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ON THE PRINCIPLE OF DUALITY IN LORENTZ SPACES

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ABSTRACT. A characterization of the spaces dual to weighted Lorentz spaces are given by means of reverse Hölder inequalities (Theorems 2.1, 2.2). This principle of duality is then applied to characterize weight functions for which the identity operator, the Hardy-Littlewood maximal operator and the Hilbert transform are bounded on weighted Lorentz spaces.

1. **Introduction.** Let $v(t) \geq 0$ be a locally integrable function on $\mathbf{R}^+ = (0,\infty)$. The weighted Lorentz spaces $\Lambda_p(v)$ and $\Gamma_p(v)$, $p > 0$, with weight v, consist of measurable functions f on $\mathbf{R}^n$, for which

$$\|f\|^*_{p,v} \equiv \left\{\int_0^\infty [f^*(t)]^p v(t)\,dt\right\}^{1/p},$$

respectively

$$\|f\|^{**}_{p,v} \equiv \left\{\int_0^\infty \left[t^{-1}\int_0^t f^*\right]^p v(t)\,dt\right\}^{1/p},$$

(with the usual interpretations when $p = \infty$) are finite. Here

$$f^*(t) = \inf\Big\{s : m\big(\{x \in \mathbf{R}^n : |f(x)| > s\}\big) \leq t\Big\}, \quad t > 0,$$

is the decreasing rearrangement of f with respect to Lebesgue measure.

These spaces were introduced by G. G. Lorentz [14]. In particular, he established the equivalence of $\|f\|^*_{p,v}$ and $\|f\|^{**}_{p,v}$ for non-increasing v, and has shown that for $1 < p < \infty$, $\Lambda^*_p(v)$, the space dual to $\Lambda_p(v)$, has norm

$$\|g\|_{\Lambda^*_p(v)} = \left\{\int_0^\infty [g^0(t)/v(t)]^{p'} v(t)\,dt\right\}^{1/p'}, \tag{1.1}$$

where here and in the sequel $p' = p/(p-1)$ is the conjugate index of p, and g^0 Halperin's level function ([11]). These level functions have recently been applied in the study of Hardy-type inequalities ([18], [22], [23]). In fact in [18], the extension of (1.1) to arbitrary Borel measures permitted the characterization of weights (measures) for which the Hardy operator is bounded in weighted Lebesgue spaces in the index range $0 < q < p$, $p > 1$.

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Mapping properties of operators on Lorentz spaces (with power weights) are particularly important in the theory of interpolation (*cf.* [3], [4], [13], [20], [26]) and it is natural therefore to seek characterizations of weight functions for which classical operators defined on the cone of monotone functions are bounded in weighted Lebesgue and Lorentz spaces. There is an extensive list where characterizations for a variety of operators are given. See for example [1], [2], [5], [9], [12], [21], [22], [23] and the literature cited there. E. T. Sawyer ([17]) in particular, established an explicit reverse Hölder inequality for non-increasing functions of the form

$$\sup_{0\le f\downarrow} \frac{\int_0^\infty fg}{\{\int_0^\infty f^p v\}^{1/p}} \approx \left\{\int_0^\infty \left[\int_0^x g\right]^{p'} \left[\int_0^x v\right]^{-p'} v(x)\,dx\right\}^{1/p'} + \frac{\int_0^\infty g}{\{\int_0^\infty v\}^{1/p}}, \tag{1.2}$$

$1 < p < \infty, g \ge 0$, and used it in conjunction with the classical reverse Hölder inequality

$$\sup_{0\le f} \frac{\int_0^\infty fg}{\{\int_0^\infty f^p v\}^{1/p}} = \left\{\int_0^\infty g^{p'} v^{1-p'}\right\}^{1/p'}, \quad 1 < p < \infty$$

to characterize weights u, v, for which the Hardy-Littlewood maximal operator and the Hilbert transform are bounded from $\Lambda_p(v)$ to $\Lambda_q(u)$, $1 < p,\ q < \infty$. This principle of duality then permits the reduction of inequalities for non-increasing functions to estimates in dual spaces for arbitrary functions by a change in weights. In particular, (1.1) may be replaced by the explicit expression

$$\Lambda_p^*(v) = \Gamma_{p'}\left(\left[x^{-1}\int_0^x v\right]^{-p'} v\right), \quad 1 < p < \infty, \tag{1.3}$$

provided $\int_0^\infty v = \infty$. This is important since the map $g \longrightarrow g^0$, where g^0 is the level function of (1.1) has only an implicit form.

If $0 < q < p < \infty$, then by Hölder's inequality

$$\left\{\int_0^\infty f^q w\right\}^{1/q} \le C\left\{\int_0^\infty f^p v\right\}^{1/p} \tag{1.4}$$

where $C = \{\int_0^\infty [w^{1/q} v^{-1/p}]^r\}^{1/r}$, $1/r = 1/q - 1/p$, is sharp. Of course (1.4) expresses the boundedness of the identity operator $i\colon L_v^p \longrightarrow L_w^q$, with $\|i\| = C$. The analogue of (1.4) for $0 \le f\downarrow$ was obtained by V. D. Stepanov [24], namely

$$\sup_{0\le f\downarrow} \frac{\{\int_0^\infty f^q w\}^{1/q}}{\{\int_0^\infty f^p v\}^{1/p}} \approx \sup_{t>0}\left\{\int_0^t w\right\}^{1/q}\left\{\int_0^t v\right\}^{-1/p}, \quad 0 < p \le q < \infty \tag{1.5}$$

and

$$\sup_{0\le f\downarrow} \frac{\{\int_0^\infty f^q w\}^{1/q}}{\{\int_0^\infty f^p v\}^{1/p}} \approx \frac{\{\int_0^\infty w\}^{1/q}}{\{\int_0^\infty v\}^{1/p}} + \left\{\int_0^\infty \left[\int_0^x w\right]^{r/q}\left[\int_0^x v\right]^{-r/p} v(x)\,dx\right\}^{1/r} \tag{1.6}$$

if $0 < q < p < \infty$, $1/r = 1/q - 1/p$. Clearly, with $q = 1$, $w = g$, (1.6) implies (1.2), and, with $v = 1$ it yields a variant of Hölder's inequality for non-increasing f. Similarly, it follows from (1.5) with $0 < p < 1 = q$ and $w = g$, that $\Lambda_p^*(v) = \Gamma_\infty([x^{-p}\int_0^x v]^{-1/p})$.

This suggests the study of reverse Hölder inequalities of the form (1.5) and (1.6), where however the suprema are taken over even more restrictive function classes. It is the purpose of this paper to establish such reverse Hölder inequalities where the suprema are taken over the class of quasi-concave ([4]) functions

$$\Omega_{0.1} = \{f(x) \geq 0, f(x)\uparrow, x^{-1}f(x)\downarrow, x \in \mathbf{R}^+\}, \tag{1.7}$$

and where the weights are replaced by positive Borel measures (Theorem 2.1). From this we are able (via Theorem 2.2) to study mapping properties of classical operators in weighted Γ-spaces. Specifically, a principle of duality is established (Theorem 2.2) which makes it possible to characterize weights for which the classical operators are bounded on weighted Γ-spaces and thus extends the recent work of Sawyer [17] to Γ-spaces.

This paper is divided into three sections. In the next section we provide a discretization method which leads to the reverse Hölder inequality for functions in $\Omega_{0.1}$ (Theorem 2.1) and yields subsequent duality results. Such discretization method and the construction of the associated sequences seem to be introduced first by K. I. Oskolkov [16] and was modified and applied by G. Kalyabin, V. Kolyada, I. Netrusov, M. L. Gol'dman and others in the study of function spaces, while the method and its variants in the theory of interpolation were utilized, among others, by J. Brudnij, N. Krugljak, S. Janson and V. Ovchinnikov. Here we follow the work of Gol'dman.

The principle of duality in weighted Γ-spaces (Theorem 2.2) leads in Section 3 to the characterization of weights for which the identity operator, the Hardy-Littlewood maximal operator and the Hilbert transform are bounded on weighted Γ-spaces.

Throughout, we adhere to the convention that uncertainties of the form $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are taken as zero. The notation $A \approx B$ indicates that A/B is bounded above and below by positive constants, $A \ll B$ means that there is a constant C, depending on the involved parameters only, such that $A \leq CB$, and $0 \leq f\downarrow$ indicates that f is non-negative and non-increasing or essentially decreasing, *i.e.*, $f(x) \leq Cf(y)$ holds, for $C \geq 1$ and $0 < y < x < \infty$. Similarly for $f\uparrow$. The function δ_μ denotes the Dirac δ-function concentrated at the point μ. $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}^+$, etc. denote as usual the natural numbers, the integers and the positive real numbers, respectively. Other notations and definitions will be introduced as needed.

2. **The discretization method and main result.**

We now describe the method of discretization and construction of the associated sequences to obtain the converse Hölder inequality for functions in $\Omega_{0.1}$. We follow here M. L. Gol'dman's work [8], [9], [10].

