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S-MAXIMAL SUBGROUPS OF $\pi_1(M^3)$

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Let M be a compact, connected, irreducible 3-manifold. Let S be a closed, connected, 2-manifold other than the 2-sphere or projective plane. Let f be a map of S into M such that

$$f_*:\pi_1(S) \to \pi_1(M)$$

is an injection. Suppose for every closed, connected surface S_1 and every map $g:S_1 \to M$ such that

(1) $g_*:\pi_1(S_1) \to \pi_1(M)$ is an injection,

(2) $g_*\pi_1(S_1) \supset f_*\pi_1(S)$,

$g_*\pi_1(S_1) = f_*\pi_1(S)$. Then we shall say that the subgroup $f_*\pi_1(S)$ is a *surface maximal* or *S-maximal subgroup* of $\pi_1(M)$. We may also say that the map f is S-maximal.

Let M be an irreducible 3-manifold which does not admit any embedding of the projective plane. Then we shall say that M is *p^2-irreducible*. Throughout this paper all spaces will be simplicial complexes and all maps will be piecewise linear.

It is the purpose of this paper to prove the following:

THEOREM 1. *Let M be a compact, connected, p^2-irreducible 3-manifold. Let S be a closed, connected 2-manifold, not the 2-sphere or projective plane. Let $f:(S,x_0) \to (M,x)$ be an embedding such that $f_*:\pi_1(S) \to \pi_1(M)$ is an injection but f_* is not S-maximal. Then M has a 3-submanifold N, bounded by $f(S)$, which is homeomorphic to a twisted line bundle. Furthermore, if $g_*:\pi_1(S_1,x_1) \to \pi_1(M,x)$ is an injection and*

$$\pi_1(M,x) \supset g_*\pi_1(S_1,x_1) \underset{\neq}{\supset} f_*\pi_1(S,x_0),$$

then N may be chosen so that

$$\pi_1(N,x) = g_*\pi_1(S_1,x_1) \subset \pi_1(M,x).$$

COROLLARY. *Let $f:S \to M$ be an embedding such that f_* is $1-1$. If $f(S)$ does not separate a regular neighborhood of itself in M, f_* is S-maximal.*

Proof. A surface which does not separate a regular neighborhood of itself in M cannot bound a 3-submanifold in M.

We shall denote the boundary, closure, and interior of a subspace X of a space Y by bd(X), cl(X), and int(X), respectively. When X is a subset of a

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space Y, we shall denote the natural inclusion map from X into Y by ρ and the induced homomorphism from $\pi_1(X)$ into $\pi_1(Y)$ by ρ_*.

We give below an outline for the proof of Theorem 1. Let (M^*, P) be the covering space of M associated with $g_*\pi_1(S_1, x_1) \subset \pi_1(M, x)$. Let $f_1:S \to M^*$ be a map such that $pf_1 = f$. Then we will show that:

I. There is an embedding $g_1:S_1 \to M^*$ such that

$$(pg_1)_*(\pi_1(S_1, x_1)) = g_*\pi_1(S_1, x_1) \subset \pi_1(M, x).$$

II. There is a compact, connected 3-submanifold N_1^* of M^* containing $g_1(S_1)$ and $f_1(S)$ such that $\rho_*:\pi_1(N_1^*) \to \pi_1(M^*)$ is an isomorphism and N_1^* is is homeomorphic to a twisted line bundle except perhaps for a fake cell.

III. There is a compact 3-submanifold N^* of N_1^* such that

(1) $bd(N^*) = p^{-1}f(S) \cap N^*$;

(2) $p|bd(N^*)$ is a homeomorphism;

(3) $\rho_*:\pi_1(N^*) \to \pi_1(N_1^*)$ is an isomorphism;

(4) $p|N^*$ is a homeomorphism;

(5) N^* is a twisted line bundle over S_1.

The desired result follows as $p(N^*)$ will be the 3-submanifold of M^* which Theorem 1 requires.

We digress to prove a number of lemmas useful in the proof of Theorem 1.

Definition. Let F be a closed 2-sided surface embedded in a 3-manifold M. Suppose that no component of F is a 2-sphere or projective plane. If for each component F_0 of F, $\rho_*:\pi_1(F_0) \to \pi_1(M)$ is an injection, we shall say that F is *incompressible in M.*

LEMMA 1. *Let M be a p^2-irreducible 3-manifold. Then $\pi_2(M) = 0$.*

Proof. If $\pi_2(M) \neq 0$, it follows from [**3**, Theorem 1.1] that there is an embedding in M either of the projective plane or of a 2-sphere which fails to bound a 3-ball. Either of the above contradicts our assumption that M is p^2-irreducible.

LEMMA 2. *Let M_1 be a connected 3-submanifold of the 3-manifold M. Assume that M_1 is a closed subset of M and that* $\mathrm{cl}(M - M_1) \cap M_1$ *is incompressible in M. Let l be a loop contained in M_1. If l is homotopic to a point in M, then l is homotopic to a point in M_1.*

Proof. This is [**4**, Lemma 1.2].

Throughout the remainder of this paper I will denote [0, 1].

