, \ldots\}$*, given the sequence of the manifest distribution* $\{\mathcal{F}_k, \mathcal{F}_{k+1}, \ldots, \}$*, under Conditions 1–4,*

$$\lim_{n\to\infty} \max_{1\le i\le n} \sup_{\theta\in(0,1)} \left|P_{n,i}(\theta) - P^*_{n,i}(\theta)\right| = 0.$$

It is important to note that the asymptotic equivalence in Theorem 1 holds over the entire interval (0, 1) even though that Condition 3 only assumes bounded derivatives on compact subsets of (0, 1). Intuitively, this is achievable because the value of $P_{n,i}(\theta)$ has limited variation near the two end points of (0, 1), as specified by Condition 4. However, we emphasize that the proof is not a simple application of Condition 4. The relaxations introduced by Conditions 3–4 necessitate the development of novel theoretical techniques. Due to the relaxations introduced by Conditions 3–4, many arguments in Douglas (2001) are no longer applicable. We need to develop novel theoretical techniques that adapt to different ranges of θ. Please also see the detailed proofs and more technical discussions in Remark 2.

2.1. Proof

We first provide an outline for the proof of Theorem 1. To prove Theorem 1, we will split the domain of θ into three parts and define three terms B_1, B_2, and B_3 below. Upper bounds of B_1 and B_2 can be obtained by Condition 4. We then derive an upper bound of B_3 in Theorem 2, which is proved in two steps. Step 1 introduces a grid of $\theta_k \in (0, 1)$ (which becomes finer as n increases) and shows that ICC evaluated at any θ can be well approximated by a θ_k on the grid. Step 2 establishes that P and P^* evaluated on the grids are close.

Proof. To prove Theorem 1, it suffices to prove that for any $\epsilon > 0$, there exists $N_\epsilon > 0$ such that when $n \geqslant N_\epsilon$, $\max_{1\le i\le n} \sup_{\theta\in(0,1)} |P_{n,i}(\theta) - P^*_{n,i}(\theta)| \leqslant C\epsilon$ where $C > 0$ is a universal constant. Given any $\epsilon > 0$, let l_ϵ and u_ϵ be defined as in Condition 4. Then we have $\max_{1\leqslant i\leqslant n} \sup_{\theta\in(0,1)} |P_{n,i}(\theta) - P^*_{n,i}(\theta)| \leqslant \max\{B_1, B_2, B_3\}$, where

$$B_1 = \max_{1\leqslant i\leqslant n} \sup_{\theta\in(0,l_\epsilon]} \left|P_{n,i}(\theta) - P^*_{n,i}(\theta)\right|, \qquad B_2 = \max_{1\leqslant i\leqslant n} \sup_{\theta\in[u_\epsilon,1)} \left|P_{n,i}(\theta) - P^*_{n,i}(\theta)\right|$$
$$B_3 = \max_{1\leqslant i\leqslant n} \sup_{\theta\in(l_\epsilon,u_\epsilon)} \left|P_{n,i}(\theta) - P^*_{n,i}(\theta)\right|.$$

By Condition 4,

$$\begin{aligned} B_1 &= \max_{1\leqslant i\leqslant n} \sup_{\theta\in(0,l_\epsilon]} \left|(P_{n,i}(\theta) - \kappa_{n,i}) - (P^*_{n,i}(\theta) - \kappa_{n,i})\right| \\ &\leqslant \max_{1\leqslant i\leqslant n} \sup_{\theta\in(0,l_\epsilon]} \left\{\left|P_{n,i}(\theta) - \kappa_{n,i}\right| + \left|P^*_{n,i}(\theta) - \kappa_{n,i}\right|\right\} \leqslant 2\epsilon, \\ B_2 &\leqslant \max_{1\leqslant i\leqslant n} \sup_{\theta\in[u_\epsilon,1)} \left\{\left|P_{n,i}(\theta) - \gamma_{n,i}\right| + \left|P^*_{n,i}(\theta) - \gamma_{n,i}\right|\right\} \leqslant 2\epsilon. \end{aligned}$$

We next establish Theorem 2 showing that there exists $N_\epsilon > 0$ such that when $n \geqslant N_\epsilon$, $B_3 \leqslant \epsilon$. Then the proof of Theorem 1 is finished.

Theorem 2. *Assume Conditions 1–4. For any given* $\epsilon > 0$ *and* $\alpha < \beta \in (0, 1)$*, there exists* $N_{\epsilon,\alpha,\beta}$ *such that when* $n \geqslant N_{\epsilon,\alpha,\beta}$*,*

$$\max_{1 \le i \le n} \sup_{\theta \in (\alpha,\beta)} \left| P_{n,i}(\theta) - P^*_{n,i}(\theta) \right| < \epsilon.$$