DEFINITION 2.1. a) The positive Borel measure $d\beta$ on $\mathbf{R}^+$ satisfies the nondegeneracy conditions if for $p > 0$

$$\int_0^\infty \left(\frac{s}{s+1}\right)^p d\beta(s) < \infty, \quad \int_0^1 d\beta(s) = \int_1^\infty s^p\, d\beta(s) = \infty. \tag{2.1}$$

b) The fundamental function of a positive Borel measure (say $d\beta$) is defined by

$$\rho(t) = \rho_{\beta,p}(t) \equiv \left\{\int_0^\infty \left(\frac{s}{s+t}\right)^p d\beta(s)\right\}^{1/p}, \quad p > 0,\ t > 0 \tag{2.2}$$

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$$\approx t^{-1}\left\{\int_0^\infty \min(s^p, t^p)\, d\beta(s)\right\}^{1/p}.$$

It is clear that if $\delta \in (-1, p-1), p > 0$, then $d\beta(s) \equiv s^{-p+\delta}\, ds$ satisfies (2.1). Moreover, if a fundamental function of a Borel measure satisfies (2.1), then via standard limiting theorems

$$\lim_{t\to\infty} \rho(t) = \lim_{t\to\infty} \frac{1}{t\rho(t)} = \lim_{t\to 0} t\rho(t) = \lim_{t\to 0} \frac{1}{\rho(t)} = 0. \tag{2.3}$$

DEFINITION 2.2. a) A positive sequence $\{a_k\}_{k\in\mathbf{Z}}$ is said to be *strongly increasing*, respectively, strongly decreasing, if

$$\inf_{k\in\mathbf{Z}} \frac{a_{k+1}}{a_k} > 1, \quad \text{respectively,} \ \sup_{k\in\mathbf{Z}} \frac{a_{k+1}}{a_k} < 1,$$

and we write $a_k \uparrow\uparrow$, respectively, $a_k \downarrow\downarrow$.

b) A sequence $\{\lambda_k\}_{k\in\mathbf{Z}}$ of positive numbers is said to discretize the fundamental function ρ, if

(i) $\lambda_0 = 1, 0 < \lambda_k \uparrow\uparrow$,

(ii) $\rho(\lambda_k) \downarrow\downarrow$ and $\lambda_k\rho(\lambda_k) \uparrow\uparrow, k \in \mathbf{Z}$,

(iii) there is a decomposition $\mathbf{Z} = \mathbf{Z}_1 \cup \mathbf{Z}_2, \mathbf{Z}_1 \cap \mathbf{Z}_2 = \emptyset$, such that for $t \in [\lambda_k, \lambda_{k+1}] \equiv \Delta_k$

$$\rho(\lambda_k) \approx \rho(t) \quad \text{if } k \in \mathbf{Z}_1 \text{ and } \lambda_k\rho(\lambda_k) \approx t\rho(t) \quad \text{if } k \in \mathbf{Z}_2,$$

and the constants of equivalence are independent of $k \in \mathbf{Z}$.

Following [9] (see also [16]) we construct by recurrence the sequence $\{\mu_k\}_{k\in\mathbf{Z}}$ as follows: $\mu_0 = 1$ and for $a > 1$ fixed, let

$$\mu_{k+1} = \left\{t : \min\left\{\frac{\rho(\mu_k)}{\rho(t)}, \frac{t\rho(t)}{\mu_k\rho(\mu_k)}\right\} = a\right\} \quad \text{if } k \geq 0 \tag{2.4}$$

$$\mu_{k-1} = \left\{t : \min\left\{\frac{\rho(t)}{\rho(\mu_k)}, \frac{\mu_k\rho(\mu_k)}{t\rho(t)}\right\} = a\right\} \quad \text{if } k \leq 0,$$

where ρ is a fundamental function satisfying (2.3).

LEMMA 2.1. *Let $a > 1$, then the sequence $\{\mu_k\}_{k\in\mathbf{Z}}$ defined by (2.4) discretizes the fundamental function ρ.*

PROOF. Define $\mathbf{Z}_1$ and $\mathbf{Z}_2$ by

$$\mathbf{Z}_1 = \{k \in \mathbf{Z} : \rho(\mu_k) = a\rho(\mu_{k+1})\} \tag{2.5}$$

$$\mathbf{Z}_2 = \{k \in \mathbf{Z} \setminus \mathbf{Z}_1 : \mu_{k+1}\rho(\mu_{k+1}) = a\mu_k\rho(\mu_k)\}.$$

Clearly $\mathbf{Z}_1 \cap \mathbf{Z}_2 = \emptyset$ and a straightforward argument using (2.4) shows that $\mathbf{Z}_1 \cup \mathbf{Z}_2 = \mathbf{Z}$.

Since $t\rho(t)$ is strictly increasing and $\rho(t)$ strictly decreasing, it follows from (2.4) that $\mu_{k+1} > \mu_k, k \in \mathbf{Z}$. Hence $\mu_{k+1}\rho(\mu_{k+1}) \geq \mu_k\rho(\mu_k)$ and therefore, if $k \in \mathbf{Z}_1$, $\mu_{k+1}/\mu_k \geq$

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$\rho(\mu_k)/\rho(\mu_{k+1}) = a$. Also, since $\rho(\mu_k) \geq \rho(\mu_{k+1})$, $k \in \mathbf{Z}$, it follows that for $k \in \mathbf{Z}_2$, $\mu_{k+1}/\mu_k \geq [\mu_{k+1}\rho(\mu_{k+1})]/[\mu_k\rho(\mu_k)] = a$. This shows that $\mu_k \uparrow\uparrow$.

Next, by (2.4), if $k \in \mathbf{Z}$

$$[\mu_{k+1}\rho(\mu_{k+1})]/[\mu_k\rho(\mu_k)] \geq a > 1, \quad \rho(\mu_{k+1})/\rho(\mu_k) \leq 1/a < 1, \tag{2.6}$$

so that $\rho(\mu_k) \downarrow\downarrow$ and $\mu_k\rho(\mu_k) \uparrow\uparrow$. Finally, since $\rho(t) \downarrow$ and $t\rho(t) \uparrow$, then for $k \in \mathbf{Z}_1$, $t \in [\mu_k, \mu_{k+1}]$, $\rho(t) \leq \rho(\mu_k) = a\rho(\mu_{k+1}) \leq a\rho(t)$ and for $k \in \mathbf{Z}_2$, $t\rho(t) \leq \mu_{k+1}\rho(\mu_{k+1}) = a\mu_k\rho(\mu_k) \leq at\rho(t)$, so the result follows.

For the next result we require the following notation, which is also used throughout: If $d\beta$ is a positive Borel measure and $0 < p < \infty$, then we write

$$\rho_1(t) = \left\{\int_t^\infty d\beta\right\}^{1/p}, \quad \rho_2(t) = t^{-1}\left\{\int_0^t s^p\, d\beta(s)\right\}^{1/p}, \tag{2.7}$$
$$\rho_m(t) = \max\bigl(\rho_1(t), \rho_2(t)\bigr).$$

If ρ is the fundamental function (2.2) then clearly

$$\rho_m(t)/2 \leq \rho(t) \leq 2\rho_m(t). \tag{2.8}$$

We also write for $0 < p < \infty$ and β a positive Borel measure

$$L_{p,\beta} = \left\{f \in \Omega_{0,1} : \|f\|_{p,\beta} \equiv \left\{\int_0^\infty f^p\, d\beta\right\}^{1/p} < \infty\right\}$$

where $\Omega_{0,1}$ is the class of quasi concave functions given by (1.7).

LEMMA 2.2. *Let ρ be the fundamental function (2.2) and $\{\mu_k\}_{k\in\mathbf{Z}}$ the sequence defined by (2.4) with $a > 4$. Then for all $f \in \Omega_{0,1}$ and $0 < p < \infty$,*

$$\|f\|_{p,\beta} \approx \left\{\sum_{k\in\mathbf{Z}} [f(\mu_k)\rho(\mu_k)]^p\right\}^{1/p}. \tag{2.9}$$

PROOF. Let $\mathbf{Z}_1$ and $\mathbf{Z}_2$ be given by (2.5) and $\Delta_k = [\mu_k, \mu_{k+1}]$, then we obtain the upper bound for (2.9) as follows:

$$\|f\|_{p,\beta}^p = \int_0^\infty f^p\, d\beta = \sum_{k\in\mathbf{Z}} \int_{\Delta_k} f^p\, d\beta = \sum_{k\in\mathbf{Z}_1} + \sum_{k\in\mathbf{Z}_2} \equiv I_1 + I_2,$$

respectively. For $k \in Z_1$, $\rho(\mu_k) = a\rho(\mu_{k+1})$ and since $f \uparrow$, (2.7) and (2.8) imply that

$$\begin{aligned} I_1 &= \sum_{k\in\mathbf{Z}_1} \int_{\mu_k}^{\mu_{k+1}} f^p\, d\beta \leq \sum_{k\in\mathbf{Z}_1} f^p(\mu_{k+1}) \int_{\mu_k}^\infty d\beta = \sum_{k\in\mathbf{Z}_1} f^p(\mu_{k+1})\rho_1^p(\mu_k) \\ &\leq 2^p \sum_{k\in\mathbf{Z}_1} f^p(\mu_{k+1})\rho^p(\mu_k) = (2a)^p \sum_{k\in\mathbf{Z}_1} [f(\mu_k)\rho(\mu_k)]^p. \end{aligned}$$

Similarly, for $k \in \mathbf{Z}_2$, $\mu_{k+1}\rho(\mu_{k+1}) = a\mu_k\rho(\mu_k)$, and since $t^{-1}f(t) \downarrow$

$$\begin{aligned} I_2 &= \sum_{k\in\mathbf{Z}_2} \int_{\mu_k}^{\mu_{k+1}} f^p\, d\beta \le \sum_{k\in\mathbf{Z}_2} [f(\mu_k)/\mu_k]^p \int_0^{\mu_{k+1}} s^p\, d\beta(s) \\ &= \sum_{k\in\mathbf{Z}_2} [f(\mu_k)/\mu_k]^p [\mu_{k+1}\rho_2(\mu_{k+1})]^p \le 2^p \sum_{k\in\mathbf{Z}_2} [f(\mu_k)/\mu_k]^p [\mu_{k+1}\rho(\mu_{k+1})]^p \\ &= (2a)^p \sum_{k\in\mathbf{Z}_2} [f(\mu_k)(\mu_k)]^p \end{aligned}$$

from which the upper bound follows with bound $2a$.