LEMMA 3. *Let S_1 and S_2 be closed, connected surfaces other than the 2-sphere or projective plane. Let $f:S_1 \to S_2 \times I$ be an embedding such that $f_*:\pi_1(S_1) \to \pi_1(S_2 \times I)$ is $1 - 1$. Then f_* is an isomorphism.*

Proof. This is [**4**, Lemma 1.3] except that S_j is not required to be orientable for $j = 1, 2$. The proof is identical to that of 1.3 in [**4**].

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By a *fake cell* we shall mean a homotopy cell which may not be a cell. Let S be a closed surface. We shall say that a 3-manifold M is a *fake* $S \times I$ if one can obtain an $S \times I$ from M by replacing a fake cell in M with a 3-ball. We define a *fake twisted line bundle* similarly.

Observation 1. If M is a compact connected 3-manifold and $\pi_2(M) = 0$, one can replace a single fake cell with a 3-ball to obtain an irreducible 3-manifold.

If M is orientable, it follows from [**6**, Generalization 1] that there are only finitely many disjoint, prime homotopy cells which are not 3-balls.

We can find a fake cell in M which contains all of these homotopy 3-cells and remove this fake cell from M.

If M is non-orientable, there can again only be finitely many disjoint homotopy cells which are not 3-balls since otherwise the orientable double cover of M would contain more than finitely many of these homotopy cells. The observation follows.

LEMMA 4. *Let N be a compact, connected 3-manifold with connected, incompressible, non-vacuous boundary. Let S_1 be a closed, connected surface not the 2-sphere or projective plane. Suppose $\pi_1(N) \cong \pi_1(S_1)$. Then N is a fake twisted line bundle and $\rho_*\pi_1(\mathrm{bd}(N))$ is of index two in $\pi_1(N)$.*

Proof. There are no embeddings of the projective plane in N since there are no elements of order 2 in $\pi_1(N)$. It follows from [**3**, 1.1] that $\pi_2(N) = 0$. We have observed that one can obtain an irreducible 3-manifold N_1 from N by replacing a fake cell with a ball. Thus we may assume that N_1 is p^2-irreducible.

Since there is a continuous map from N_1 into $S_1 \times \{0\} \subset S_1 \times I$ which induces an isomorphism from $\pi_1(M)$ to $\pi_1(S_1 \times I)$, it follows from [**5**, Theorem A Corollary] that N_1 is a twisted line bundle. If one splits N_1 along its zero section, one sees that $\mathrm{bd}(N_1)$ is a double cover of the zero section.

The desired result follows immediately.

LEMMA 5. *Let M be a p^2-irreducible 3-manifold. Let N be a compact 3-submanifold of M such that $\mathrm{bd}(N) \subset \mathrm{int}(M)$ and $\mathrm{bd}(N)$ is incompressible in M. Then N is p^2-irreducible.*

Proof. Since M contains no embedded projective planes, N will not contain any embedded projective planes. Suppose there is a 2-sphere S^2 embedded in N and that S^2 does not bound a ball in N. But we have assumed that S^2 bounds a ball C in M; and C will contain a component of $\mathrm{bd}(N)$. This is impossible since $\mathrm{bd}(N)$ was assumed to be incompressible in M.

LEMMA 6 (Kneser's Lemma). *Let M be a 3-manifold. Let F be a closed two-sided surface embedded in M. Suppose there is a component S of F such that $\rho_*:\pi_1(S) \to \pi_1(M)$ is not an injection. Then there exists a disk D embedded in M such that $D \cap F = \mathrm{bd}(D)$ and $\mathrm{bd}(D)$ is not nullhomotopic in F.*

Proof. Case 1. Suppose S separates M into 3-submanifolds M_1 and M_2. It is a consequence of [**2**, 4.2] that $\rho_*:\pi_1(S) \to \pi_1(M_j)$ is not an injection for $j = 1$

or 2. We assume that $\rho_*:\pi_1(S) \to \pi_1(M_1)$ is not $1-1$. Then the loop theorem in [7] guarantees the existence of a disk D_1 embedded in M_1 such that $D_1 \cap S = \text{bd}(D_1)$ and $\text{bd}(D_1)$ is not nullhomotopic in S. We may assume that $D_1 \cap F$ is a collection of disjoint loops and pick D_1 so that the number of loops in $D_1 \cap F$ is a minimum. Suppose there is a loop $l \subset D_1 \cap (F - S)$ which is nullhomotopic in F. Then l bounds a disk $D_0 \subset F$. We can choose a disk $\bar{D} \subset D_0$ so that $D_1 \cap \bar{D} = \text{bd}(D)$. But now it is easy to reduce the number of loops in $D_1 \cap F$ by a simple cutting argument. Thus every loop in $D_1 \cap F$ may be taken to be nontrivial in F. It is now easy to choose a disk $D \subset D_1$ such that $D \cap F = \text{bd}(D)$ and $\text{bd}(D)$ is not nullhomotopic on F.

Case 2. Suppose S does not separate M. Let $M_/$ be the 3-manifold obtained by cutting M along S. Let S_1 and S_2 be the two boundary components of $M_/$ which come from S.

