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GENERATORS OF MONOTHETIC GROUPS

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A topological group G is called *monothetic* if it contains a dense cyclic subgroup. An element x of G is called a *generator* of G if x generates a dense cyclic subgroup of G. We denote by $E(G)$ the set of generators of G; the complement of $E(G)$ in G, consisting of the "non-generators" of G, we write as $N(G)$. Throughout this paper we consider only locally compact abelian (LCA) groups satisfying the T_2 separation axiom (note that a monothetic group is automatically abelian). In [**1**] certain problems of measurability concerning the set $E(G)$ are discussed. In this paper we shall consider some algebraic and topological properties of the sets $E(G)$ and $N(G)$.

The LCA groups which we shall mention often are the integers $\mathbf{Z}$, the cyclic groups $\mathbf{Z}(n)$, the quasicyclic groups $\mathbf{Z}(p^\infty)$, the additive group of the rational numbers $\mathbf{Q}$ taken discrete, the circle group $\mathbf{T}$, and the group $\mathbf{J}_p$ of p-adic integers with its usual compact topology. Information on all these groups can be found in [**2**]. If G is an LCA group, we denote by $\hat{G}$ the character group of G. If $\gamma \in \hat{G}$ we write ker γ for the kernel of γ; the trivial character is written 1. If two groups G and H are topologically isomorphic, we write $G \cong H$. We shall have occasion to write the operation of G both multiplicatively and additively; in the former case, the identity element is written as e, and in the latter case as 0. We shall make constant use of the fact that a locally compact monothetic group is either topologically isomorphic with $\mathbf{Z}$ or is compact (see [**2**, 9.2]). Our last preliminary will be the statement of a well-known result:

LEMMA 1. *If G is* LCA, $N(G) = \{x \in G: x \in \ker \gamma$ *for some* $\gamma \neq 1$ *in* $\hat{G}\}$.

Proof. See, for example, [**2**, 25.11].

Our first result will be the determination of necessary and sufficient conditions for $E(G)$ to form a dense subset of G. Since $E(\mathbf{Z})$ is certainly not dense in $\mathbf{Z}$, we may restrict our attention to compact groups. We first observe that, if G is not connected, it has a proper open subgroup U. Since $E(G)$ is contained in the complement of U, it is clear that $E(G)$ cannot be dense in G. It is not difficult to show that the converse is true:

THEOREM 1. *Let G be a monothetic* LCA *group. Then $E(G)$ is dense in G if and only if G is connected.*

Proof. One direction has already been indicated. For the converse, assume that G is connected and note that if $x \in E(G)$, then $x^n \in E(G)$ for all integers

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$n \neq 0$. To see this, assume that $x^n \in N(G)$ for some $n \neq 0$. Then by Lemma 1, there exists $\gamma \neq 1$ in $\hat{G}$ such that $\gamma(x^n) = 1$. Hence $\gamma^n(x) = 1$, and so, again by Lemma 1, $\gamma^n = 1$, contradicting the fact that $\hat{G}$ is torsion-free. If (x) denotes the cyclic subgroup of G generated by x, we have now shown that $(x) - \{e\} \subseteq E(G)$. But (x) is dense in G, and since G is not discrete, $(x) - \{e\}$ is dense in G also. Hence $E(G)$ is dense in G, completing the proof.

Later on, in the proof of Theorem 3, we shall have a use for Theorem 1. For the present, we ask the analogous question concerning $N(G)$. We first show that, if G is not totally disconnected, then $N(G)$ is dense in G. To see this, suppose that $\gamma \in \hat{G}$ has infinite order. Then $\bigcup_{n=1}^{\infty} \ker(\gamma^n) \subseteq N(G)$, by Lemma 1. On the other hand, a simple duality argument shows that $\bigcup_{n=1}^{\infty} \ker(\gamma^n)$ is a dense subgroup of G, since the annihilator in $\hat{G}$ of this subgroup is just $\bigcap_{n=1}^{\infty}(\gamma^n) = \{1\}$, where (γ^n) denotes the cyclic subgroup of $\hat{G}$ generated by γ^n. Hence, if G is not totally disconnected, it has a character of infinite order and so $N(G)$ is dense in G, by the above argument. Thus it remains only to examine the totally disconnected monothetic groups G for which $N(G)$ is dense. Since $N(\mathbf{Z})$ is not dense in $\mathbf{Z}$, we may restrict our attention to the compact totally disconnected monothetic groups, for which we have a simple structure theorem:

LEMMA 2. *A compact totally disconnected monothetic group has the form* $\prod_{p \in P} A_p$ *(i.e., the (full) direct product of the groups* A_p *under the product topology, where* P *is the set of primes) where* A_p *is either trivial,* $\mathbf{Z}(p^{r_p})$ *(where* r_p *is a positive integer), or* $\mathbf{J}_p$.