To prove the lower bound we consider cases: With the notation (2.7) let

(a) $\mathbf{Z}_a \equiv \{k \in \mathbf{Z} : \rho_m(\mu_k) = \rho_1(\mu_k)\}$

and

(b) $\mathbf{Z}_b \equiv \{k \in \mathbf{Z}, \rho_m(\mu_k) = \rho_2(\mu_k)\}$.

Clearly, $\mathbf{Z}_a \cup \mathbf{Z}_b = \mathbf{Z}$. Now if $k \in \mathbf{Z}_a$, define λ_k by $\rho(\lambda_k) = a^{-1}\rho(\mu_k)$. Then $\lambda_k > \mu_k$ since ρ is strictly decreasing. Also if $k \in \mathbf{Z}_1 \cap \mathbf{Z}_a$, then $a^{-1}\rho(\mu_k) = \rho(\mu_{k+1}) = \rho(\lambda_k)$, so that $\lambda_k = \mu_{k+1}$, while if $k \in \mathbf{Z}_2 \cap \mathbf{Z}_a$, (2.6) shows that $\rho(\lambda_k) = a^{-1}\rho(\mu_k) \ge \rho(\mu_{k+1})$. Therefore for all $k \in \mathbf{Z}_a$, $\mu_k < \lambda_k \le \mu_{k+1}$ and

$$\begin{aligned} \int_{\Delta_k} f^p\, d\beta &= \int_{\mu_k}^{\mu_{k+1}} f^p\, d\beta \ge f^p(\mu_k) \int_{\mu_k}^{\lambda_k} d\beta = f^p(\mu_k)[\rho_1^p(\mu_k) - \rho_1^p(\lambda_k)] \\ &\ge f^p(\mu_k)[\rho_m^p(\mu_k) - \rho_m^p(\lambda_k)] \ge f^p(\mu_k)[2^{-p}\rho^p(\mu_k) - 2^p\rho^p(\lambda_k)] \\ &= [f(\mu_k)\rho(\mu_k)]^p[2^{-p} - (2/a)^p]. \end{aligned}$$

If $k \in \mathbf{Z}_b$, define γ_k by $\gamma_k\rho(\gamma_k) = a^{-1}\mu_k\rho(\mu_k)$. Since $t\rho(t)$ is strictly increasing, $\gamma_k < \mu_k$. Now if $k-1 \in \mathbf{Z}_1 \cap \mathbf{Z}_b$ then by (2.6), $\gamma_k\rho(\gamma_k) = a^{-1}\mu_k\rho(\mu_k) \ge \mu_{k-1}\rho(\mu_{k-1})$ so that $\mu_{k-1} \le \gamma_k$. If $k-1 \in \mathbf{Z}_2 \cap \mathbf{Z}_b$ then again by (2.6), $\gamma_k\rho(\gamma_k) = a^{-1}\mu_k\rho(\mu_k) = \mu_{k-1}\rho(\mu_{k-1})$ so that $\gamma_k = \mu_{k-1}$. Therefore for all $k-1 \in \mathbf{Z}_b$, $\mu_{k-1} \le \gamma_k < \mu_k$ and

$$\begin{aligned} \int_{\Delta_{k-1}} f^p\, d\beta &\ge \int_{\gamma_k}^{\mu_k} f^p\, d\beta \ge [f(\mu_k)/\mu_k]^p \int_{\gamma_k}^{\mu_k} s^p\, d\beta(s) = [f(\mu_k)/\mu_k]^p[\mu_k^p\rho_2^p(\mu_k) - \gamma_k^p\rho_2^p(\gamma_k)] \\ &\ge [f(\mu_k)/\mu_k]^p[\mu_k^p\rho_m^p(\mu_k) - \gamma_k^p\rho_m^p(\gamma_k)] \\ &\ge [f(\mu_k)/\mu_k]^p[\mu_k^p 2^{-p}\rho^p(\mu_k) - \gamma_k^p 2^p\rho^p(\gamma_k)] = [f(\mu_k)\rho(\mu_k)]^p b \end{aligned}$$

where $b = 2^{-p} - (2/a)^p$. Combining these two results we get

$$\begin{aligned} \sum_{k\in\mathbf{Z}} [f(\mu_k)\rho(\mu_k)]^p &= \sum_{k\in\mathbf{Z}_a} + \sum_{k\in\mathbf{Z}_b} \le b^{-1} \sum_{k\in\mathbf{Z}_a} \int_{\Delta_k} f^p\, d\beta + b^{-1} \sum_{k\in\mathbf{Z}_b} \int_{\Delta_{k-1}} f^p\, d\beta \\ &\le b^{-1} \int_0^\infty f^p\, d\beta, \end{aligned}$$

from which the lower bound for (2.9) follows.

Before we can give the main result of this section, the following elementary proposition is required:

PROPOSITION 2.1. *Let $\{a_k\}_{k\in\mathbf{Z}}$, $\{\sigma_k\}_{k\in\mathbf{Z}}$ and $\{\tau_k\}_{k\in\mathbf{Z}}$ be non-negative sequences, and $0 < p < \infty$.*

a) If $\sigma_k \uparrow\uparrow$, then $\{\sum_{k\in\mathbf{Z}}[\sum_{m\ge k} a_m]^p\sigma_k^p\}^{1/p} \ll \{\sum_{m\in\mathbf{Z}}[a_m\sigma_m]^p\}^{1/p}$

and

b) if $\tau_k \downarrow\downarrow$ then $\{\sum_{k\in\mathbf{Z}}[\sum_{m\le k} a_m]^p\tau_k^p\}^{1/p} \ll \{\sum_{m\in\mathbf{Z}}[a_m\tau_m]^p\}^{1/p}$.

PROOF. Let $\sigma = \inf_k \sigma_{k+1}/\sigma_k > 1$. If $0 < p \le 1$, Jensen's inequality shows that

$$\sum_{k\in\mathbf{Z}}\Big[\sum_{m\ge k} a_m\Big]^p \sigma_k^p \le \sum_{k\in\mathbf{Z}}\Big[\sum_{m\ge k} a_m^p\Big]\sigma_k^p = \sum_{m\in\mathbf{Z}} a_m^p \sum_{k\le m}\sigma_k^p$$
$$\le (1+\sigma^{-p}+\sigma^{-2p}+\cdots)\sum_{m\in\mathbf{Z}}[a_m\sigma_m]^p = \frac{\sigma^p}{\sigma^p-1}\sum_{m\in\mathbf{Z}}[a_m\sigma_m]^2.$$

If $1 < p < \infty$ one may use the discrete version of the weighted Hardy's inequality (*cf.* [18]) or the following direct argument: By Hölder's inequality

$$\sum_{m\ge k} a_m \le \Big\{\sum_{m\ge k} a_m^p\sigma_m\Big\}^{1/p}\Big\{\sum_{m\ge k}\sigma_m^{-p'/p}\Big\}^{1/p'}$$
$$\le \frac{\sigma^{1/p}}{(\sigma^{p'/p}-1)^{1/p'}}\Big\{\sum_{m\ge k} a_m^p\sigma_m\Big\}^{1/p}\sigma_k^{-1/p},$$

and therefore

$$\Big\{\sum_{k\in\mathbf{Z}}\Big[\sum_{m\ge k} a_m\Big]^p\sigma_k^p\Big\} \le \frac{\sigma}{(\sigma^{p'/p}-1)^{p/p'}}\sum_{k\in\mathbf{Z}}\sigma_k^{p-1}\sum_{m\ge k} a_m^p\sigma_m$$
$$= \frac{\sigma}{(\sigma^{p'/p}-1)^{p/p'}}\sum_{m\in\mathbf{Z}}[a_m\sigma_m]^p$$
$$\big(1+(\sigma_{m-1}/\sigma_m)^{p-1}+(\sigma_{m-2}/\sigma_m)^{p-1}+\cdots\big)$$
$$= \frac{\sigma^p}{(\sigma^{p'/p}-1)(\sigma^{p-1}-1)}\sum_{m\in\mathbf{Z}}[a_m\sigma_m]^p$$

which proves a).

The argument to prove b) is analogous and hence omitted.

THEOREM 2.1. *Let $d\beta$ and $d\gamma$ be positive Borel measures and $d\beta$ satisfies the non-degeneracy condition (2.1). If $p > 0$ and $\{\mu_k\}$ is the discretizing sequence of the fundamental function $\rho_{\beta,p}$ of (2.2), then for $0 < q < p < \infty$,*

$$J \equiv \sup_{f\in\Omega_{0,1}}\frac{\{\int_0^\infty f^q\,d\gamma\}^{1/q}}{\{\int_0^\infty f^p\,d\beta\}^{1/p}} \approx \Big\{\sum_{k\in\mathbf{Z}}\big[\rho_{\gamma,q}(\mu_k)/\rho_{\beta,p}(\mu_k)\big]^r\Big\}^{1/r} \equiv E, \tag{2.10}$$

where $1/r = 1/q - 1/p$. If $0 < p \le q < \infty$, then

$$J \approx \sup_{t>0}[\rho_{\gamma,q}(t)/\rho_{\beta,p}(t)] \equiv \mathcal{E}. \tag{2.11}$$

PROOF. We establish the result in the following sequence: First we assume that $d\gamma$ also satisfies the non-degeneracy condition (2.1). Then we prove the upper bound for (2.10), (2.11), then the lower bounds for (2.11) and then for (2.10). Finally, we remove the nondegeneracy assumption from $d\gamma$.