Proof. Consider an arbitrary item $i \in \{1, \ldots, n\}$ and $\theta \in (\alpha, \beta)$. For each integer k such that $\alpha < k/(n-1) < \beta$, define θ_k and θ^*_k to satisfy

$$\bar{P}_{n,-i}(\theta_k) = \bar{P}^*_{n,-i}\left(\theta^*_k\right) = k/(n-1), \tag{5}$$

where we define the functions

$$\bar{P}_{n,-i}(\theta) = \sum_{j \neq i} P_{n,i}(\theta)/(n-1), \qquad \text{and} \qquad \bar{P}^*_{n,-i}(\theta) = \sum_{j \neq i} P^*_{n,i}(\theta)/(n-1),$$

representing the means of the ICCs of their respective sequences. Note that θ_k and θ^*_k depend on n and i, but this is suppressed in the notation for simplicity of presentation. For any given $\theta \in (\alpha, \beta)$, select the integer k such that $\theta_k \in (\alpha, \beta)$ and $|\theta - \theta_k|$ is minimized. (The definition of θ_k depends on a given θ in the analysis, and this is not emphasized for notational simplicity.) Then

$$\begin{aligned} &|P_{n,i}(\theta) - P^*_{n,i}(\theta)| \\ &\quad \leqslant |P_{n,i}(\theta) - P_{n,i}(\theta_k)| + |P_{n,i}(\theta_k) - P^*_{n,i}(\theta^*_k)| + |P^*_{n,i}(\theta^*_k) - P^*_{n,i}(\theta)|. \end{aligned}$$

The following proof consists of two main steps showing that

$$\begin{aligned} &\text{Step 1:} \quad \max\{|P_{n,i}(\theta) - P_{n,i}(\theta_k)|,\ |P^*_{n,i}(\theta^*_k) - P^*_{n,i}(\theta)|\} \leqslant \frac{4M_{\alpha\beta}}{m_{\alpha\beta} n}, \\ &\text{Step 2:} \quad |P_{n,i}(\theta_k) - P^*_{n,i}(\theta^*_k)| \leqslant \epsilon, \end{aligned}$$

respectively, where $m_{\alpha\beta}$ and $M_{\alpha\beta}$ are constants specified as in Condition 3. It is worth mentioning that $M_{\alpha\beta}$ and $m_{\alpha\beta}$ are constants that depend on (α, β) but are independent with (n, i).

Step 1.

As θ and $\theta_k \in (\alpha, \beta)$,

$$|P_{n,i}(\theta) - P_{n,i}(\theta_k)| \leqslant \sup_{\eta \in (\alpha,\beta)} |P'_{n,i}(\eta)| \times |\theta - \theta_k| \leqslant M_{\alpha\beta} \times |\theta - \theta_k|. \tag{6}$$

By the definition in (5) and Condition 4, there exist $(l_{\alpha,\beta}, u_{\alpha,\beta})$ independent with (n, i) such that $0 < l_{\alpha,\beta} < \theta_{k-1}, \theta_{k+1} < u_{\alpha,\beta} < 1$ when n is sufficiently large. By Condition 3, $P_{n,i}(\theta)$ and $\bar{P}_{n,-i}(\theta)$ are non-decreasing functions with respect to θ on a fixed interval $(l_{\alpha,\beta}, u_{\alpha,\beta})$. Therefore, we know $\theta_{k-1} < \theta_k < \theta_{k+1}$. Moreover, by $\theta, \theta_k \in (\alpha, \beta)$, we have

$$\alpha \leqslant \lambda_{k-} < \theta_k < \lambda_{k+} \leqslant \beta,$$

where we define $\lambda_{k-} = \max\{\theta_{k-1}, \alpha\}$ and $\lambda_{k+} = \min\{\theta_{k+1}, \beta\}$. As k is the integer that minimizes $|\theta - \theta_k|$,

$$|\theta - \theta_k| \leqslant |\lambda_{k-} - \lambda_{k+}| \leqslant \frac{|\bar{P}_{n,-i}(\lambda_{k-}) - \bar{P}_{n,-i}(\lambda_{k+})|}{\inf_{\eta\in[\alpha,\beta]} |\bar{P}'_{n,-i}(\eta)|}. \tag{7}$$

By Condition 3, there exists a constant $m_{\alpha\beta} > 0$ independent with (n, i) such that $\inf_{\eta\in(\alpha,\beta)} |P'_{n,i}(\eta)| > m_{\alpha\beta}$. Thus,

$$\begin{aligned}(7) &\leqslant \frac{1}{m_{\alpha\beta}} |\bar{P}_{n,-i}(\lambda_{k-}) - \bar{P}_{n,-i}(\lambda_{k+})| \\ &\leqslant \frac{1}{m_{\alpha\beta}} |\bar{P}_{n,-i}(\theta_{k-1}) - \bar{P}_{n,-i}(\theta_{k+1})| = \frac{2}{m_{\alpha\beta}(n-1)} \leqslant \frac{4}{m_{\alpha\beta} n}\end{aligned}$$

where the second inequality is obtained by the monotonicity of $\bar{P}_{n,-i}(\theta)$ under Condition 4. In summary, we have $(6) \leqslant 4M_{\alpha\beta}/(m_{\alpha\beta} n)$. The same upper bound can be obtained for $|P^*_{n,i}(\theta^*_k) - P^*_{n,i}(\theta)|$ following a similar analysis, and thus, Step 1 is proved.