Proof. See [**2**, 25.16].

We shall show that if $N(G)$ is not dense in G, then it is already a closed subset of G, and that this occurs in relatively few monothetic groups.

THEOREM 2. *Let* G *be a compact monothetic group. If* G *is not totally disconnected, then* $N(G)$ *is dense in* G. *If* $N(G)$ *is not dense in* G, *then* $N(G)$ *is closed in* G *and this occurs if and only if* G *is the direct product of a finite number of the groups* A_p *described in Lemma* 2.

Proof. The first assertion has already been shown above. If $N(G)$ is not dense in G, then G is totally disconnected and has the form given in Lemma 2. Now it is shown in [**2**, 25.27] that for such groups G, $E(G) = \prod_{p \in P} E(A_p)$. It is clear from the definition of the product topology that $N(G)$ will be dense in G unless all but a finite number of the factors A_p are trivial, so G is the direct product of a finite number of the groups A_p. Finally, we must show that in this case, $N(G)$ is indeed closed. To see this, we observe that $E(A_p)$ is open in A_p for each p. This is obvious when A_p has the form $\mathbf{Z}(p^{r_p})$ and for $A_p = \mathbf{J}_p$ it is shown, again in [**2**, 25.27], that $N(J_p)$ is the open and closed subgroup of sequences with zero in the first coordinate, so $E(\mathbf{J}_p)$ is open in $\mathbf{J}_p$. Hence $E(G)$ is open in G, so $N(G)$ is closed. This completes the proof.

We can also deduce from Lemma 2 and from the remarks in the proof of Theorem 2 that $N(G)$ is open in G whenever G is a compact totally disconnected monothetic group (alternatively, we could see this by observing that $\ker \gamma$ is open in G for each γ in $\hat{G}$; then apply Lemma 1). Our next result shows that $N(G)$ is open in G only when G is totally disconnected.

THEOREM 3. *Let G be monothetic. Then $N(G)$ is open in G if and only if G is totally disconnected.*

Proof. We remarked above that if G is totally disconnected, then $N(G)$ is open in G. The converse appears more difficult to establish. We show that if the identity component C of G is not trivial, then $N(G)$ is not open in G. We do this by finding an element x_0 in $N(G)$ and a net $\{x_i\}_{i\in I}$ where $x_i \in E(G)$ for each i in the index set I, such that $\lim x_i = x_0$. We may of course assume that G is compact.

We first observe that an element x in G is a generator of G if and only if x, considered as a character of the discrete group $\hat{G}$, is one-one (this follows from Lemma 1). We may assume that G is not connected, since otherwise $E(G)$ is dense in G, by Theorem 1, so $N(G)$ cannot be open. Thus we are assuming that $\{e\} \subset C \subset G(\{e\} \neq C \neq G)$. Let $B(\hat{G})$ and $B(\mathbf{T})$ denote the torsion subgroups of $\hat{G}$ and the circle group $\mathbf{T}$, respectively. Since $B(\hat{G})$ may be considered as a subgroup of $B(\mathbf{T})$ [**2**, 24.32] and since $B(\mathbf{T})$ is divisible, we may extend the identity mapping from $B(\hat{G})$ into $B(\mathbf{T})$ to a homomorphism $f: \hat{G} \rightarrow B(\mathbf{T})$ (see [**2**, A.7]). Since $\hat{G}$ is not a torsion group, it is clear that f is not a one-one character of $\hat{G}$. Now f, being a character of $\hat{G}$, may be identified as an element x_0 of G. It is clear that x_0 is a non-generator of G with the property that $\gamma(x_0) \neq 1$ for every non-trivial $\gamma \in \hat{G}$ of finite order.

We next observe that C is a monothetic group [**2**, 25.14]. Hence, by Theorem 1, $E(C)$ is dense in C, so we may find a net $\{y_i\}_{i\in I}$ with $y_i \in E(C)$ for each $i \in I$, such that $\lim y_i = e$. If we set $x_i = x_0 y_i$, then clearly $\lim x_i = x_0$ and it only remains to show that x_i is in $E(G)$ for each $i \in I$. We do this by appealing to Lemma 1; that is, we show that if y is any element in $E(C)$, then $\gamma(x_0 y) \neq 1$ for each $\gamma \neq 1$ in $\hat{G}$.