Since $d\gamma$ satisfies the nondegeneracy condition (2.1), there is a discretizing sequence $\{\lambda_\ell\}_{\ell\in\mathbf{Z}}$ of the fundamental function $\rho_{\gamma,q}$, and hence by Lemma 2.2, $\|f\|_{q,\gamma} \approx$

$\{\sum_{\ell\in\mathbf{Z}}[f(\lambda_\ell)\rho_{\gamma,q}(\lambda_\ell)]^q\}^{1/q}$, where the constants of equivalence depend only on a and q. In fact for the upper estimate of (2.10) we have

$$\|f\|_{q,\gamma}^q \le (2a)^q \sum_{\ell\in\mathbf{Z}}[f(\lambda_\ell)\rho_{\gamma,q}(\lambda_\ell)]^q \equiv (2a)^q[J_1+J_2],$$

where

$$J_i = \sum_{k\in\mathbf{Z}_i}\ \sum_{\mu_k\le\lambda_\ell<\mu_{k+1}} [f(\lambda_\ell)\rho_{\gamma,q}(\lambda_\ell)]^q \quad i=1,2$$

and $\mathbf{Z}_i$ are the sets defined in (2.5) with discretizing sequence $\{\mu_k\}$ of the fundamental function $\rho_{\beta,p}$. Now if $k\in\mathbf{Z}_1$, then $\mu_k\le\lambda_\ell<\mu_{k+1}$, so $f(\lambda_\ell)\le f(\mu_{k+1})$ and $\rho_{\gamma,q}(\lambda_\ell)\le\rho_{\gamma,q}(\mu_k)$. Since Lemma 2.1 applies with discretizing sequence $\{\lambda_\ell\}$ and fundamental function $\rho_{\gamma,q}$ it follows that $\rho_{\gamma,q}(\lambda_\ell)\downarrow\downarrow$ and (2.6) holds with $\{\mu_k\}$ replaced by $\{\lambda_\ell\}$. Hence

$$\begin{aligned}
J_1 &= \sum_{k\in\mathbf{Z}_1}\ \sum_{\mu_k\le\lambda_\ell<\mu_{k+1}} [f(\lambda_\ell)\rho_{\gamma,q}(\lambda_\ell)]^q \\
&\le \sum_{k\in\mathbf{Z}_1}[f(\mu_{k+1})\rho_{\gamma,q}(\mu_k)]^q(1+a^{-q}+a^{-2q}+\cdots) \\
&= \frac{a^q}{a^q-1}\sum_{k\in\mathbf{Z}_1}[f(\mu_{k+1})\rho_{\beta,p}(\mu_{k+1})]^q[\rho_{\gamma,q}(\mu_k)/\rho_{\beta,p}(\mu_{k+1})]^q.
\end{aligned}$$

If $0<q<p<\infty$, $1/r=1/q-1/p$ we apply Hölder's inequality with exponents p/q and r/q and Lemma 2.2 to obtain

$$\begin{aligned}
J_1 &\le \frac{a^q}{a^q-1}\Big\{\sum_{k\in\mathbf{Z}_1}[f(\mu_{k+1})\rho_{\beta,p}(\mu_{k+1})]^p\Big\}^{q/p}\Big\{\sum_{k\in\mathbf{Z}_1}[\rho_{\gamma,q}(\mu_k)/\rho_{\beta,p}(\mu_{k+1})]^r\Big\}^{q/r} \\
&\ll \|f\|_{p,\beta}^q a^q\Big\{\sum_{k\in\mathbf{Z}_1}\Big[\frac{\rho_{\gamma,q}(\mu_k)}{\rho_{\beta,p}(\mu_k)}\Big]^r\Big\}^{q/r},
\end{aligned}$$

where we used the fact that $\rho_{\beta,p}(\mu_{k+1})=a^{-1}\rho_{\beta,p}(\mu_k)$ if $k\in\mathbf{Z}_1$.

The estimate for J_2 is similar: By (2.6)

$$\begin{aligned}
J_2 &= \sum_{k\in\mathbf{Z}_2}\ \sum_{\mu_k\le\lambda_\ell<\mu_{k+1}} [f(\lambda_\ell)/\lambda_\ell]^q[\lambda_\ell\rho_{\gamma,q}(\lambda_\ell)]^q \\
&\le \sum_{k\in\mathbf{Z}_2}[f(\mu_k)/\mu_k]^q[\mu_{k+1}\rho_{\gamma,q}(\mu_{k+1})]^q(1+a^{-q}+a^{-2q}+\cdots) \\
&= \frac{a^q}{a^q-1}\sum_{k\in\mathbf{Z}_2}[f(\mu_k)\rho_{\beta,p}(\mu_k)]^q\Big[\frac{\mu_{k+1}\rho_{\gamma,q}(\mu_{k+1})}{\mu_k\rho_{\beta,p}(\mu_k)}\Big]^q.
\end{aligned}$$

Again by Hölder's inequality with exponents, p/q and r/q, Lemma 2.2 and the fact that $\mu_k\rho_{\beta,p}(\mu_k)=a^{-1}\mu_{k+1}\rho_{\beta,p}(\mu_{k+1})$ for $k\in\mathbf{Z}_2$ yields

$$\begin{aligned}
J_2 &\le \frac{a^q}{a^q-1}\Big\{\sum_{k\in\mathbf{Z}_2}[f(\mu_k)\rho_{\beta,p}(\mu_k)]^p\Big\}^{q/p}\Big\{\sum_{k\in\mathbf{Z}_2}\Big[\frac{\mu_{k+1}\rho_{\gamma,q}(\mu_{k+1})}{\mu_k\rho_{\beta,p}(\mu_k)}\Big]^r\Big\}^{q/r} \\
&\ll \|f\|_{p,\beta}^q a^q\Big\{\sum_{k\in\mathbf{Z}_2}\Big[\frac{\rho_{\gamma,q}(\mu_{k+1})}{\rho_{\beta,p}(\mu_{k+1})}\Big]^r\Big\}^{q/r}.
\end{aligned}$$

Therefore, if $0 < q < p < \infty$

$$J \ll \Big\{ \sum_{k \in \mathbf{Z}} [\rho_{\gamma,q}(\mu_k)/\rho_{\beta,p}(\mu_k)]^r \Big\}^{1/q}$$

and the upper bound for (2.10) follows.

The upper bound for (2.11) is similar. In fact, since $0 < p \leq q < \infty$ we apply instead of Hölder's inequality above, Jensen's inequality and obtain

$$\begin{aligned}
\|f\|_{q,\gamma} \ll J_1^{1/q} + J_2^{1/q} &\leq \sup_{k \in \mathbf{Z}_1} [\rho_{\gamma,q}(\mu_k)/\rho_{\beta,p}(\mu_{k+1})] \Big\{ \sum_{k \in \mathbf{Z}_1} [f(\mu_{k+1})\rho_{\beta,p}(\mu_{k+1})]^q \Big\}^{1/q} \\
&\quad + \sup_{k \in \mathbf{Z}_2} \Big[\frac{\mu_{k+1}\rho_{\gamma,q}(\mu_{k+1})}{\mu_k \rho_{\beta,p}(\mu_k)} \Big] \Big\{ \sum_{k \in \mathbf{Z}_2} [f(\mu_k)\rho_{\beta,p}(\mu_k)]^q \Big\}^{1/q} \\
&\ll \sup_{k \in \mathbf{Z}} [\rho_{\gamma,q}(\mu_k)/\rho_{\beta,p}(\mu_k)] \Big\{ \sum_{k \in \mathbf{Z}} [f(\mu_k)\rho_{\beta,p}(\mu_k)]^p \Big\}^{1/p} \\
&\ll \sup_{t>0} [\rho_{\gamma,q}(t)/\rho_{\beta,p}(t)] \|f\|_{p,\beta}
\end{aligned}$$

by Lemma 2.2.

The lower bound of (2.11) follows from the inequality

$$\Big\{ \int_0^\infty f^q \, d\gamma \Big\}^{1/q} \leq J \Big\{ \int_0^\infty f^p \, d\beta \Big\}^{1/p}, \quad f \in \Omega_{0,1}, \tag{2.12}$$

where J is given in (2.10). For $t > 0$ fixed, define f_t by $f_t(s) = s/t$ if $0 < s \leq t$ and $f_t(s) = 1$ if $s > t$. Then $f_t \in \Omega_{0,1}$ and by (2.2) and (2.12)

$$\rho_{\gamma,q}(t) \leq \Big\{ \int_0^\infty f_t^q \, d\gamma \Big\}^{1/q} \leq J \Big\{ \int_0^\infty f_t^p \, d\beta \Big\}^{1/p} \leq 2J\rho_{\beta,p}(t).$$

It follows that $\sup_{t>0}[\rho_{\gamma,q}(t)/\rho_{\beta,p}(t)] \leq 2J$.

