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ON THE CLOSURE OF THE CONVEX HULL OF A SET

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Let Y be a linear space over the complex plane C, and let F be a mapping on the complex linear space $Y \oplus C$ into subsets of C with the following properties: for $y \in Y$, λ and $\mu \in C$, $F(y+\mu)$ is a nonempty and bounded subset of C, $F(\lambda y+\mu)=\lambda F(y)+\mu$ and $F(\mu)=\{\mu\}$. We shall write $f(y+\mu)=\sup\{|\lambda+\mu| : \lambda \in F(y)\}$, the radius of $F(y+\mu)$, $y \in Y$ and $\mu \in C$. The convex hull (resp. the closure) of a subset M of C is denoted by conv M (resp. $\bar{M}$).

THEOREM.

(1) $$\lim_{t\to\infty} f(y+t)-t = \sup \operatorname{Re} F(y);$$

(2) $$[\operatorname{conv} F(y)]^- = \bigcap_{\lambda\in C}\{\mu : \mu \in C \quad \textit{and} \quad |\mu-\lambda| \le f(y-\lambda)\};$$

(3) $$[\operatorname{conv} F(y)]^- = [\operatorname{conv} F(y')]^- \quad \textit{iff} \quad f(y-\mu) = f(y'-\mu), \qquad \forall \mu \in C.$$

Proof. (1) For $F(y+t)=F(y)+t, f(y+t) \ge \operatorname{Re}(\mu+t)=\operatorname{Re}\mu+t$, $\mu \in F(y)$. Hence $f(y+t)-t \ge \sup \operatorname{Re} F(y)$. Since $|z+t|-\operatorname{Re}(z+t)\to 0$ as $t\to+\infty$, uniformly for z in bounded sets. Then if $\varepsilon>0$.

$$\operatorname{Sup}\ \operatorname{Re}\ F(y) \le f(y+t)-t \le \sup\ \operatorname{Re}\ F(y)+\varepsilon$$

for t sufficiently large, from which it follows that the required limit exists and has the right value.

(2) If $\mu \in [\operatorname{conv} F(y)]^-$ and $\lambda \in C$, then $\mu-\lambda \in [\operatorname{conv} F(y)]^- -\lambda = [\operatorname{conv} F(y-\lambda)]^-$ and hence $|\mu-\lambda| \le f(y-\lambda)$. Thus μ is in the right-hand side. If $\mu \notin [\operatorname{conv} F(y)]^-$, by a preliminary translation and rotation, we may assume without loss of generality that $[\operatorname{conv} F(y)]^-$ lies in the left half-plane $\operatorname{Re} z \le 0$, and that $\mu>0$. For large $t>0$, $f(y+t)-t<\mu$ by (1), i.e., $|\mu+t|>f(\mu+t)$ and hence μ is not in the right-hand side.

(3) If $[\operatorname{conv} F(y)]^-=[\operatorname{conv} F(y')]^-$, then $[\operatorname{conv} F(y-\mu)]^-=[\operatorname{conv} F(y'-\mu)]^-$ for all $\mu \in C$ and the necessity follows. The sufficiency follows easily from (2).

As for applications, consider the complex linear space $B(X)$ of bounded linear operators on a Hilbert space X. Let $\sigma(T)$, $\sigma_e(T)$, $W(T)$ and $W_e(T)$ denote respectively the spectrum, the essential spectrum (here, suppose dim $X=\infty$ in order that $\sigma_e(T)\ne\phi$ [2]), the numerical range and the essential numerical range [3] of $T \in B(X)$. Also let $r(T)$, $r_e(T)$, $w(T)$ and $w_e(T)$ denote respectively the corresponding radii. Now, applying the theorem above and using some elementary facts in the

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theory of numerical ranges, we have the following:

$$\lim_{t\to\infty} r(T+t)-t = \sup \operatorname{Re} \sigma(T); \lim_{t\to\infty} r_e(T+t)-t \tag{1}$$
$$= \sup \operatorname{Re} \sigma_e(T); \lim_{t\to\infty} w(T+t)-t = \sup \operatorname{Re} W(T)$$

(this one should be compared with a remarkable result of Lumer [3, Lemma 2]: $\lim_{t\to\infty} \|T+t\|-t=\sup \operatorname{Re} W(T)$), and $\lim_{t\to\infty} w_e(T+t)-t=\sup \operatorname{Re} W_e(T)$.

$$\operatorname{conv} \sigma(T) = \bigcap_{\lambda\in C} \{\mu : \mu \in C \quad \text{and} \quad |\mu-\lambda| \le r(T-\lambda)\}; \tag{2}$$

and

$$\operatorname{conv} \sigma_e(T) = \bigcap_{\lambda\in C} \{\mu : \mu \in C \quad \text{and} \quad |\mu-\lambda| \le r_e(T-\lambda)\};$$

$$[W(T)]^- = \bigcap_{\lambda\in C} \{\mu : \mu \in C \quad \text{and} \quad |\mu-\lambda| \le w(T-\lambda)\} \text{ [1, 3]},$$

and

$$W_e(T) = \bigcap_{\lambda\in C} \{\mu : \mu \in C \quad \text{and} \quad |\mu-\lambda| \le w_e(T-\lambda)\}.$$

(3) For example, conv $\sigma(T)=[W(T)]^-$, i.e., T is convexoid, iff $r(T-\mu)= w(T-\mu)$ for all $\mu \in C$ [1].

We can find more applications. For example, similar results apply to the numerical range in an arbitrary Banach algebra with identity in the sense of [3].

References

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