There are two cases to consider. If $\gamma \neq 1$ in $\hat{G}$ has finite order, then certainly $\gamma(C) = \{1\}$. Hence $\gamma(x_0 y) = \gamma(x_0)\gamma(y) = \gamma(x_0) \neq 1$, by the way in which x_0 was constructed. If, on the other hand, γ in $\hat{G}$ has infinite order, we first observe that $\gamma(y)$ has infinite order for any $y \in E(C)$ (for if $(\gamma(y))^n = 1$ for some positive integer n, then $\gamma^n(y) = 1$ and since γ^n, restricted to C, is a non-trivial character of C, it would follow from Lemma 1 that $y \notin E(C)$). On the other hand, $\gamma(x_0) \in B(\mathbf{T})$ for each $\gamma \in \hat{G}$, again by the construction of x_0. Hence if $\gamma(x_0 y) = 1$, we would conclude that $\gamma(y) \in B(\mathbf{T})$, contradicting the fact that $\gamma(y)$ has infinite order.

In summary, $x_0 y \in E(G)$ for each $y \in E(C)$ and so x_0 is the limit of the net of generators $x_i = x_0 y_i$, so $N(G)$ is not open in G. This completes the proof.

Remark. In Theorem 2 we found that if $N(G)$ is not dense in G then it is closed. From Theorem 3, however, we see that if $E(G)$ is not dense (i.e., if G is not connected) it does not follow that $E(G)$ is closed (i.e., that G is totally disconnected). It would be of interest to find an explicit description of the closure of the set $E(G)$ for an arbitrary monothetic group G.

We now consider some algebraic properties of the generators and non-generators of a monothetic group. We first observe that the set of non-generators of the circle coincides with the torsion subgroup of the circle; in particular, then, $N(\mathbf{T})$ is a subgroup of $\mathbf{T}$. We are led to two questions:

(1) Which monothetic groups G have the property that $N(G)$ is a subgroup of G?

(2) For which monothetic groups G is every element of infinite order a generator of G?

The answers to these two questions will be the substance of our last two theorems.

THEOREM 4. *Let G be a monothetic* LCA *group. Then $N(G)$ is a subgroup of G if and only if G is one of the following:*

(1) *$\mathbf{Z}(p^n)$ where p is a prime and n a non-negative integer,*

(2) *$\mathbf{J}_p$ for some prime p,*

(3) *a compact connected group of dimension one.*

Proof. It is obvious that $N(\mathbf{Z}(p^n))$ is a subgroup of $\mathbf{Z}(p^n)$. Moreover, it is shown in [**2**, 25.27] that $N(\mathbf{J}_p)$ is an open and closed subgroup of $\mathbf{J}_p$. If G is of type (3), then the rank of $\hat{G}$ is one, by [**2**, 24.28], so $\hat{G}$ is a subgroup of $\mathbf{Q}$ (see [**2**, A.15 and A.16]). If x and y are in $N(G)$, there are non-trivial characters γ_1 and γ_2 in $\hat{G}$ such that $\gamma_1(x) = \gamma_2(y) = 1$. Since $\hat{G} \subseteq \mathbf{Q}$, there exist nonzero integers m and n such that $\gamma_1^m = \gamma_2^n$. Then we have $\gamma_1^m(xy) = 1$, so xy is also in $N(G)$. Since the inverse of a non-generator is always a non-generator, we conclude that $N(G)$ is a subgroup of G if G is of type (3).

For the converse, we note first that if $N(G)$ is a subgroup of G, then G must be compact, since $N(\mathbf{Z})$ is not a subgroup of $\mathbf{Z}$. It is, moreover, easy to see that G must be indecomposable (i.e., G cannot be written as the direct sum of two of its proper closed subgroups). Hence the discrete group $\hat{G}$ is algebraically indecomposable, so either $\hat{G} \cong \mathbf{Z}(p^n)$, $\hat{G} \cong \mathbf{Z}(p^\infty)$, or else $\hat{G}$ is torsion-free [**3**, Theorem 10]. Thus, either $G \cong \mathbf{Z}(p^n)$, $G \cong \mathbf{J}_p$, or else G is compact and connected. It remains only to show that, in the last case, the rank of $\hat{G}$ is one [**2**, 24.28]. We shall show that, if the rank of $\hat{G}$ exceeds one, we can find two characters on $\hat{G}$, neither of which is one-one, but whose product is one-one; this will mean that we have found two non-generators of G whose product is a generator of G, so $N(G)$ is not a subgroup of G.