For the lower bound for (2.10) some auxiliary observation is needed. From the definitions of $\{\lambda_\ell\}$, $\mathbf{Z}_1$, $\mathbf{Z}_2$ of (2.5) and since $\rho_{\beta,p}(t) \downarrow$, $\rho_{\gamma,q}(\lambda_\ell) \downarrow\downarrow$ and $t\rho_{\beta,p}(t) \uparrow$, $\lambda_\ell \rho_{\gamma,q}(\lambda_\ell) \uparrow\uparrow$ one obtains

$$\begin{aligned}
\sum_{\ell \in \mathbf{Z}} [\rho_{\gamma,q}(\lambda_\ell)/\rho_{\beta,p}(\lambda_\ell)] &= \sum_{k \in \mathbf{Z}_1} \sum_{\mu_k \leq \lambda_\ell < \mu_{k+1}} [\rho_{\gamma,q}(\lambda_\ell)/\rho_{\beta,p}(\lambda_\ell)]^r \\
&\quad + \sum_{k \in \mathbf{Z}_1} \sum_{\mu_k \leq \lambda_\ell < \mu_{k+1}} [\rho_{\gamma,q}(\lambda_\ell)/\rho_{\beta,p}(\lambda_\ell)]^r \\
&\leq a \sum_{k \in \mathbf{Z}_1} [\rho_{\beta,p}(\mu_k)]^{-r} \sum_{\mu_k \leq \lambda_\ell < \mu_{k+1}} [\rho_{\gamma,q}(\lambda_\ell)]^r \\
&\quad + a \sum_{k \in \mathbf{Z}_2} [\mu_{k+1}\rho_{\beta,p}(\mu_{k+1})]^{-r} \sum_{\mu_k \leq \lambda_\ell < \mu_{k+1}} [\lambda_\ell \rho_{\gamma,q}(\lambda_\ell)]^r \\
&\leq \frac{2a^{2r}}{a^r - 1} \sum_{k \in \mathbf{Z}} [\rho_{\beta,p}(\mu_k)/\rho_{\gamma,q}(\mu_k)]^r
\end{aligned}$$

and, analogously,

$$\sum_{k\in\mathbf{Z}}[\rho_{\beta,p}(\mu_k)/\rho_{\gamma,q}(\mu_k)]^r \le \frac{2a^{2r}}{a^r-1}\sum_{\ell\in\mathbf{Z}}[\rho_{\beta,p}(\lambda_\ell)/\rho_{\gamma,q}(\lambda_\ell)]^r. \tag{2.13}$$

Applying (2.13) it is clear that E (the right side of 2.10) satisfies

$$E^r \ll \sum_{\ell\in\mathbf{Z}}[\rho_{\gamma,q}(\lambda_\ell)/\rho_{\beta,p}(\lambda_\ell)]^r \equiv E_1^r + E_2^r,$$

where

$$E_i^r = \sum_{k\in\mathbf{Z}_i}\left(\int_{\mu_k}^{\mu_{k+1}} \rho_{\beta,p}^{-q}(s)\,d\gamma_0(s)\right)^{r/q}, \quad d\gamma_0(s) = \sum_{\lambda_\ell}\rho_{\gamma,q}^q(s)\delta_{\lambda_\ell}(s)\,ds,$$

$i = 1,2$, and δ_{λ_ℓ} is the Dirac δ-function at the point λ_ℓ.

Now by Lemma 2.2, with p replaced by q and β by γ it follows that

$$J \approx \sup_{f\in\Omega_{0,1}} \frac{\{\sum_\ell [f(\lambda_\ell)\rho_{\gamma,q}(\lambda_\ell)]^q\}^{1/q}}{\{\int_0^\infty f^p\,d\beta\}^{1/p}} = \sup_{f\in\Omega_{0,1}} \frac{\{\int_0^\infty f^q\,d\gamma_0\}^{1/q}}{\{\int_0^\infty f^p\,d\beta\}^{1/p}} \equiv J_0.$$

It suffices therefore to prove that $E_i \ll J_0, \quad i = 1,2$.

Applying Lemma 2.2 again we find that

$$J_0 \approx \sup_{f\in\Omega_{0,1}} \left\{\int_0^\infty f^q\,d\gamma_0\right\}^{1/q}\left\{\sum_k [f(\mu_k)\rho_{\beta,p}(\mu_k)]^p\right\}^{-1/p}. \tag{2.14}$$

Denote by $B_{0,1} \equiv \{b = \{b_k\}_{k\in\mathbf{Z}} : b_k > 0,\ b_k \uparrow \text{ and } \mu_k^{-1}b_k \downarrow\}$, then $f \in \Omega_{0,1}$ implies $f(\mu_k) \in B_{0,1}$. Given $b \in B_{0,1}$ we define the extremal function f_b by

$$f_b(x) = \begin{cases} b_k\mu_k^{-1}x & \text{if } \mu_k \le x < \nu_k \\ b_{k+1} & \text{if } \nu_k \le x < \mu_{k+1}, \end{cases}$$

where $\nu_k = b_{k+1}\mu_k/b_k$, $k \in \mathbf{Z}$. Clearly, $f_b \in \Omega_{0,1}$ and for any $f \in \Omega_{0,1}$ satisfying $f(\mu_k) = b_k$, $k \in \mathbf{Z}$, the inequality $f(x) \le f_b(x)$ holds. To see this, let $x \in [\nu_k, \mu_{k+1})$, then $f_b(x) = b_{k+1} = f(\mu_{k+1}) \ge f(x)$, while $x \in [\mu_k, \nu_k)$ implies $f_b(x) = b_k\mu_k^{-1}x \ge b_{k+1}x/\mu_{k+1} = x\mu_{k+1}^{-1}f(\mu_{k+1}) \ge xf(x)/x = f(x)$.

This together with (2.14) shows that

$$J_0 = \sup_{b\in B_{0,1}} \left\{\int_0^\infty f_b^q\,d\gamma_0\right\}^{1/q}\left\{\sum_k [b_k\rho_{\beta,p}(\mu_k)]^p\right\}^{-1/p}.$$

Writing

$$A_i \equiv \left\{\sum_{k\in\mathbf{Z}_i}\int_{\mu_k}^{\mu_{k+1}} f_b^q\,d\gamma_0\right\}^{1/q}, \quad J_{0,i} \equiv \sup_{b\in B_{0,1}} A_i\left\{\sum_k [b_k\rho_{\beta,p}(\mu_k)]^p\right\}^{-1/p}$$

$i = 1,2$, then $J_0 \approx J_{0,1} + J_{0,2}$. Now if $x \in [\mu_k, \mu_{k+1})$, then $b_{k+1}\mu_{k+1}^{-1}x \le f_b(x) \le b_k\mu_k^{-1}x$ and denoting d_k by

$$d_k \equiv \left\{\int_{\mu_k}^{\mu_{k+1}} x^q\,d\gamma_0(x)\right\}^{1/q}, \quad \text{if } k \in \mathbf{Z}_2 \text{ and } d_k = 0 \quad \text{if } k \in \mathbf{Z}_1$$

then

$$\Big\{\sum_k [b_{k+1}\mu_{k+1}^{-1}\, d_k]^q\Big\}^{1/q} \le A_2 \le \Big\{\sum_k [b_k\mu_k^{-1}\, d_k]^q\Big\}^{1/q},$$

which shows that

$$J_{0.2} \ge \sup_{b\in B_{0.1}} \Big\{\sum_k [b_{k+1}\mu_{k+1}^{-1}\, d_k]^q\Big\}^{1/q} \Big\{\sum_k [b_k\rho_{\beta,p}(\mu_k)]^p\Big\}^{-1/p} \equiv E_{0.2}.$$

If

$$E_{1.2} \equiv \sup_{a_k\ge 0} \Big\{\sum_k [a_{k+1}\mu_{k+1}^{-1}\, d_k]^q\Big\}^{1/q} \Big\{\sum_k [a_k\rho_{\beta,p}(\mu_k)]^p\Big\}^{-1/p}$$

then obviously $E_{0.2} \le E_{1.2}$. To obtain a reverse estimate let $b_k \equiv \mu_k \sum_{m\in\mathbf{Z}} a_m/(\mu_k+\mu_m)$. Then $b_k \uparrow$ and $b_k \approx \sum_{m\le k} a_m + \mu_k \sum_{m>k} a_m\mu_m^{-1}$, and since the lower bound in this equivalence is $1/2$ it follows that $b_k \ge a_k/2$, $k \in \mathbf{Z}$. Moreover $\mu_k^{-1} b_k \downarrow$ so that $b = \{b_k\} \in B_{0.1}$. Recall that $\rho_{\beta,p}(\mu_k) \uparrow\uparrow$ and $\mu_k\rho_{\beta,p}(\mu_k) \downarrow\downarrow$ so by Proposition 2.1

$$\begin{aligned}\Big\{\sum_k [b_k\rho_{\beta,p}(\mu_k)]^p\Big\}^{1/p} &\ll \Big\{\sum_k \rho_{\beta,p}(\mu_k)\Big[\sum_{m\le k} a_m\Big]^p\Big\}^{1/p} \\ &\quad + \Big\{\sum_k [\mu_k\rho_{\beta,p}(\mu_k)]^p\Big[\sum_{m>k} a_m\mu_m^{-1}\Big]^p\Big\}^{1/p} \\ &\ll \Big\{\sum_k [a_k\rho_{\beta,p}(\mu_k)]^p\Big\}^{1/p}.\end{aligned}$$

and therefore $J_{0.2} \ge E_{0.2} \gg E_{1.2}$. But by the usual reverse Hölder's inequality and the fact that $\mu_{k+1}\rho_{\beta,p}(\mu_{k+1}) = a\mu_k\rho_{\beta,p}(\mu_k)$ if $k \in \mathbf{Z}_2$ we obtain that

$$\begin{aligned}E_{1.2} &= \sup_{a_k\ge 0} \frac{\{\sum_k [a_{k+1}\mu_{k+1}^{-1}\, d_k]^q\}^{1/q}}{\{\sum_k [a_k\rho_{\beta,p}(\mu_k)]^p\}^{1/p}} = \Big\{\sum_{k\in\mathbf{Z}} \Big[\frac{d_k}{\mu_{k+1}\rho_{\beta,p}(\mu_k)}\Big]^r\Big\}^{1/r} \\ &\gg \Bigg\{\sum_{k\in\mathbf{Z}_2} \Big\{\int_{\mu_k}^{\mu_{k+1}} \frac{d\gamma_0(s)}{\rho_{\beta,p}^q(s)}\Big\}^{r/q}\Bigg\}^{1/r} = E_2.\end{aligned}$$

Similarly, using the fact that $\rho_{\beta,p}(\mu_k) = a\rho_{\beta,p}(\mu_{k+1})$ if $k \in \mathbf{Z}_1$, one obtains that $J_{0.1} \gg E_1$. It follows that $J_0 \gg E_i$, $i = 1, 2$ and the lower bound for (2.10) follows.

To complete the proof of the theorem it remains to show that the nondegeneracy assumption on the measure $d\gamma$ can be removed.