Step 2. Define $\bar{Y}_{n,-i} = \sum_{j\neq i} Y_{n,i}/(n-1)$ and the event $\mathcal{E}_{n,k} = \{\bar{Y}_{n,-i} = k/(n-1)\}$. Let $\delta \in (0, 1/2)$ be a fixed small number. Define an interval $I_\delta = (\delta, 1-\delta)$. Then we have

$$\begin{aligned}&|P_{n,i}(\theta_k) - P^*_{n,i}(\theta^*_k)| \\ &\leqslant |P_{n,i}(\theta_k) - P(Y_{n,i} = 1,\ \Theta \in I_\delta \mid \mathcal{E}_{n,k})| + |P(Y_{n,i} = 1,\ \Theta \in I_\delta \mid \mathcal{E}_{n,k}) - P^*_{n,i}(\theta^*_k)| \\ &\leqslant A_1 + A_2 + A^*_1 + A^*_2,\end{aligned}$$

where we define

$$\begin{aligned}A_1 &= \left|P_{n,i}(\theta_k)\left\{1 - P(\Theta \in I_\delta \mid \mathcal{E}_{n,k})\right\}\right| \\ A_2 &= \left|P_{n,i}(\theta_k)\, P\left(\Theta \in I_\delta \mid \mathcal{E}_{n,k}\right) - P\left(Y_{n,i} = 1, \Theta \in I_\delta \mid \mathcal{E}_{n,k}\right)\right| \\ A^*_1 &= \left|P^*_{n,i}\left(\theta^*_k\right)\left\{1 - P(\Theta \in I_\delta \mid \mathcal{E}_{n,k})\right\}\right| \\ A^*_2 &= \left|P^*_{n,i}\left(\theta^*_k\right) P\left(\Theta \in I_\delta \mid \mathcal{E}_{n,k}\right) - P\left(Y_{n,i} = 1, \Theta \in I_\delta \mid \mathcal{E}_{n,k}\right)\right|.\end{aligned}$$

We point out that (A_1, A^*_1, A_2, A^*_2) depend on (n, i, k, δ), but this is suppressed in the notation for simplicity. By Lemma 2 in the appendix,

$$A_1 + A^*_1 \leqslant \frac{2n\delta \exp(-n\tilde{C}_{\alpha\beta,1})}{\tilde{C}_{\alpha\beta,2}}. \tag{8}$$

As the exponential term converges to 0 faster than polynomial n^{-r} for any $r > 0$, for any $\epsilon > 0$, there exists $N_{\epsilon,\alpha,\beta,\delta}$ such that for $n \geqslant N_{\epsilon,\alpha,\beta,\delta}$, $A_1 + A^*_1 \leqslant \epsilon/2$.

We next prove that for any fixed $\epsilon > 0$ and $\delta > 0$, when n is sufficiently large,

$$A_2 + A^*_2 \leqslant \epsilon/2. \tag{9}$$

In particular, define an interval $I_{k,\eta} = (\theta_k - n^{-\eta}, \theta_k + n^{-\eta})$ with $0 < \eta < 1/2$. Then

$$\begin{aligned} A_2 = &\ \left| P_{n,i}(\theta_k)\, P\left(\Theta \in I_\delta \mid \mathcal{E}_{n,k}\right) - P\left(Y_{n,i} = 1, \Theta \in I_\delta \mid \mathcal{E}_{n,k}\right)\right| \\ \leqslant &\ \left| P_{n,i}(\theta_k)\, [P(\Theta \in I_\delta \mid \mathcal{E}_{n,k}) - P(\Theta \in I_\delta \cap I_{k,\eta} \mid \mathcal{E}_{n,k})]\right| \\ &+ \left| P_{n,i}(\theta_k) P\left(\Theta \in I_\delta \cap I_{k,\eta} \mid \mathcal{E}_{n,k}\right) - P(Y_{n,i} = 1, \Theta \in I_\delta \cap I_{k,\eta} \mid \mathcal{E}_{n,k})\right| \\ &+ \left| P(Y_{n,i} = 1, \Theta \in I_\delta \cap I_{k,\eta} \mid \mathcal{E}_{n,k}) - P(Y_{n,i} = 1, \Theta \in I_\delta \mid \mathcal{E}_{n,k})\right| \\ \leqslant &\ A_{21} + A_{22} + A_{23}, \end{aligned}$$

where we define

$$\begin{aligned} A_{21} = &\ \left| P\left(\Theta \in I_\delta \mid \mathcal{E}_{n,k}\right) - P\left(\Theta \in I_\delta \cap I_{k,\eta} \mid \mathcal{E}_{n,k}\right)\right| = P\left(\Theta \in I_\delta \cap I_{k,\eta}^c \mid \mathcal{E}_{n,k}\right), \\ A_{22} = &\ \int |P_{n,i}(\theta_k) - P_{n,i}(\theta)|\, f_{n,k,-i}(\theta)\, I(\theta \in I_\delta \cap I_{k,\eta})\, \mathrm{d}\theta, \\ A_{23} = &\ P(Y_{n,i} = 1, \Theta \in I_\delta \cap I_{k,\eta}^c \mid \mathcal{E}_{n,k}), \end{aligned}$$

where in the definition of A_{22}, we let $f_{n,k,-i}(\theta)$ denote the probability density of Θ conditioning on $\mathcal{E}_{n,k}$. Since $A_{23} \leqslant A_{21}$ and A_2 and A_2^* can be analyzed similarly, to prove (9), it suffices to show $A_{21} \leqslant \epsilon/12$ and $A_{22} \leqslant \epsilon/12$ below.