We define a covering space $(\tilde{M}, q)$ of M as follows: Let $M_/^i$ be homeomorphic to $M_/$ for i an integer. Let S_j^i be the embedding of S_j in $M_/^i$ for i an integer and $j = 1, 2$. Let $\tilde{M}$ be the space formed by pasting S_1^i to S_2^{i+1} via the natural homeomorphism. Let $q:\tilde{M} \to M$ be the map which is the natural homeomorphism on $M_/^i - (S_1^i \cup S_2^i)$ and which identifies S_1^i and S_2^i in the natural way.

Now $\rho_*:\pi_1(S_1^0) \to \pi_1(\tilde{M})$ is not $1-1$. Furthermore, S_1^0 separates $\tilde{M}$ into submanifolds M_1 and M_2. As was shown above, we can find a disk D_1 embedded in M_1 such that $D_1 \cap S_1^0 = \text{bd}(D_1)$ and $\text{bd}(D_1)$ is nontrivial in S_1^0.

It is easy to use a general position argument and then a cutting argument to find a disk $\bar{D}_1$ which meets $\bigcup_{i=-\infty}^{\infty} S_1^i$ only in essential loops. One can then find a subdisk $\bar{D}$ of $\bar{D}_1$ which meets $\bigcup_{i=-\infty}^{\infty} S_1^i$ in a single loop. Now $q(\bar{D})$ is a disk embedded in M such that $q(\bar{D}) \cap S = \text{bd}(q(\bar{D}))$ and $\text{bd}(q(\bar{D}))$ is nontrivial in S. The remainder of the proof of Case 2 is the same as that of Case 1.

LEMMA 7. *Let M be a 3-manifold and S a closed, two sided surface embedded in M. Let (M^*, p) be a covering space of M (not necessarily compact). Let R be a connected, compact 3-submanifold of M^* such that*

(1) $R \cap p^{-1}(S) = \text{bd}(R)$;

(2) *The number of components in* $\text{bd}(R)$ *is the same as the number of components in* S.

Then $(R, p|R)$ *is a finite covering space of* $p(R)$.

Proof. It follows from the definition of covering space that $p|R$ is a local homeomorphism. Since the number of components in $\text{bd}(R)$ is the same as the number of components in S, and R is compact, each component of $\text{bd}(R)$ is a finite covering of one component of S. Since S is two sided and $p^{-1}(S) \cap \text{int}(R)$ is empty, $S = \text{bd}(p(R))$. Let y_0 be a point in $\text{bd}(R)$ and z_0 a point in R but not in $p^{-1}p(y_0)$. Let α_0 be a path from y_0 to z_0. Then for each point z_1 in $p^{-1}p(z_0)$ there is a unique path α_1 such that $p(\alpha_0) = p(\alpha_1)$ and α_1 has one endpoint in $p^{-1}p(z_0)$ and one endpoint in $p^{-1}p(y_0)$. It follows that $p^{-1}p(z_0)$

and $p^{-1}p(y_0)$ are of the same cardinality and thus $(R, p|R)$ is a finite covering of $p(R)$.

LEMMA 8. *In the lemma above, if* $p|\text{bd}(R)$ *is a homeomorphism,* $p|R$ *is a homeomorphism.*

Proof. This is an immediate consequence of Lemma 7.

LEMMA 9. *Let M be a 3-manifold. Suppose $\pi_2(M) = 0$. Then every 2-sphere embedded in M bounds a homotopy cell in M.*

Proof. Let S be a sphere embedded in M. Let $(\tilde{M}, p)$ be the universal cover of M. Let $\tilde{S}$ be a sphere embedded in $\tilde{M}$ such that $p\tilde{S} = S$. Since $\tilde{S}$ is homotopic to a point in $\tilde{M}$, it bounds a finite chain in $C_3(\tilde{M}, Z_2)$. Thus $\tilde{S}$ bounds a compact 3-submanifold B of $\tilde{M}$. It is a consequence of Van Kampen's theorem that B is simply connected. It is well known that this implies that B is a homotopy 3-cell.

Now $B \cap p^{-1}(S)$ is a finite collection of 2-spheres. By the argument above each of these 2-spheres bounds a homotopy cell in B. It is possible to choose a sphere in $p^{-1}(S) \cap B$ which bounds a homotopy cell B_1 such that $p^{-1}(S) \cap B_1 = \text{bd}(B_1)$. It now follows from Lemma 8 that $p|B_1$ is a homeomorphism, and $p(B_1)$ is a homotopy cell bounded by S.