Suppose that

$$\int_0^\infty \Big(\frac{s}{s+1}\Big)^q d\gamma(s) = \infty$$

then the function $f(s) = \frac{s}{s+1}$, is in $\Omega_{0.1}$ and it is easily seen that J and, consequently, E of (2.10) are not finite. Therefore the estimate (2.10) is satisfied in this case. Similarly $\mathcal{E} = \infty$ so that (2.11) also holds.

Now let

$$\int_0^\infty \Big(\frac{s}{s+1}\Big)^q d\gamma(s) < \infty, \quad \int_0^1 d\gamma(s) < \infty \text{ or } \int_0^\infty s^q\, d\gamma(s) < \infty.$$

If $0 < p \leq q < \infty$, then $f_t(s) \equiv s/(s+t)$, $t > 0$, is in $\Omega_{0.1}$, so that $J \geq \mathcal{E}$ of (2.11) is always satisfied. We need to show only that $J \ll \mathcal{E}$. To prove this, we assume $\mathcal{E} < \infty$, for otherwise there is nothing to prove, and define $d\gamma_\varepsilon$, $\varepsilon > 0$, by $d\gamma_\varepsilon(s) = d\gamma(s) + \varepsilon\, d\beta(s)$, where $d\beta(s)$ satisfies (2.1). It follows then that $d\gamma_\varepsilon$ satisfies the non-degeneracy condition (2.1) with q instead of p. Moreover, $\rho_{\beta,q}(t) \leq \rho_{\beta,p}(t)$, $t > 0$, when $p \leq q$ and therefore by what we have proved

$$J \equiv \sup_{f\in\Omega_{0.1}} \frac{\{\int_0^\infty f^q\, d\gamma\}^{1/q}}{\{\int_0^\infty f^p\, d\beta\}^{1/p}} \leq \sup_{f\in\Omega_{0.1}} \frac{\{\int_0^\infty f^q\, d\gamma_\varepsilon\}^{1/q}}{\{\int_0^\infty f^p\, d\beta\}^{1/p}} \ll \sup_{t>0}[\rho_{\gamma_\varepsilon,q}(t)/\rho_{\beta,p}(t)] \leq \mathcal{E} + \varepsilon$$

and the required part of (2.11) easily follows as $\varepsilon \longrightarrow 0$.

If $0 < q < p < \infty$ we define $d\gamma_\varepsilon$ by

$$d\gamma_\varepsilon(s) = d\gamma(s) + \varepsilon\alpha(s)\, d\beta(s),$$

where $\alpha \geq 0$ is to be defined. If J and E are defined as in (2.10) then J_ε and E_ε are defined by (2.10) with $d\gamma$ replaced by $d\gamma_\varepsilon$ and $J \leq J_\varepsilon$, $E \leq E_\varepsilon$ always holds. We wish to find α so that $d\gamma_\varepsilon$ satisfies the non-degeneracy conditions.

By Hölder's inequality with exponents p/q, r/q

$$\Big(\int_0^\infty f^q \alpha\, d\beta\Big)^{1/q} \leq \Big(\int_0^\infty \alpha^{r/q}\, d\beta\Big)^{1/r}\Big(\int_0^\infty f^p\, d\beta\Big)^{1/p},$$

so that α should satisfy

(i) $\int_0^\infty \alpha^{r/q}\, d\beta < \infty$, for then

$$\begin{aligned}&\int_0^\infty \Big(\frac{s}{s+1}\Big)^q d\gamma_\varepsilon(s)\\ &\quad\leq \int_0^\infty \Big(\frac{s}{s+1}\Big)^q d\gamma(s) + \varepsilon\Big\{\int_0^\infty \Big(\frac{s}{s+1}\Big)^p d\beta(s)\Big\}^{1/p}\Big\{\int_0^\infty \alpha^{r/q}\, d\beta\Big\}^{1/r}\end{aligned}$$

is finite. Since we require that

$$\infty = \int_0^1 d\gamma_\varepsilon(s) = \int_0^1 d\gamma(s) + \varepsilon\int_0^1 \alpha(s)\, d\beta(s)$$

and

$$\infty = \int_1^\infty s^q\, d\gamma_\varepsilon(s) = \int_1^\infty s^q\, d\gamma(s) + \varepsilon\int_1^\infty s^q\alpha(s)\, d\beta(s)$$

the function α must also satisfy

(ii) $\int_0^1 \alpha\, d\beta = \infty$ and (iii) $\int_0^\infty s^q\alpha(s)\, d\beta(s) = \infty$.

To this end we construct $\alpha(s)$ on the intervals $[0, 1]$ and $(1, \infty)$ separately. First consider $[0, 1]$ and use that $\int_0^1 d\beta(s) = \infty$, but $\int_0^1 s^p\, d\beta(s) < \infty$.

Let

$$\xi = \inf\Big\{\nu \in (0, p] : \int_0^1 s^\nu\, d\beta(s) < \infty\Big\} \quad \text{and} \quad \tilde{\xi} \in \big(\xi, \min(p, \xi r/q)\big),$$

then define for $s \in [0,1]$, $\alpha(s) = s^{\tilde{\xi}}q/r$. Clearly,

$$\int_0^1 \alpha^{r/q}\,d\beta = \int_0^1 s^{\tilde{\xi}}\,d\beta(s) < \infty$$

and

$$\int_0^1 \alpha\,d\beta = \int_0^1 s^{\tilde{\xi}_{q/r}}\,d\beta(s) = \infty, \quad \text{since } \tilde{\xi}q/r < \xi.$$

Next, assume that $\int_1^\infty d\beta < \infty$ but $\int_1^\infty s^p\,d\beta(s) = \infty$.
Let

$$\zeta = \sup\Big\{\nu \in (0,p) : \int_1^\infty s^{\nu}\,d\beta(s) < \infty\Big\} \quad \text{and} \quad \tilde{\zeta} \in \Big(\Big(\frac{\zeta}{q} - 1\Big), \zeta\Big),$$

then define for $s \in [1,\infty)$, $\alpha(s) = s^{\tilde{\zeta}q/r}$. Clearly

$$\int_1^\infty \alpha^{r/q}\,d\beta = \int_1^\infty s^{\tilde{\zeta}}\,d\beta(s) < \infty$$

and

$$\int_1^\infty s^q\alpha(s)\,d\beta(s) = \int_1^\infty s^{q+\tilde{\zeta}q/r}\,d\beta(s) = \infty$$

since $q+\tilde{\zeta}q/r > \zeta$. Hence the α so defined satisfies (i), (ii) and (iii) and $d\gamma_\varepsilon$ satisfies the nondegeneracy conditions. By what we have proved, $J_\varepsilon \approx E_\varepsilon$ where

$$J_\varepsilon = \sup_{f\in\Omega_{0,1}} \frac{\{\int_0^\infty f^q\,d\gamma + \varepsilon\int_0^\infty f^q\alpha\,d\beta\}^{1/q}}{\{\int_0^\infty f^p\,d\beta\}^{1/p}} \approx J + \varepsilon^{1/q}C < \infty.$$

Therefore, $J+\varepsilon^{1/q}C \approx J_\varepsilon \approx E_\varepsilon \geq E$ and hence $J \gg E$. On the other hand $\lim_{\varepsilon\to 0} E_\varepsilon = E$ and therefore $J + \varepsilon^{1/q}C \ll E$ which implies $J \ll E$. This completes the proof of the theorem.

By taking $q = 1$ and $d\beta(s) = s^{-p}v(s)\,ds$, $0 < p < \infty$, where $v \geq 0$ is a weight function satisfying

$$\int_0^\infty \frac{v(s)}{(s+1)^p}\,ds < \infty, \quad \int_0^1 s^{-p}v(s)\,ds = \int_1^\infty v(s)\,ds = \infty, \tag{2.15}$$

in Theorem 2.1, we see that $d\beta$ satisfies the nondegeneracy conditions. Moreover the corresponding fundamental function

$$\rho_{\beta,p}(t) = \Big\{\int_0^\infty \frac{v(s)\,ds}{(s+t)^p}\Big\}^{1/p} \equiv t^{-1}V^{1/p}(t) \tag{2.16}$$

has discretizing sequence $\{\mu_k\}_{k\in\mathbf{Z}}$. Under these notations we obtain a principle of duality formulated by the following reverse Hölder inequality:

THEOREM 2.2. *(a) Let* $1 < p < \infty$ *and* $0 \leq g \downarrow$ *on* $\mathbf{R}^+$. *Suppose* v *satisfies (2.15), then*

$$I \equiv \sup_{0\leq f\downarrow} \frac{\int_0^\infty fg}{\{\int_0^\infty [x^{-1}\int_0^x f]^p v(x)\,dx\}^{1/p}} \approx \Big\{\int_0^\infty \Big[\int_0^x g\Big]^{p'} \mathcal{V}(x)\,dx\Big\}^{1/p'}, \tag{2.17}$$

where

$$\mathcal{V}(x) = \sum_k V^{-p'/p}(x)\delta_{\mu_k}(x). \tag{2.18}$$

Here V is defined by (2.16) and δ_{μ_k} is the Dirac δ-function.

(b) If $0 < p \leq 1$ and $g \geq 0$, then

$$I \approx \sup_{t>0} \Big(\int_0^t g\Big) V^{-1/p}(t). \tag{2.19}$$

PROOF. (a) We may assume that $\lim_{x\to\infty} g(x) = 0$, for if $\lim_{x\to\infty} g(x) = C > 0$, then with $f = \chi_{[0,t]}$, $t > 0$

$$I \geq \frac{Ct}{t\{t^{-p}\int_0^t v + \int_t^\infty x^{-p}v(x)\,dx\}^{1/p}} \approx \frac{C}{\rho_{\beta,p}(t)}$$

and hence $I \geq C\sup_{t>0} \frac{1}{\rho_{\beta,p}(t)} = \infty$, where $\rho_{\beta,p}$ is given by (2.16).