First, $A_{21} = A_{21,num}/P(\mathcal{E}_{n,k})$, where we define $A_{21,num} = P(\Theta \in I_\delta \cap I_{k,\eta}^c, \mathcal{E}_{n,k})$ satisfying

$$A_{21,num} = P\left(|\Theta - \theta_k| > n^{-\eta},\ \Theta \in I_\delta,\ \bar{Y}_{n,-i} = \frac{k}{n-1}\right). \tag{10}$$

By $\bar{P}_{n,-i}(\theta_k) = k/(n-1)$ and $\bar{Y}_{n,-i} = k/(n-1)$ in (10), we know $\theta_k = \bar{P}_{n,-i}^{-1}(\bar{Y}_{n,-i})$, where $\bar{P}_{n,-i}^{-1}$ represents the inverse of the function $\bar{P}_{n,-i}(\cdot)$. Then

$$\begin{aligned} (10) =&P\left(\left|\Theta - \bar{P}_{n,-i}^{-1}(\bar{Y}_{n,-i})\right| > n^{-\eta},\ \Theta \in I_\delta,\ \bar{Y}_{n,-i} = \frac{k}{n-1}\right) \\ =& \int P\left\{\bar{P}_{n,-i}^{-1}(\bar{Y}_{n,-i}) > \theta + n^{-\eta} \text{ or } < \theta - n^{-\eta},\ \bar{Y}_{n,-i} = \frac{k}{n-1} \mid \Theta = \theta\right\} \mathrm{I}(\theta \in I_\delta)\mathrm{d}\theta \\ \leqslant& \int P\left\{\bar{Y}_{n,-i} > \bar{P}_{n,-i}(\theta + n^{-\eta}) \text{ or } < \bar{P}_{n,-i}(\theta - n^{-\eta}),\ \bar{Y}_{n,-i} = \frac{k}{n-1}\ \middle|\ \Theta = \theta\right\} \mathrm{I}(\theta \in I_\delta)\mathrm{d}\theta \\ \leqslant& \int P\left\{\left|\bar{Y}_{n,-i} - \bar{P}_{n,-i}(\theta)\right| > |\bar{P}'_{n,-i}(\theta_\eta)| n^{-\eta} \mid \Theta = \theta\right\} \mathrm{I}(\theta \in I_\delta)\mathrm{d}\theta \end{aligned} \tag{11}$$

where the last inequality is obtained by the intermediate value theorem, $\bar{P}'_{n,-i}(\cdot)$ represents the first-order derivative of $\bar{P}_{n,-i}(\cdot)$, and θ_η is between $\theta - n^{-\eta}$ and $\theta + n^{-\eta}$. When n is sufficiently large, $\theta_\eta \in (\delta/2, 1 - \delta/2)$ given $\theta \in (\delta, 1 - \delta)$. Thus, by Condition 3, there exists a constant $m_{\delta/2} > 0$ independent with (n, i) such that $|P'_{n,-i}(\theta_\eta)| > m_{\delta/2}$. Therefore,

$$(11) \leqslant \int P\left\{\left|\bar{Y}_{n,-i} - \bar{P}_{n,-i}(\theta)\right| > m_{\delta/2} n^{-\eta} \mid \Theta = \theta\right\} \mathrm{I}(\theta \in I_\delta)\mathrm{d}\theta.$$

By Lemma 3, i.e., Hoeffding's inequality of bounded variables, we have

$$(11) \leqslant 2 \int e^{-2(n-1)n^{-2\eta} m_{\delta/2}^2} \, \mathrm{I}(\theta \in I_\delta) \mathrm{d}\theta \leqslant 2\delta e^{-2(n-1)n^{-2\eta} m_{\delta/2}^2}.$$

Combining the above inequality with Lemma 1, we have

$$A_{21} = \frac{A_{21,num}}{P(\mathcal{E}_{n,k})} \leqslant 2\delta e^{-(n-1)n^{-2\eta} m_{\delta/2}^2} \frac{n}{\tilde{C}_{\alpha\beta}}.$$

When $0 < \eta < 1/2$, the exponential term converges to 0 faster than n^{-r} for any $r > 0$. Thus, for any $\epsilon > 0$, there exists $N_{\epsilon,\alpha,\beta,\delta}$ such that for $n \geqslant N_{\epsilon,\alpha,\beta,\delta}$, $A_{21} \leqslant \epsilon/12$.

Second, by $\theta_k \in (\alpha, \beta)$, when n is sufficiently large, $I_{k,\eta} \subseteq (\alpha/2, (1+\beta)/2)$. Therefore,

$$\sup_{\theta \in I_{k,\eta}} \left| P_{n,i}(\theta) - P_{n,i}(\theta_k) \right| \leqslant \sup_{\eta \in I_{k,\eta}} |P'_{n,i}(\eta)| \sup_{\theta \in I_{k,\eta}} |\theta - \theta_k| \leqslant M_{\alpha\beta,2} n^{-\eta},$$

where $M_{\alpha\beta,2}$ is a constant that is independent of (n, i) by Condition 3. It follows that

$$\begin{aligned} A_{22} &\leqslant \int M_{\alpha\beta,2} n^{-\eta} f_{n,k,-i}(\theta) \, \mathrm{I}(\theta \in I_\delta \cap I_{k,\eta}) \mathrm{d}\theta \\ &\leqslant M_{\alpha\beta,2} n^{-\eta} \times P(\Theta \in I_\delta \cap I_{k,\eta} \mid \mathcal{E}_{n,k}) \\ &\leqslant M_{\alpha\beta,2} n^{-\eta} \leqslant \epsilon/12, \end{aligned}$$

when n is sufficiently large.

Remark 2. Although Theorem 2 concerns a strict subset of (0, 1), its proof markedly differs from Theorem 1 in Douglas (2001), since Assumptions 4–5 in Douglas (2001) cannot be directly applied. Notably, in Step 2 of the preceding proof, an interval I_δ is introduced to establish the necessary inequalities, which is not required in Douglas (2001).