LEMMA 10. *Let M be a 3-manifold with nonvacuous disconnected boundary. Let F_1 be a component of* $\text{bd}(M)$. *Let F be a closed, connected surface, not the 2-sphere or projective plane. Let $(\tilde{M}, q)$ be a covering space of $\tilde{M}$ such that $\tilde{M}$ is a fake $F \times I$. Then M is a fake $F_1 \times I$.*

Proof. Since $\text{bd}(\tilde{M})$ is compact, the covering is of finite index. Since $\text{bd}(M)$ is disconnected, the components of $\text{bd}(\tilde{M})$ are mapped to distinct components of $\text{bd}(M)$ by q. Let $F_0 = q^{-1}(F_1)$. Then $(F_0, q|F_0)$ is a k-fold covering of F_1 and $q_*\pi_1(F_0)$ is of index k in $\pi_1(F_1) \subset \pi_1(M)$. Since $\text{bd}(\tilde{M})$ is incompressible in $\tilde{M}$, $\text{bd}(M)$ is incompressible in M. But now $\rho_*\pi_1(F_1) \to \pi_1(M)$ is an isomorphism since $\rho_*\pi_1(F_0) = \pi_1(\tilde{M})$. It follows from Observation 1 that one can replace a fake cell in M with a 3-ball and obtain an irreducible 3-manifold. The lemma follows from 3.1 in [**1**].

LEMMA 11. *Let F be a closed, connected surface not the 2-sphere or projective plane. Let M be a 3-manifold such that $\pi_1(M) \cong \pi_1(F)$ and $\pi_2(M) = 0$. Let F_1 and F_2 be disjoint, closed, connected, two sided surfaces, other than the 2-sphere or projective plane, embedded in M. Suppose F_1 and F_2 are incompressible in M. Then $F_1 \cup F_2$ bounds a fake $F_1 \times I$ embedded in M.*

Proof. Since $\rho_*\pi_1(F_1) \to \pi_1(M) \cong \pi_1(F)$ is an injection, it follows from [**5**, Theorem 1] that F_1 is the cover of F associated with $\rho_*\pi_1(F_1) \subset \pi_1(M) \cong \pi_1(F)$. Since F_1 is compact, this cover is of finite index and $\pi_1(F_1)$ is of finite index in $\pi_1(M)$. Let A be the subgroup of $\pi_1(M)$ associated with the orientable double cover of M if M is not orientable and $\pi_1(M)$, otherwise. Now $A_0 = \rho_*(\pi_1(F_1)) \cap A$ is of finite index in $\pi_1(M)$ since $\rho_*(\pi_1(F_1))$ and A are each of finite index in $\pi_1(M)$. Let $(\tilde{M}, q)$ be the cover of M associated with A_0. Let $(\tilde{F}_1, q_1)$ be the cover of F_1 associated with $\rho_*^{-1}(\rho_*\pi_1(F_1) \cap A_0) = \rho_*^{-1}(A_0)$.

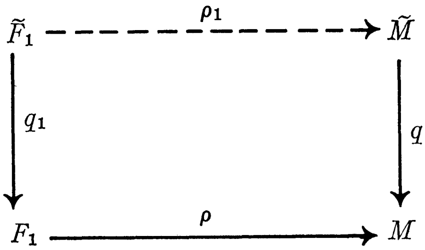

Figure 1

It is easily shown that there is an embedding ρ_1 making the diagram in Figure 1 commutative. Note $\rho_{1*}:\pi_1(\tilde{F}_1) \to \pi_1(\tilde{M})$ is an isomorphism. Let $\tilde{F}_2$ be a component of $q^{-1}(F_2)$. Since $\rho_1\tilde{F}_1$ and $\tilde{F}_2$ are two-sided closed surfaces embedded in an orientable 3-manifold, they are orientable. We claim that both $\rho_1\tilde{F}_1$ and $\tilde{F}_2$ separate $\tilde{M}$. We see this as follows: Let λ be a simple loop meeting either $\rho_1\tilde{F}_1$ or $\tilde{F}_2$ in a single point and crossing $\rho_1\tilde{F}_1$ or $\tilde{F}_2$ at that point. Since $\rho_{1*}:\pi_1(F_1) \to \pi_1(\tilde{M})$ is an isomorphism, λ is homotopic to a loop λ_1 which lies in a regular neighborhood of $\rho_1\tilde{F}_1$ so that λ_1 does not meet $\rho_1\tilde{F}_1$ or $\tilde{F}_2$. This is impossible as the intersection number of λ and $\rho_1\tilde{F}_1$ or $\tilde{F}_2$ is one while that of λ_1 and $\rho_1\tilde{F}_1$ or $\tilde{F}_2$ is zero. We observe that $\rho_1\tilde{F}_1 \cup \tilde{F}_2$ bounds a 3-submanifold R of $\tilde{M}$. It is an easy consequence of Lemma 2 that $\rho_*:\pi_1(R) \to \pi_1(\tilde{M})$ is an isomorphism since R contains $\rho_1\tilde{F}_1$. Also $\rho_{1*}:\pi_1(\tilde{F}_1) \to \pi_1(R)$ is an isomorphism. It follows that every loop in $\tilde{F}_2 \subset R$ is homotopic in R to a loop in $\rho_1\tilde{F}_1$. Consider the proof of 5.1 in [**8**]. In this proof Waldhausen produces a 2-sphere and concludes that since his 3-manifold is irreducible the 2-sphere bounds a ball. We observe that the construction of this 2-sphere was independent of his assumption of irreducibility. Thus we can construct the same 2-sphere in R.