Similarly, since by Lemma 2.1, $\rho_{\beta,p}(\mu_k) \downarrow\downarrow$ the right side of (2.17) is

$$\begin{aligned}
C\Big\{\int_0^\infty x^{p'} \sum_k V^{-p'/p}(x)\delta_{\mu_k}(x)\Big\}^{1/p'} &= C\Big\{\sum_k [\mu_k^{-1} V^{1/p}(\mu_k)]^{-p'}\Big\}^{1/p'} \\
&= C\Big\{\sum_k \rho_{\beta,p}^{-p'}(\mu_k)\Big\}^{1/p'} = \infty.
\end{aligned}$$

Hence (2.17) holds in this case.

Now it is known[1] (*cf.* [4, p. 117]) that $F \in \Omega_{0,1}$, if and only if, $F(x) \approx \int_0^x f$ with $f\downarrow$. Hence the left side of (2.17) becomes

$$I \approx \sup_{F\in\Omega_{0,1}} \frac{\int_0^\infty F(-dg)}{\{\int_0^\infty F(x)x^{-p}v(x)\,dx\}^{1/p}}.$$

By Theorem 2.1 with $q = 1$, $d\gamma = -dg$ and $d\beta(x) = x^{-p}v(x)\,dx$, this is equivalent to

$$\begin{aligned}
\Big\{\sum_k [\rho_{\gamma,1}(\mu_k)/\rho_{\beta,p}(\mu_k)]^{p'}\Big\}^{1/p'} &= \Big\{\sum_k \Big[\mu_k^{-1}\int_0^{\mu_k} g\Big]^{p'} [\mu_k^{-1} V^{1/p}(\mu_k)]^{-p'}\Big\}^{1/p'} \\
&= \Big\{\int_0^\infty \Big[\int_0^x g\Big]^{p'} \mathcal{V}(x)\Big\}^{1/p'}.
\end{aligned}$$

Note that in this case

$$\begin{aligned}
\rho_{\gamma,1}(\mu_k) &\approx \mu_k^{-1}\int_0^{\mu_k} -s\,dg(s) + \int_{\mu_k}^\infty -dg(s) \\
&= \mu_k^{-1}\Big[-sg(s)\big|_0^{\mu_k} + \int_0^{\mu_k} g(s)\,ds\Big] + g(\mu_k) = \mu_k^{-1}\int_0^{\mu_k} g.
\end{aligned}$$

Hence (2.17) follows.

Part (b) of the theorem follows from the proof of [25, Theorem 3.3].

[1] We are grateful to Professor L.-E. Persson for drawing our attention to this fact and for the discussion we had on this topic.

COROLLARY 2.1. *Suppose* $0 < p < \infty$, $g \geq 0$, $v \geq 0$ *measurable on* $\mathbf{R}^+$ *such that*

$$\int_0^\infty \Big(\frac{s}{s+1}\Big)^p v(s)\,ds < \infty, \quad \int_0^1 v(s)\,ds = \int_1^\infty s^p v(s)\,ds = \infty.$$

If $1 < p < \infty$ *and* $\mathcal{V} = \sum_k \Big[\int_0^\infty \min(s^p, x^p) v(s)\,ds\Big]^{-p/p'} \delta_{\mu_k}(x)$ *with* $\{\mu_k\}$ *the discretizing sequence for* $\rho_{\beta,p}(t)$ *with* $d\beta(s) = v(s)\,ds$, *then*

$$\sup_{f\in\Omega_{0,1}} \frac{\int_0^\infty fg}{\{\int_0^\infty f^p v\}^{1/p}} \approx \Big\{\int_0^\infty \Big[\int_0^\infty \min(s,x)g(s)\,ds\Big]^{p'} \mathcal{V}(x)\,dx\Big\}^{1/p'}. \tag{2.20}$$

If $0 < p \leq 1$, *then*

$$\sup_{f\in\Omega_{0,1}} \frac{\int_0^\infty fg}{\{\int_0^\infty f^p v\}^{1/p}} \approx \sup_{t>0} \frac{\int_0^\infty \min(t,s)g(s)}{\{\int_0^\infty \min(t^p, s^p)v(s)\,ds\}^{1/p}}.$$

PROOF. Let $q = 1$, $d\gamma(s) = g(s)\,ds$, $d\beta(s) = v(s)\,ds$ in Theorem 2.1, then the result follows since

$$\rho_{\beta,p}(t) \approx t^{-1}\Big\{\min \int_0^\infty (s^p, t^p)v(s)\,ds\Big\}^{1/p}$$

and

$$\rho_{g,1}(t) \approx t^{-1} \int_0^\infty \min(s,t)g(s)\,ds.$$

Observe that if T is an integral operator defined on $\Omega_{0,1}$ then the characterization of weight functions u, v for which $\|Tf\|_{q,u} \leq C\|f\|_{p,v}$, $1 < p, q < \infty$, holds is equivalent by (2.20) to the problem of characterizing the weight functions $\mathcal{V}$ and u for which

$$\Big\{\int_0^\infty \Big[\int_0^\infty \min(s,x)(T^*g)(s)\,ds\Big]^{p'} \mathcal{V}(x)\,dx\Big\}^{1/p'} \leq C\Big\{\int_0^\infty g^{q'} u^{1-q'}\Big\}^{1/q'}$$

holds. Here T^* denotes the adjoint of T and $g \geq 0$, arbitrary. This problem is in some instances easier to solve, specifically if one permits additional conditions on $\rho_{g,1}$ (*cf.* [9, Remark 4]).

3. **Applications.** We now make use of the results of Section 2 to study mapping properties of the identity operator, the Hardy-Littlewood maximal operator and the Hilbert transform on weighted Γ-Lorentz spaces. Specifically we extend to Γ-spaces some of the results proved in [17] for weighted Λ-spaces.

We assume throughout this section that the weight function $v(x) \geq 0$ satisfies the nondegeneracy condition (2.15) and that V and $\mathcal{V}$ are defined by

$$V(t) = t^p \int_0^\infty \frac{v(s)}{(s+t)^p}\,ds, \quad \mathcal{V} = \sum_k V(t)^{-p/p'} \delta\mu_k(t) \tag{3.1}$$

respectively (*cf.* (2.16), (2.18)).

Observe that in the assertions below the norm of the dual space $\Gamma_p^*(v)$ with $0 < p < \infty$ is expressed by

$$\|g\|_{\Gamma_p^*(v)} = \sup_{\|f\|_{\Gamma_p(v)}\leq 1} \Big[\int_0^\infty f(x)g(x)\,dx\Big]$$

(see for example [3]).

The following formulation for the weighted dual Γ-space holds:

THEOREM 3.1. *If v satisfies (2.15) and V, $\mathcal{V}$ are given by (3.1) where $\{\mu_k\}$ is the discretizing sequence of $t^{-1}V^{1/p}(t)$, then*

$$\Gamma_p^*(v) = \Gamma_{p'}\big(t^{p'}\mathcal{V}(t)\big), \quad 1 < p < \infty$$

and

$$\Gamma_p^*(v) = \Gamma_\infty\big(tV^{-1/p}(t)\big), \quad 0 < p \le 1.$$

PROOF. Apply [3, Theorem 4.1, Chapter 7], (2.17) and (2.19) of Theorem 2.2.

The weight characterizations for which the identity operator is bounded on weighted Λ-spaces has been given in [2], [17], [24], [25] for various ranges of indices. The extensions of these results to weighted Γ-spaces in [8], [9], [10], [25] follows now directly from Theorem 2.1 in the next result.

As in the case of V in (3.1) or (2.16) we write $d\gamma(s) = s^{-q}w(s)\,ds$, $w \ge 0$, and define

$$\rho_{\gamma.q}(t) = \left\{\int_0^\infty \frac{w(s)}{(s+1)^q}\,ds\right\}^{1/q} \equiv t^{-1}W^{1/q}(t). \tag{3.2}$$

Also the operator norm of the identity operator on Γ is denoted by $\|i\|_{p\to q}$.

THEOREM 3.2. *(a) If $0 < p \le q < \infty$, then $i\colon \Gamma_p(v) \longrightarrow \Gamma_q(w)$ is bounded, if and only if*

$$\|i\|_{p\to q} \approx \mathcal{E}_{p.q} \equiv sup_{t>0}\frac{W^{1/q}(t)}{V^{1/p}(t)} < \infty.$$

(b) If $0 < q < p < \infty$, $1/r = 1/q - 1/p$, then

$$\|i\|_{p\to q} \approx E_{p.q} \equiv \left\{\sum_k [\rho_{\gamma.q}(\mu_k)/\rho_{\beta.p}(\mu_k)]^r\right\}^{1/r} < \infty,$$

where $\{\mu_k\}_{k\in\mathbf{Z}}$ is the discretizing sequence for $\rho_{\beta.p}$.

PROOF. Since $f \in \Omega_{0.1}$ has the form $f(x) = \int_0^x g$, where $g\downarrow$ (*cf.* [4, p. 117]), the results follow from (2.11) and (2.10) of Theorem 2.1.

Now we consider mapping properties of the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x\in Q\subset\mathbf{R}^n} \frac{1}{|Q|}\int_Q |f(y)|\,dy \tag{3.3}$$

on weighted Λ-spaces. Here Q are cubes in $\mathbf{R}^n$ with sides parallel to the coordinate axes.