3. Discussions

Nonparametric IRT models provide a versatile framework and play an important role in ensuring robust measurement against model misspecification and in assessing parametric IRT model fit. In practical applications, it is often imperative to consider a large space of functions that embraces popular parametric IRT models. This requirement, however, considerably complicates the study of model identifiability. In this paper, we show that the assumptions restricting the model class in Douglas (2001) can be substantially relaxed, and we establish asymptotic identifiability for an extended model class that includes many popular parametric IRT models.

The result implies that as the number of items increases, an IRT in the extended model class can be uniquely identified, which provides a solid theoretical foundation for assessing the model fit. For instance, existing literature has proposed to assess the goodness-of-fit of a parametric IRT model by measuring its discrepancy to a nonparametric IRT on the same data (Douglas & Cohen, 2001; Lee et al., 2009) . When the measured discrepancy passes a suitable significant threshold, one can infer that the parametric and nonparametric IRTs are significantly different, indicating the inadequacy of the parametric IRT model fit. Our asymptotic identifiability suggests that this approach is applicable to popular 2PL, 3PL, and 4PL models asymptotically, as they are included in the extended model class. In future, it would also be an interesting research direction to develop appropriate goodness-of-fit test under finite number of items with rigorous theoretical guarantee.

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Appendix

We present all the lemmas in Section A and provide their proofs in Section B. The proofs of propositions are provided in Section C.

A. Lemmas

Lemma 1. *Consider an integer k such that $\theta_k \in (\alpha, \beta)$. There exist constants $\tilde{C}_{\alpha\beta} > 0$ and $N_{\alpha\beta}$ independent with (n, i) such that when $n \geqslant N_{\alpha\beta}$,*

$$P\left(\bar{Y}_{n,-i} = \frac{k}{n-1}\right) \geqslant \tilde{C}_{\alpha\beta} n^{-1}.$$

Lemma 2. *Under the conditions of Theorem 2, there exist constants $\tilde{C}_{\alpha\beta,1}$ and $\tilde{C}_{\alpha\beta,2}$ independent with (n, i) such that*

$$\left|1 - P\left(\Theta \in I_\delta \mid \mathcal{E}_{n,k}\right)\right| \leqslant \frac{n\delta \exp(-n\tilde{C}_{\alpha\beta,1})}{\tilde{C}_{\alpha\beta,2}}.$$

Lemma 3. (Hoeffding inequality of bounded variables) *For any (n, i), and $m > 0$,*

$$P(|\bar{Y}_{n,-i} - \bar{P}_{n,-i}(\theta)| > m \mid \Theta = \theta) \leqslant 2\exp[-2(n-1)m^2].$$

B. Proofs of Lemmas

B1. Proof of Lemma 1 Let $\mu_\theta = \sum_{j\neq i} P_{n,j}(\theta)$ and $\sigma_\theta^2 = \sum_{j\neq i} P_{n,j}(\theta)(1 - P_{n,j}(\theta))$, which are the mean and variance of $(n-1)\bar{Y}_{n,-i}$ conditioning on $\Theta = \theta$, respectively. By applying a bound on the normal approximation for the distribution of a sum of independent Bernoulli variables (Mikhailov, 1994)

$$\left| P\left[(n-1)\bar{Y}_{n,-i} = k \mid \Theta = \theta\right] - \frac{1}{\sigma_\theta\sqrt{2\pi}} e^{-\frac{(k-\mu_\theta+1/2)^2}{2\sigma_\theta^2}} \right| \leq \frac{c}{\sigma_\theta^2} \tag{12}$$

for some universal constant c.

Define the intervals $I_{n,1/2} = (\theta_k - n^{-1/2}, \theta_k + n^{-1/2})$ and $\tilde{I}_{n,1/2} = I_{n,1/2} \cap (\alpha/2, (1+\beta)/2)$. For $\theta \in \tilde{I}_{n,1/2}$ and $\theta_k \in (\alpha, \beta)$,

$$\frac{|k - \mu_\theta|}{n-1} = |\bar{P}_{n,-i}(\theta_k) - \bar{P}_{n,-i}(\theta)| \leqslant M_{\alpha\beta,2}|\theta_k - \theta| < M_{\alpha\beta,2}n^{-1/2}, \tag{13}$$

where in the second inequality, $M_{\alpha\beta,2}$ is a constant that is independent with (n, i) by Condition 3. Moreover, by Condition 4, there exist positive constants $L_{\alpha\beta}$ and $U_{\alpha\beta}$ that are independent with (n, i) such that

$$L_{\alpha\beta} < \sigma_\theta^2/(n-1) < U_{\alpha\beta}. \tag{14}$$

Therefore, by (12), (13), and (14), for $\theta \in \tilde{I}_{n,1/2}$,

$$\begin{aligned} P\left(\bar{Y}_{n,-i} = \frac{k}{n-1} \,\middle|\, \theta\right) &> \frac{1}{\sqrt{2\pi\sigma_\theta^2}} e^{-\frac{(k-\mu_\theta+1/2)^2}{2\sigma_\theta^2}} - \frac{c}{\sigma_\theta^2} \\ &> \frac{1}{\sqrt{2U_{\alpha\beta}(n-1)\pi}} e^{-\frac{\left(M_{\alpha\beta,2}(n-1)n^{-1/2}+\frac{1}{2}\right)^2}{2L_{\alpha\beta}(n-1)}} - \frac{c}{L_{\alpha\beta}(n-1)} \\ &> \frac{\tilde{C}_{\alpha\beta}}{2n^{1/2}}, \end{aligned}$$

where $\tilde{C}_{\alpha\beta} > 0$ is a constant that is independent with (n, i). Therefore,

$$P\left(\bar{Y}_{n,-i} = \frac{k}{n-1}\right) \geqslant \int_{\theta \in \tilde{I}_{n,1/2}} P\left(\bar{Y}_{n,-i} = \frac{k}{n-1} \,\middle|\, \theta\right) \mathrm{d}\theta > \int_{\theta \in \tilde{I}_{n,1/2}} \frac{\tilde{C}_{\alpha\beta}}{2n^{1/2}} \mathrm{d}\theta. \tag{15}$$