Throughout this part of the proof we use the more convenient additive notation. If the rank of $\hat{G}$ exceeds one, let us partition a maximal independent subset M of $\hat{G}$ into two disjoint non-empty subsets M_1 and M_2. Note that since $\hat{G}$ is isomorphic to a subgroup of the circle, the cardinality of M does not exceed the power of the continuum. Let (M_i) denote the subgroup of G generated by M_i for $i = 1, 2$. Let $D(M_i)$ be the minimal divisible extension of (M_i) [**2**, A.15] and note that M_i is a maximal independent subset of $D(M_i)$ for $i = 1, 2$ (see the proof of [**2**, A.16]). Define f_1: $\hat{G} \to D(M_1)$ by setting f_1 equal to the identity mapping on (M_1) and zero on (M_2) and then extending this mapping to all of $\hat{G}$ by the divisibility of $D(M_1)$ [**2**, A.7]. Similarly, define f_2: $\hat{G} \to D(M_2)$ with f_2 the identity on (M_2) and zero on (M_1). We then consider both f_1 and f_2 to be homomorphisms from $\hat{G}$ into $H = D(M_1) \oplus D(M_2)$, the external direct sum of $D(M_1)$ and $D(M_2)$. Now since the rank of H does not exceed the power of the continuum, we may consider H to be a subgroup of the circle [**2**, 15.13] and so the functions f_1 and f_2 may be identified with characters of $\hat{G}$. They are obviously not one-one.

Our proof will be completed by showing that the pointwise sum $f_1 + f_2$ is one-one. Let c be a member of $\hat{G}$ and suppose that $(f_1 + f_2)(c)$ is the zero $(0, 0)$ of H. Since $f_1(c)$ has the form $(m_1, 0)$ and $f_2(c)$ has the form $(0, m_2)$, we conclude that $f_1(c) = f_2(c) = (0, 0)$. Since M is a maximal independent set in G we may write $nc = m_1 + m_2$, where n is a non-zero integer, and $m_i \in (M_i)$ for $i = 1, 2$. A direct computation shows that $(0, 0) = (f_1 + f_2)(nc) = (m_1, m_2)$, so $m_1 = m_2 = 0$ and hence $nc = 0$. Since $\hat{G}$ is torsion-free, it follows that $c = 0$, so $f_1 + f_2$ is one-one. This completes the proof.

COROLLARY. *Let G be an infinite monothetic* LCA *group. The following are equivalent:*

(1) *$N(G)$ is a closed subgroup of G,*
(2) *$N(G)$ is an open subgroup of G,*
(3) *$G \cong \mathbf{J}_p$ for some prime p.*

Proof. If (1) holds, Theorem 2 and the previous theorem imply that (3) must hold. We have already remarked that (3) $\Rightarrow$ (2) $\Rightarrow$ (1), which completes the proof.

We conclude our findings by answering our second question given above.

THEOREM 5. *Let G be an infinite monothetic* LCA *group. If every element of infinite order in G is a generator of G, then $G \cong \mathbf{T}$.*

Proof. Since $G \cong \mathbf{Z}$ is impossible, we assume that G is compact. Now G cannot be a torsion group, since then G would be finite. Let $x \in G$ have infinite order. Then x^n has infinite order for every positive integer n. This implies that $\hat{G}$ is torsion-free, since if $\gamma \neq 1$ in $\hat{G}$ had finite order n, then $\gamma(x^n) = 1$, whence x^n is not a generator of G, a violation of hypothesis. Since G is compact, G must be connected. Now a compact connected monothetic group is solenoidal [**2**, 25.14 and 25.18]. Hence there is a continuous homomorphism f: $\mathbf{R} \to G$ having dense image, where $\mathbf{R}$ denotes the additive group of real numbers with the usual topology.

We next show that f cannot be one-one. If f were one-one, then $f(\mathbf{R})$ would be a torsion-free connected subgroup of G. Hence if $\gamma \neq 1$ is in $\hat{G}$, we know that γ is one-one on $f(\mathbf{R})$, since every element of $f(\mathbf{R})$ except $f(0)$ must, by hypothesis, be a generator of G. On the other hand, $\gamma(f(\mathbf{R})) = \mathbf{T}$, since $\mathbf{T}$ has no proper connected subgroups. Thus we obtain an algebraic isomorphism between $\mathbf{R}$ and $\mathbf{T}$, which is absurd. Hence f is not one-one.

Finally, the transpose map $f^*: \hat{G} \to \mathbf{R}$ is one-one, but does not have dense image, by the preceding paragraph and [**2**, 24.41(b)]. Since all non-dense subgroups of $\mathbf{R}$ are isomorphic to $\mathbf{Z}$, we conclude that $\hat{G} \cong \mathbf{Z}$, so $G \cong \mathbf{T}$. This completes the proof.

References

1. P. Halmos and H. Samelson, *On monothetic groups*, Proc. Nat. Acad. Sci. U.S.A. *28* (1942), 254/258.

2. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. I (Academic Press, New York, 1963).

3. I. Kaplansky, *Infinite abelian groups* (University of Michigan Press, Ann Arbor, 1954).

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