It follows from Lemma 9 that a 2-sphere in $\tilde{M}$ bounds a fake cell and thus by Waldhausen's proof that R is a fake $\tilde{F}_1 \times I$.

We observe that our proof was independent of the components of $q^{-1}(F_1)$ and $q^{-1}(F_2)$ which we chose. Thus if $L = \text{int}(R) \cap q^{-1}(F_2)$ is non-empty, we can reduce the number of components in L by picking a different $\tilde{F}_2$. Similarly, one can reduce the number of components in $q^{-1}(F_1) \cap \text{int}(R)$. It follows that we may assume that $q^{-1}(F_1) \cup F_2) \cap R = \text{bd}(R)$. Thus by Lemma 8, $(R, q|R)$ covers $q(R)$. It follows from Lemma 10 that $q(R)$ is a fake $F_1 \times I$.

THEOREM 2. *Let M be a compact, connected, p^2-irreducible 3-manifold. Let S be a closed, connected 2-manifold, not the 2-sphere or projective plane. Let $g:(S, x_0) \to (M, x)$ be a map such that $g_*:\pi_1(S, x_0) \to \pi_1(M, x)$ is $1-1$. Then there exists a covering space (M^*, p) of M and an embedding $g_1:S \to M^*$ such that*

$$(pg_1)_*\pi_1(S, x_0) = g_*\pi_1(S, x_0) \subset \pi_1(M, x).$$

Proof. Let (M^*, p) be the covering space of M associated with $g_*\pi_1(S,x_0) \subset \pi_1(M, x)$. Let $g_4:S \to M^*$ be a map such that $pg_4 = g:S \to M$. Let $g_3:S \to M^*$ be a map homotopic to g_4 such that $\text{cl}\{z:\{z\} \neq g_3^{-1}g_3(z)\}$ is a 1-complex. Let R_0 be a regular neighborhood of $g_3(S)$. We propose to modify R_0 to obtain a compact connected 3-submanifold R^* of M^* such that

(a) $\text{bd}(R^*)$ is incompressible in M^*;

(b) $\rho_*:\pi_1(R^*) \to \pi_1(M^*)$ is an isomorphism. Given R_k, for k an integer, such that

$$\text{cl}(\text{bd}(R_k) - (\text{bd}(R_k) \cap \text{bd}(R_0)))$$

is a collection of disjoint disks (possibly empty), we define R_{k+1} as follows:

(1) If for every component F of $\text{bd}(R_k)$ $\rho_*:\pi_1(F) \to \pi_1(M_*)$ is an injection, $R_{k+1} = R_k$.

(2) Otherwise, we let D_{k+1} be a disk embedded in M^* such that $D_{k+1} \cap \text{bd}(R_k) = \text{bd}(D_{k+1})$ and $\text{bd}(D_{k+1})$ is not nullhomotopic in $\text{bd}(R_k)$. It follows immediately from Lemma 6 that such a disk exists. We may assume that $\text{bd}(D_{k+1}) \subset \text{int}(\text{bd}(R_0) \cap \text{bd}(R_k))$ since $\text{cl}(\text{bd}(R_k) - \text{bd}(R_0))$ is a collection of disjoint disks in $\text{bd}(R_k)$. We may also assume that D_{k+1} is in general position with respect to $g_3(S_1)$ and the portion of $\text{bd}(R_0)$ not contained in $\text{bd}(R_k)$. Then if $D_{k+1} \subset R_k$, we remove a regular neighborhood of D_{k+1} from R_k to obtain R_{k+1}. Otherwise we add a regular neighborhood of D_{k+1} to R_k to obtain R_{k+1}. Thus if there is a component F of $\text{bd}(R_k)$ such that $\rho_*:\pi_1(F) \to \pi_1(M^*)$ is not an injection, the total genus of $\text{bd}(R_{k+1})$ is less than the total genus of $\text{bd}(R_k)$.

Since the total genus of $\text{bd}(R_0)$ is finite, there exists a positive integer n such that $R_k = R_{k+1}$ for $k \geqq n$.

Since $\pi_2(M^*) = 0$, it follows from Lemma 4 that every sphere in $\text{bd}(R_n)$ bounds a homotopy cell in M^*. We define R_n^* to be the union of R_n with the collection of homotopy cells bounded by 2-spheres in $\text{bd}(R_n)$. We observe that $\text{bd}(R_n^*)$ is incompressible in M^*. By construction, $g_3^{-1}(g_3(S) \cap \bigcup_{k=1}^n D_k)$ is a collection of disjoint simple loops in S. Since $g_{3*}:\pi_1(S) \to \pi_1(M^*)$ is an injection, each of these simple loops is nullhomotopic in S. It follows that we can find a disk $D \subset S$ such that

$$D \supset g_3^{-1}(g_3(S) \cap \bigcup_{k=1}^n D_k).$$

We let R^* be the component of R_n^* which contains $g_3(S - D)$. We note that $g_3\text{bd}(D)$ is nullhomotopic in M^*; and thus by Lemma 2, $g_3\text{bd}(D)$ is nullhomotopic in R^* since $\text{bd}(R^*)$ is incompressible. We define a map $g_2:S \to R^*$

(1) by $g_2|S - D = g_3|S - D$, and

(2) by using the nullhomotopy of $g_3\text{bd}(D)$ in R^* to extend g_2 to D. Since $\pi_2(M^*) = 0$, g_2 and g_3 are homotopic. Thus $g_{2*}:\pi_1(S) \to \pi_1(M^*)$ is an isomorphism. It is an easy consequence of Lemma 2 that $g_{2*}:\pi_1(S) \to \pi_1(R^*)$ is an isomorphism.