There exists $N_{\alpha\beta}$ independent with i such that when $n \geqslant N_{\alpha\beta}$, $\tilde{I}_{n,1/2} = I_{n,1/2}$ by $\theta_k \in (\alpha, \beta)$. Then by (15),

$$P\left(\bar{Y}_{n,-i} = \frac{k}{n-1}\right) > \int_{\theta \in I_{n,1/2}} \frac{\tilde{C}_{\alpha\beta}}{2n^{1/2}} \mathrm{d}\theta = \tilde{C}_{\alpha\beta} n^{-1}.$$

B2. Proof of Lemma 2 To prove Lemma 2, we note that

$$1 - P(\Theta \in I_\delta \mid \mathcal{E}_{n,k}) = P\left(0 < \Theta < \delta \mid \mathcal{E}_{n,k}\right) + P\left(1 - \delta < \Theta < 1 \mid \mathcal{E}_{n,k}\right).$$

We next derive an upper bound of $P\left(0 < \Theta < \delta \mid \mathcal{E}_{n,k}\right)$, and a similar upper bound can be obtained for $P\left(1 - \delta < \Theta < 1 \mid \mathcal{E}_{n,k}\right)$ following a similar analysis.
By $P(0 < \Theta < \delta \mid \mathcal{E}_{n,k}) = P(0 < \Theta < \delta,\ \mathcal{E}_{n,k})/P(\mathcal{E}_{n,k})$, and the lower bound of $P(\mathcal{E}_{n,k})$ in Lemma 1, it suffices to derive an upper bound of $P(0 < \Theta < \delta,\ \mathcal{E}_{n,k})$ below. In particular,

$$P\left(0 < \Theta < \delta,\ \mathcal{E}_{n,k}\right) = \int P\left(\bar{Y}_{n,-i} = \frac{k}{n-1} \mid \Theta = \theta\right) \mathrm{I}(0 < \theta < \delta)\mathrm{d}\theta. \tag{16}$$

By $\bar{P}_{n,-i}(\theta_k) = k/(n-1)$,

$$\bar{Y}_{n,-i} = \frac{k}{n-1} \quad \Leftrightarrow \quad \bar{Y}_{n,-i} - \bar{P}_{n,-i}(\theta) = [\bar{P}_{n,-i}(\theta_k) - \bar{\kappa}_{-i}] - [\bar{P}_{n,-i}(\theta) - \bar{\kappa}_{-i}], \tag{17}$$

where we define $\bar{\kappa}_{-i} = \sum_{j \neq i} \kappa_j/(n-1)$. By $0 < \alpha/2 < \alpha < \theta_k < \beta$ and Conditions 3 and 4, we have $\bar{P}_{n,-i}(\theta_k) \geqslant \bar{P}_{n,-i}(\alpha)$ and $\bar{P}_{n,-i}(\alpha/2) \geqslant \bar{\kappa}_{-i}$. Therefore,

$$\bar{P}_{n,-i}(\theta_k) - \bar{\kappa}_{-i} \geqslant \bar{P}_{n,-i}(\alpha) - \bar{P}_{n,-i}(\alpha/2) = \bar{P}'_{n,-i}(\tilde{\alpha})\alpha/2 \geqslant \tilde{m}_\alpha \alpha/2, \tag{18}$$

where the second equation is obtained by the intermediate value theorem with $\tilde{\alpha} \in (\alpha/2, \alpha)$, and the last inequality is obtained by Condition 3 with $\tilde{m}_\alpha$ being a constant independent with (n, i). Let $\tilde{C}_\alpha = \tilde{m}_\alpha \alpha/2$. By Condition 4, there exists $\delta_\alpha > 0$ such that for $\theta < \delta \leqslant \delta_\alpha$,

$$\bar{P}_{n,-i}(\theta) - \bar{\kappa}_{-i} < \bar{P}_{n,-i}(\delta) - \bar{\kappa}_{-i} < \tilde{C}_\alpha/2. \tag{19}$$

Combining (17), (18), and (19),

$$\begin{aligned} & P\left(\bar{Y}_{n,-i} = \frac{k}{n-1} \mid \Theta = \theta\right) \\ & \quad = P\left(\bar{Y}_{n,-i} - \bar{P}_{n,-i}(\theta) = [\bar{P}_{n,-i}(\theta_k) - \bar{\kappa}_{-i}] - [\bar{P}_{n,-i}(\theta) - \bar{\kappa}_{-i}] \mid \Theta = \theta\right) \\ & \quad \leqslant P\left(\bar{Y}_{n,-i} - \bar{P}_{n,-i}(\theta) \geqslant \tilde{C}_\alpha/2 \mid \Theta = \theta\right) \\ & \quad \leqslant 2\exp[-(n-1)\tilde{C}_\alpha^2/2], \end{aligned}$$

where the last inequality follows by Lemma 3. By the above inequality and (16),

$$P\left(0 < \Theta < \delta \mid \mathcal{E}_{n,k}\right) \leqslant \frac{2\delta \exp[-(n-1)\tilde{C}_\alpha^2/2]}{P(\mathcal{E}_{n,k})} \leqslant \frac{2n\delta \exp[-(n-1)\tilde{C}_\alpha^2/2]}{\tilde{C}_{\alpha\beta}},$$

where the second inequality is obtained by Lemma 1. A similar upper bound can be obtained for $P(1-\delta < \Theta < 1 \mid \mathcal{E}_{n,k})$ too. Lemma 2 is proved.

C. Proofs of Propositions

C1. Proof of Proposition 1 Let $x = \Phi^{-1}(\theta)$. We can equivalently write $P'_{n,i}(\theta) = h_{n,i}(x)$, where

$$h_{n,i}(x) = \frac{a_{n,i}\phi[a_{n,i}(x - b_{n,i})]}{\phi(x)} = a_{n,i} \exp\left[-\frac{1}{2}(a_{n,i}^2 - 1)x^2 + a_{n,i}^2 b_{n,i}\left(x - \frac{1}{2}\right)\right].$$

Note that $\theta \downarrow 0$ and $\theta \uparrow 0$ correspond to $x \to -\infty$ and $x \to +\infty$, respectively. Let $p_+ = \lim_{\theta\to 1} P'_{n,i}(\theta) = \lim_{x\to+\infty} h_{n,i}(x)$ and $p_- = \lim_{\theta\to 0} P'_{n,i}(\theta) = \lim_{x\to-\infty} h_{n,i}(x)$. As $h_{n,i}(x) = 0$ when $a_{n,i} = 0$, it suffices to consider $a_{n,i} \neq 0$ below. In particular,

$$(p_-, p_+) = \begin{cases} (+\infty, +\infty) & \text{when } 0 < a_{n,i}^2 < 1 \\ (0, 0) & \text{when } a_{n,i}^2 > 1 \\ (0, +\infty) & \text{when } a_{n,i}^2 = 1, b_{n,i} > 0 \\ (+\infty, 0) & \text{when } a_{n,i}^2 = 1, b_{n,i} < 0 \\ (1, 1) & \text{when } a_{n,i}^2 = 1, b_{n,i} = 0. \end{cases}$$

This suggests that $P'_{n,i}(\theta)$ cannot be uniformly bounded on (0, 1) when $a_{n,i}^2 \neq 1$ or $b_{n,i} \neq 0$.
On the other hand, when $\theta \in [\alpha, \beta]$, $x \in [\Phi^{-1}(\alpha), \Phi^{-1}(\beta)]$ with the two end points bounded away from $-\infty$ and $+\infty$. Therefore, when $a_{n,i} \in [1/C, C]$ and $|b_{n,i}| \in [1/C', C']$, there exist $0 < m_{\alpha\beta} < M_{\alpha\beta} < \infty$ such that $P'_{n,i}(\theta) = h_{n,i}(x) \in (m_{\alpha\beta}, M_{\alpha\beta})$. Thus Condition 3 is satisfied.
We next prove that Condition 4 is also satisfied. When $\epsilon \in (0, 1/2)$, $\Phi^{-1}(\epsilon) < 0$, and then we set $l_\epsilon = \Phi\left[C_a\Phi^{-1}(\epsilon) - C_b\right]$. When $\theta \in [0, l_\epsilon]$,

$$\begin{aligned} P_{n,i}(\theta) &\leqslant P_{ni}(l_\epsilon) = \Phi\left[a_{ni}(\Phi^{-1}(l_\epsilon) - b_{ni})\right] \leqslant \Phi\{a_{ni}[C_a\Phi^{-1}(\epsilon) - C_b + |b_{ni}|]\} \\ &\leqslant \Phi\{a_{ni}[C_a\Phi^{-1}(\epsilon) - C_b + C_b]\} \leqslant \Phi\left[\frac{1}{C_a}C_a\Phi^{-1}(\epsilon)\right] = \epsilon. \end{aligned}$$

When $\epsilon \in (1/2, 1)$, $\Phi^{-1}(\epsilon) > 0$, and we set $l_\epsilon = \Phi\left[C_a^{-1}\Phi^{-1}(\epsilon) - C_b\right]$. Then we can obtain $P_{n,i}(\theta) \leqslant \epsilon$ similarly to the above analysis. Following similar analysis, we can construct u_ϵ so that $1 - P_{n,i}(\theta) \leqslant \epsilon$ for $\theta \in [u_\epsilon, 1]$. In brief, Condition 4 is satisfied.

C2. Proof of Proposition 2 Let $x = \Phi^{-1}(\theta)$. We can equivalently write $P'_{n,i}(\theta) = h_{n,i}(x)$, where

$$\begin{aligned} h_{n,i}(x) &= (d_{n,i} - c_{n,i})a_{n,i}\frac{g'\left[a_{n,i}(x - b_{n,i})\right]}{\phi(x)} \\ &= (d_{n,i} - c_{n,i})a_{n,i}\frac{e^{x^2/2}/[e^{a_{n,i}(x-b_{n,i})} + e^{-a_{n,i}(x-b_{n,i})}]}{1 + 2/[e^{a_{n,i}(x-b_{n,i})} + e^{-a_{n,i}(x-b_{n,i})}]} \end{aligned}$$