Now we claim that R^* is a fake line bundle over S. This can be seen as follows: If $\text{bd}(R^*)$ is not connected, it follows from Lemma 11 that two components of $\mathrm{bd}(R^*)$ bound a fake $S \times I$ in R^*. Thus R^* is a fake $S \times I$. If $\mathrm{bd}(R^*)$ is connected, it follows from Lemma 4 that R^* is a fake twisted line bundle over S. We may assume that R^* contains a point y in $p^{-1}(x)$. Now we can find an embedding g_1 of S in R such that y is in $g_1(S)$ and $g_{1*}:\pi_1(S) \rightarrow \pi_1(M^*)$ is an isomorphism. This completes the proof of the theorem since $(pg_1)_*:\pi_1(S, x_0) \rightarrow \pi_1(M, x)$ is an isomorphism onto $g_*\pi_1(S, x_0) \subset \pi_1(M, x)$.

Proof of Theorem 1. Let $g:(S_1, x_1) \rightarrow (M, x)$ be a map such that $g_*:\pi_1(S_1, x_1) \rightarrow \pi_1(M, x)$ is an injection and

$$g_*: \pi_1(S_1, x_1) \underset{\neq}{\supset} f_*\pi_1(S, x_0).$$

Let (M^*, p) be the covering space of M associated with $g_*\pi_1(S_1, x_1) \subset \pi_1(M, x)$. It follows from Theorem 2 that there is an embedding $g_1:S_1 \rightarrow M^*$ such that $(pg_1)_*\pi_1(S_1, x_1) = g_*\pi_1(S_1, x_1) \subset \pi_1(M, x)$. Let $f_1:S \rightarrow M^*$ be an embedding such that $pf_1 = f$. It follows from a general position argument that we can find a small motion of g_1 so that $L = g_1(S_1) \cap f_1(S)$ will be a 1-manifold, i.e., a collection of simple loops. Suppose some loop $\lambda \subset L$ is nullhomotopic in M^*. Since g_{1*} and f_{1*} are injections, $l_1 = g_1^{-1}(\lambda)$ and $l_2 = f_1^{-1}(\lambda)$ are nullhomotopic on S_1 and S, respectively. Let D_1 be the disk contained in S_1 bounded by l_1. It is easy to choose λ so that $D_1 \cap g_1^{-1}(L) = l_1$. Let $D_2 \subset S$ be the disk bounded by l_2. It follows from Lemma 1 that $\pi_2(M) = 0$ and thus from Lemma 9 that $g_1(D_1) \cup f_1(D_2)$ bounds a homotopy cell C in M^*. We notice that $f_1(S)$ meets a regular neighborhood of C in a disk $\bar{D}_2$. Since every loop in $g_1^{-1}(g_1(S_1) \cap \bar{D}_2)$ bounds a disk on S_1, it is not hard to define an embedding $g_2:S_1 \rightarrow M$ such that

(1) $g_2 = g_1$ except on a collection of disks on S_1;
(2) $g_{2*}:\pi_1(S_1) \rightarrow \pi_1(M^*)$ is an isomorphism;
(3) $g_2(S_1) \cap \bar{D}_2$ is empty;
(4) $g_2(S_1) \cap f_1(S) \subset L$.

Since $\pi_2(M^*) = 0$, g_1 and g_2 are homotopic and g_{2*} is an isomorphism. It follows that we may choose g_1 so that every loop in $L = g_1(S_1) \cap f_1(S)$ is nontrivial in M^*. Note that we do not require that $pg_1(x_1) = x$. The proof of the theorem breaks into two cases.

Case 1. $f_1(S) \cap g_1(S_1)$ is empty.
Case 2. $f_1(S) \cap g_1(S_1)$ is non-empty.

Case 1. If $f_1(S)$ and $g_1(S_1)$ are two sided in M^*, it follows from Lemma 11 that $f_1(S)$ and $g_1(S_1)$ bound a fake $S \times I$ embedded in M^*. This is impossible since