which is obtained by

$$g'(x) = \frac{1/(e^x + e^{-x})}{1 + 2/(e^x + e^{-x})}.$$

Note that $\theta \downarrow 0$ and $\theta \uparrow 0$ correspond to $x \to -\infty$ and $x \to +\infty$, respectively. When $a_{n,i} = 0$, $h_{n,i}(x) = 0$. When $a_{n,i} \neq 0$, $e^{a_{n,i}(x-b_{n,i})} + e^{-a_{n,i}(x-b_{n,i})} \to +\infty$ and $e^{x^2/2}$ as $|x| \to +\infty$. Moreover, as the quadratic term diverges faster than the linear term, we know $e^{x^2/2}/[e^{a_{n,i}(x-b_{n,i})} + e^{-a_{n,i}(x-b_{n,i})}] \to \infty$. Thus, $h_{n,i}(x) \to +\infty$.
On the other hand, when $\theta \in [\alpha, \beta]$, $x \in [\Phi^{-1}(\alpha), \Phi^{-1}(\beta)]$ with the two end points bounded away from $-\infty$ and $+\infty$. Therefore, under the conditions in (ii) of Proposition 2, there exist $0 < m_{\alpha\beta} < M_{\alpha\beta} < \infty$ such that $P'_{n,i}(\theta) = h_{n,i}(x) \in (m_{\alpha\beta}, M_{\alpha\beta})$. Thus, Condition 3 is satisfied.
We next prove that Condition 4 is also satisfied. When ϵ is small such that $g^{-1}(\epsilon/C_{c,d}) < 0$, we set $l_\epsilon = \Phi\left[C_a g^{-1}(\epsilon/C_{c,d}) - C_b\right]$. Then for $\theta \in [0, l_\epsilon]$,

$$\begin{aligned} P_{n,i}(\theta) - c_{n,i} \leqslant P_{n,i}(l_\epsilon) - c_{n,i} &= \left(d_{n,i} - c_{n,i}\right) g\left(a_{n,i}[C_a g^{-1}(\epsilon/C_{c,d}) - C_b - b_{n,i}]\right) \\ &\leqslant \left(d_{n,i} - c_{n,i}\right) g\left(a_{n,i}[C_a g^{-1}(\epsilon/C_{c,d}) - C_b + |b_{n,i}|]\right) \\ &\overset{(i1)}{\leqslant} \left(d_{n,i} - c_{n,i}\right) g[g^{-1}(\epsilon/C_{c,d})] \leqslant \epsilon, \end{aligned}$$

where the inequality $(i1)$ above is obtained by $a_{n,i}C_a \geqslant 1 > 0$ and $|b_{n,i}| - C_b \leqslant 0$. When ϵ is large such that $g^{-1}(\epsilon/C_{c,d}) > 0$, we set $l_\epsilon = \Phi\left[C_a^{-1}g^{-1}(\epsilon/C_{c,d}) - C_b\right]$, and then $P_{n,i}(\theta) - c_{n,i} \leqslant \epsilon$ can be obtained similarly. Following similar analysis, we can construct u_ϵ so that $d_{n,i} - P_{n,i}(\theta) \leqslant \epsilon$ for $\theta \in [u_\epsilon, 1]$. In brief, Condition 4 is satisfied.

References

Chen, Y., Li, X., Liu, J., & Ying, Z. (2021). Item response theory—a statistical framework for educational and psychological measurement. arXiv preprint arXiv:2108.08604.

Douglas, J. (1997). Joint consistency of nonparametric item characteristic curve and ability estimation. *Psychometrika, 62*, 7–28.

Douglas, J. A. (2001). Asymptotic identifiability of nonparametric item response models. *Psychometrika, 66*(4), 531–540.

Douglas, J., & Cohen, A. (2001). Nonparametric item response function estimation for assessing parametric model fit. *Applied Psychological Measurement, 25*(3), 234–243.

Falk, C. F., & Cai, L. (2016). Maximum marginal likelihood estimation of a monotonic polynomial generalized partial credit model with applications to multiple group analysis. *Psychometrika, 81*, 434–460.

Johnson, M. S. (2007). Modeling dichotomous item responses with free-knot splines. *Computational Statistics & Data Analysis, 51*(9), 4178–4192.

Lee, Y.-S., Wollack, J. A., & Douglas, J. (2009). On the use of nonparametric item characteristic curve estimation techniques for checking parametric model fit. *Educational and Psychological Measurement, 69*(2), 181–197.

Mikhailov, V. G. (1994). On a refinement of the central limit theorem for sums of independent random indicators. *Theory of Probability & Its Applications, 38*(3), 479–489.

Peress, M. (2012). Identification of a semiparametric item response model. *Psychometrika, 77*, 223–243.

Ramsay, J. O. (1991). Kernel smoothing approaches to nonparametric item characteristic curve estimation. *Psychometrika, 56*(4), 611–630.

Ramsay, J. O., & Abrahamowicz, M. (1989). Binomial regression with monotone splines: A psychometric application. *Journal of the American Statistical Association, 84*(408), 906–915.

Ramsay, J. O., & Winsberg, S. (1991). Maximum marginal likelihood estimation for semiparametric item analysis. *Psychometrika, 56*(3), 365–379.

Sijtsma, K. & Molenaar, I. W. (2002). *Introduction to nonparametric item response theory*. SAGE Publications, Inc.

Sijtsma, K. (1998). Methodology review: Nonparametric IRT approaches to the analysis of dichotomous item scores. *Applied Psychological Measurement, 22*(1), 3–31.

Van der Linden, W. J. (2018). *Handbook of item response theory*. CRC Press.

Winsberg, S., Thissen, D., & Wainer, H. (1984). Fitting item characteristic curves with spline functions. *ETS Research Report Series, 1984*(2), i–14.

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