$$f_{1*}\pi_1(S) \underset{\neq}{\subset} g_{1*}\pi_1(S_1).$$

If $g_1(S_1)$ is two sided in M^* and $f_1(S)$ is not, we let R be a regular neighborhood of $f_1(S)$. Now $\mathrm{bd}(R)$ is two sided in M^*. Since $\mathrm{bd}(R)$ is incompressible in M^*, bd(R) and $g_1(S_1)$ bound a fake $S_1 \times I$ which we denote by R_1. Now $R_1 \cup R$ is a fake twisted line bundle over S bounded by $g_1(S_1)$. It is a consequence of Lemma 4 that $g_{1*}\pi_1(S_1)$ is of index two in $\pi_1(R_1 \cup R)$. This is impossible since by Lemma 2, $\rho_*:\pi_1(R_1 \cup R) \to \pi_1(M^*)$ is an isomorphism. If neither $f_1(S)$ nor $g_1(S_1)$ separates a regular neighborhood of itself, we let R_1 and R_2 be regular neighborhoods of $f_1(S)$ and $g_1(S_1)$, respectively. Now bd$(R_1) \cup$ bd(R_2) bounds a fake product line bundle R_3 in M^* by Lemma 11. Thus $M^* = R_1 \cup R_2 \cup R_3$. This is easily seen to be impossible as $\pi_1(M^*)$ would not be isomorphic to the group of a closed surface.

If $f_1(S)$ is two sided and $g_1(S)$ fails to separate a regular neighborhood R of itself, Lemma 11 implies that $f_1(S) \cup$ bd(R) bounds a fake $S \times I$ in M^*. We denote this fake $S \times I$ by R_1. Consider $N_1^* = R_1 \cup R$. Suppose that $p^{-1}f(S) \cap N_1^*$ contains a component $F \neq f_1(S)$. We claim F is two sided in M^*.

This can be seen as follows. Let z_0 be the point in $p^{-1}(x) \cap f_1(S)$. Let R be a regular neighborhood of $f(S)$. Since $\rho_*\pi_1(R, x) \subset p_*\pi_1(M, z_0)$, $\rho:(R, x) \to (M, x)$ lifts to an embedding $\rho_1:(R, x) \to (M^*, z_0)$. Since $f_1(S)$ is two sided in $\rho_1(R)$, $f(S)$ is two sided in R and thus in M. It follows that F is two sided in M^*.

By Lemma 11, $F \cup f_1(S)$ bounds a fake $S \times I$ embedded in N_1^* which we denote by R_2. Now cl$(\mathrm{N}_1^* - R_2)$ is a deformation retract of N_1^*. Thus $\rho_*:\pi_1(\mathrm{cl}(N_1^* - R_2)) \to \pi_1(N_1^*)$ is an isomorphism. Thus

$$\rho_*\pi_1(\mathrm{cl}(N_1^* - R_2)) = \pi_1(M^*).$$

Since N_1^* is compact, there can only be a finite number of components in $p^{-1}f(S) \cap N_1^*$. Thus by an appropriate choice of F above we have that if $N^* = \mathrm{cl}(N_1^* - R_2)$,

$$N^* \cap p^{-1}f(S) = \mathrm{bd}(N^*) = F.$$

It follows from Lemma 7 that $(N^*, p|N^*)$ is a finite covering space of $p(N^*)$.

We wish to show that $(p|F)_*:\pi_1(F) \to \pi_1(f(S))$ is an isomorphism so that $p|F$ will be a homeomorphism. If $p|F$ is a homeomorphism, it will follow from Lemma 8 that $p|N^*$ is a homeomorphism. Since $p(N^*)$ is a 3-submanifold of M whose boundary is incompressible in M, it will follow from Lemma 5 that $p(N^*)$ is p^2-irreducible and thus that N^* is p^2-irreducible. But then by Lemma 4, N^* will be a twisted line bundle. Of course, this implies that $N = p(N^*)$ is a twisted line bundle which would complete the proof of Case 1. It remains to show that $p_*\pi_1(F) = \pi_1(f(S))$. Let (M^{**}, q) be the covering space of M associated with $f_*\pi_1(S, x_0) \subset \pi_1(M, x)$. Let R_2 be as above. Since $f_*\pi_1(S) \subset g_*\pi_1(S_1)$, we can find a covering map q_1 to complete the diagram in Figure 2.

We observe that there is an embedding $H:R_2 \to M^{**}$ such that $(q_1H)_* = \rho_*$. Note that both components of bd(R_2) carry the homotopy of R_2 and that $q^{-1}(x)$ meets both components of H bd(R_2) in at least one point. Let F_1 be

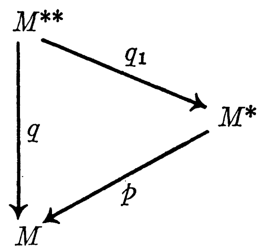

Figure 2

the component of H bd(R_2) which is contained in $q_1^{-1}(F)$ and x_3 a point in F_1 such that $q(x_3) = x$. We choose x_3 as the basepoint for M^{**}.

Since $\rho_*:\pi_1(F_1, x_3) \to \pi_1(M^{**}, x_3)$ is an isomorphism, we have that $q_*\rho_*:\pi_1(F_1, x_3) \to \pi_1(f(S), x)$ is onto and thus $p_*:\pi_1(F) \to \pi_1(f(S))$ is an isomorphism as was to be shown

Case 2. We assume that $L = f_1(S) \cap g_1(S_1)$ is a non-empty collection of disjoint simple loops and that if l is any loop in L, l is not nullhomotopic in M^*. Let R_0 be a regular neighborhood of $f_1(S) \cup g_1(S_1)$. We will modify R_0 in this proof in much the same way that we modified R_0 in the proof of Theorem 2.

We propose to modify R_0 to obtain a compact, connected 3-submanifold N_1^* of M^* such that

(a) bd(N_1^*) is incompressible in M^*;

(b) $\rho_*\pi_1(N_1^*) \to \pi_1(M^*)$ is an isomorphism;

(c) $f_1(S) \cup g_1(S_1) \subset N_1^*$.

Given R_k, for k an integer, we define R_{k+1} as follows:

(1) If for every component F of bd(R_k) $\rho_*\pi_1(F) \to \pi_1(M^*)$ is an injection $R_{k+1} = R_k$.

(2) Otherwise, we let D_{k+1} be a disk embedded in M^* such that $D_{k+1} \cap \text{bd}(R_k) = \text{bd}(D_{k+1})$ and bd(D_{k+1}) is not nullhomotopic in bd(R_k). The existence of such a disk follows from Lemma 6. We may also assume that D_{k+1} is in general position with respect to $f_1(S)$. It follows from a cutting argument that we may assume that D_{k+1} does not meet $f_1(S)$ since f_{1*} is an injection and every loop in $f_1(S) \cap D_{k+1}$ bounds a disk on $f_1(S)$. Using another general position argument we may assume that D_{k+1} meets $g_1(S_1)$ in a collection of simple closed loops. Since g_{1*} is an isomorphism, each of the simple closed loops bounds a disk D on $g_1(S_1)$. We observe that D does not meet $L = g_1(S_1) \cap f_1(S)$ since every loop in L is nontrivial in M^*. It follows by a cutting argument that $D_{k+1} \cap (f_1(S) \cup g_1(S_1))$ is empty. If $D_{k+1} \subset R_k$, we define R_{k+1} to be R_k with a regular neighborhood of D_{k+1} removed. If $D_{k+1} \cap R_k = \text{bd}(D_{k+1})$, we define R_{k+1} to be the union of R_k with a regular neighborhood of D_{k+1}. In either case the total genus of the boundary of R_{k+1} is less than the total genus of the boundary of R_k. Since the total genus of bd(R_0) is finite, there is an integer n such that $R_k = R_{k+1}$ for $k \geqq n$. Let $\bar{N}_1$ be the component of R_n which contains $g_1(S_1) \cup f_1(S)$. By Lemma 9, every

2-sphere in $\text{bd}(\bar{N}_1)$ bounds a homotopy 3-cell in M^*. We add all such homotopy cells to $\bar{N}_1$ to obtain N_1^*. It is a consequence of Lemma 2 that $\rho_*:\pi_1(N_1^*) \to \pi_1(M^*)$ is an isomorphism since $\text{bd}(N_1^*)$ is incompressible in M^*.

Suppose $\text{bd}(N_1^*)$ is disconnected. Then by Lemma 11, N_1^* is a fake $S \times I$. It is a consequence of Lemma 3 that $f_{1*}:\pi_1(S) \to \pi_1(N_1^*)$ is an isomorphism. This is impossible.

Suppose $f_1(S)$ is one sided in N_1^*. Let R be a regular neighborhood of $f_1(S)$. Then $\text{bd}(R) \cup \text{bd}(N^*)$ bounds a fake $\text{bd}(R) \times I$ by Lemma 11. It follows that R is a deformation retract of N^* and $f_{1*}:\pi_1(S) \to \pi_1(N^*)$ is an isomorphism. This is impossible since

$$f_{1*}\pi_1(S) \underset{\neq}{\subset} \pi_1(M^*).$$

Now $f_1(S)$ is two sided in N_1^*. Thus by Lemma 11, $f_1(S)$ and $\text{bd}(N_1^*)$ bound a fake $S \times I$ embedded in N_1^*. We denote this fake $S \times I$ by $\bar{N}$. Now $\text{cl}(N_1^* - \bar{N}) = N_1^{**}$ is a deformation retract of N_1^*. Thus $\rho_*\pi_1(N_1^{**}) \to \pi_1(M^*)$ is an isomorphism.

Suppose $p^{-1}f(S) \cap N_1^{**} \neq f_1(S)$. Let F be a component of $p^{-1}f(S) \cap N_1^{**}$ other than $f_1(S)$. As was shown earlier, F is two sided in M^*. By Lemma 11, $F \cup f_1(S)$ bounds a fake $S \times I$ embedded in N_1^{**}. We denote this fake $S \times I$ by $\bar{N}_1$. If we are careful in our choice of F, we can have that

$$\text{cl}(N_1^{**} - \bar{N}_1) \cap p^{-1}f(S) = F.$$

Let $N^* = \text{cl}(N_1^{**} - \bar{N}_1)$. As was shown earlier $p|F$ is a homeomorphism. Thus $p|N^*$ is a homeomorphism. As in the proof of Case 1, we see that N^* is a twisted line bundle and the theorem follows.

Note added in proof. William Jaco has obtained a result similar to our Theorem 1 in his paper *Finitely presented subgroups of 3-manifold groups*, Invent. Math. *13* (1971), 335–346.

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