As usual g^* denotes the rearrangement of g and

$$g^{**}(x) \equiv x^{-1}\int_0^x g^*(t)\,dt.$$

Hence if

$$Pf(x) = x^{-1}\int_0^x f(t)\,dt, \quad P^2f(x) = P(Pf)(x),$$

then $Pf^*(x) = f^{**}(x)$. Since f^* is decreasing it is clear that $Mf^*(x) = (Pf^*)(x)$ and since a result of C. Herz shows that $(Pf^*)(x) \leq c(Mf)^*(x)$ (*cf.* [3, Theorem 3.8, p. 122]) it follows that

$$(Mf)^*(x) \approx (Mf^*)(x) \quad \text{and} \quad (Mf)^{**}(x) \approx (P^2f^*)(x). \tag{3.4}$$

In order to establish boundedness of M on weighted Γ-spaces, it suffices therefore to establish corresponding results for the Hardy operator on decreasing functions, indeed it is equivalent to obtain embeddings for $i\colon \Gamma_p(v) \longrightarrow \Gamma_q(v)$.

THEOREM 3.3. *Let $0 < p,\ q < \infty,\ q > 1$, and $v(x) \geq 0,\ w(x) \geq 0$ be locally integrable functions with v satisfying the nondegeneracy conditions*

$$\int_0^\infty \frac{v(s)\,ds}{(s+1)^p} < \infty, \quad \int_0^1 s^{-p} v(s)\,ds = \int_1^\infty v(s)\,ds = \infty.$$

Then

(a) The inequality

$$\|Mf\|^{**}_{q,w} \leq C\|f\|^{**}_{p,v}, \quad f \in \Gamma_p(v) \tag{3.5}$$

is satisfied for $1 < p \leq q < \infty$, if and only if $A \equiv \max_{i=0,1,2,3} A_i < \infty$, where

$$A_0 = \sup_{t>0}\Big\{\int_0^t w\Big\}^{1/q}\Big\{\int_t^\infty \mathcal{V}\Big\}^{1/p'}, \quad A_1 = \sup_{t>0}\Big\{\int_0^t s^{-q}w(s)\,ds\Big\}^{1/q}\Big\{\int_t^\infty s^{p'}\mathcal{V}(s)\,ds\Big\}^{1/p'},$$

$$A_2 = \sup_{t>0}\Big\{\int_t^\infty s^{-q}\ln^q(s/t)w(s)\,ds\Big\}^{1/q}\Big\{\int_t^\infty s^{p'}\mathcal{V}(s)\,ds\Big\}^{1/p'}$$

$$A_3 = \sup_{t>0}\Big\{\int_t^\infty s^{-q}w(s)\,ds\Big\}^{1/q}\Big\{\int_t^\infty s^{p'}\ln^{p'}(t/s)\mathcal{V}(s)\,ds\Big\}^{1/p'}.$$

Moreover $A \approx C$.

(b) If $1 < q < p < \infty$, $1/r = 1/q - 1/p$, then (3.5) is satisfied, if and only if, $B = \max_{i=0,1,2,3} B_i < \infty$, where

$$B_0 = \Big\{\int_0^\infty \Big(\int_0^t w\Big)^{r/p}\Big(\int_t^\infty \mathcal{V}\Big)^{r/p'} w(t)\,dt\Big\}^{1/r},$$

$$B_1 = \Big\{\int_0^\infty \Big[\int_t^\infty s^{-q}w(s)\,ds\Big]^{r/p}\Big[\int_0^t s^{p'}\mathcal{V}(s)\,ds\Big]^{r/p'} t^{-q}w(t)\,dt\Big\}^{1/r},$$

$$B_2 = \Big\{\int_0^\infty \Big[\int_t^\infty s^{-q}w(s)\,ds\Big]^{r/p}\Big[\int_0^t s^{p'}\ln^{p'}(t/s)\mathcal{V}(s)\,ds\Big]^{r/p'} t^{-q}w(t)\,dt\Big\}^{1/r}$$

$$B_3 = \Big\{\int_0^\infty \Big[\int_t^\infty s^{-q}w(s)\,ds\Big]^{r/q}\Big[\int_0^t s^{p'}\ln^{p'}(t/s)\mathcal{V}(s)\,ds\Big]^{r/q'} t^{p'}\mathcal{V}(t)\,dt\Big\}^{1/r}.$$

Moreover $B \approx C$.

(c) If $0 < p \leq 1 < q < \infty$, then (3.5) is satisfied, if and only if, $D = \max_{i=0,1,2} D_i < \infty$, where

$$D_0 = \sup_{t>0}\Big\{\int_0^t w\Big\}^{1/q} V^{-1/p}(t),$$

$$D_1 = \sup_{t>0} \left\{ \int_t^\infty s^{-q} w(s)\, ds \right\}^{1/q} t V^{-1/p}(t)\, dt,$$

$$D_2 = \sup_{t>0} \left\{ \int_t^\infty s^{-q} \ln^q(s/t) w(s)\, ds \right\}^{1/q} t V^{-1/p}(t),$$

and again $D \approx C$.

Here $\mathcal{V}$ and V are of course defined by (3.1).

PROOF. It follows from (3.4) that (3.5) is equivalent to proving

$$\left\{ \int_0^\infty (P^2 f)^q w \right\}^{1/q} \le C \left\{ \int_0^\infty (Pf)^p v \right\}^{1/p} \quad 0 \le f \downarrow.$$

But by the usual reverse Hölder's inequality, this is equivalent to

$$\sup_{0 \le f \downarrow} \frac{\int_0^\infty (P^2 f) g}{\{\int_0^\infty (Pf)^p v\}^{1/p}} \le C \left\{ \int_0^\infty g^{q'} w^{1-q'} \right\}^{1/q'}, \quad g \ge 0. \tag{3.6}$$

Now

$$\int_0^\infty (P^2 f) g = \int_0^\infty f \tilde{P}^2 g,$$

where the adjoint $\tilde{P}$ of P is $\tilde{P}g(t) = \int_t^\infty g(s)\frac{ds}{s}$ and $\tilde{P}^2 g(t) = \int_t^\infty \ln(s/t) g(s) \frac{ds}{s}$. But clearly, $\tilde{P}^2 g \downarrow$ and therefore by Theorem 2.2(a) and (b), the left side of (3.6) is in case $1 < p < \infty$ equivalent to $\{\int_0^\infty [\int_0^x \tilde{P}^2 g]^{p'} \mathcal{V}(x)\, dx\}^{1/p'}$, and in case $0 < p \le 1$ equivalent to $\sup_{t>0} \{\int_0^t \tilde{P}^2 g\} V^{-1/p}(t)$. Now, a simple calculation shows that

$$\int_0^x \tilde{P}^2 g = x[Pg(x) + \tilde{P}g(x) + \tilde{P}^2 g(x)].$$

Therefore, it suffices to characterize the weight functions for which, in case $1 < p < \infty$, each of the following integrals

i) $\{\int_0^\infty [\int_0^x g]^{p'} \mathcal{V}(x)\, dx\}^{1/p'}$,

ii) $\{\int_0^\infty [\int_x^\infty g(t) \frac{dt}{t}]^{p'} x^{p'} \mathcal{V}(x)\, dx\}^{1/p'}$,

iii) $\{\int_0^\infty [\int_x^\infty \ln(t/x) g(t) \frac{dt}{t}]^{p'} x^{p'} \mathcal{V}(x)\, dx\}^{1/p'}$,

and in case $0 < p \le 1 < q < \infty$ the following suprema

iv) $\sup_{t>0} \left\{\int_0^t g\right\} V^{-1/p}(t)$,

v) $\sup_{t>0} \{\int_t^\infty g(s) \frac{ds}{s}\} t V^{-1/p}(t)$,

vi) $\sup_{t>0} \{\int_t^\infty \ln(s/t) g(s) \frac{ds}{s}\} t V^{-1/p}(t)$

are dominated by

$$\left\{ \int_0^\infty g^{q'} w^{1-q'} \right\}^{1/q}.$$

Here $g \ge 0$ is arbitrary and $1 < q < \infty$. But the characterizations of weights for these Hardy type operators are well known (*cf.* [15], [23]). For if $1 < p \le q < \infty$, then the estimate implied by i) holds, if and only if, $A_0 < \infty$, and the estimate for ii) holds, if and only if $A_1 < \infty$. The inequality for iii) is satisfied (see [23]) if $A_2 + A_3 < \infty$. This proves (a).

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If $1 < q < p < \infty$, $1/r = 1/q - 1/p$ the inequalities for i) and ii) are satisfied, if and only if $B_0 < \infty$, respectively $B_1 < \infty$. The estimate for iii) follows again from [23] and is satisfied if and only if $B_2 + B_3 < \infty$.

The proof of (c) requires the usual reverse Hölder's inequality. In the case iv) the estimate

$$\sup_{t>0}\Big\{\int_0^t g\Big\} V^{-1/p}(t) \le C\Big\{\int_0^\infty g^{q'} w^{1-q'}\Big\}^{1/q'}$$

expresses the boundedness of the operator $g(s) \longrightarrow g(s)\,V^{-1/p}(t)$, from $L^{q'}_{w^{1-q'}}(0,\infty)$ to $L^1(0,t)$, $t > 0$. But this is equivalent to the estimate

$$V^{-1/p}(t)\Big(\int_0^t h^q w\Big)^{1/q} \le C \sup_{0<s<t} h(s),$$

which is obviously satisfied with $C = D_0$. Similarly for (v) and (vi).

Our final result concerns the mapping properties of the Hilbert transform H defined by

$$(\mathrm{Hf})(x) = P.V.\int_{-\infty}^{\infty} \frac{f(y)\,dy}{x-y}$$

on weighted Γ-spaces.

THEOREM 3.4. *Suppose p, q, w and v are as in Theorem 3.3, then the inequality $\|\,\mathrm{Hf}\,\|^{**}_{q,w} \le C\|f\|^{**}_{p,v}$, $f \in \Gamma_p(v)$, is satisfied*

(a) In case $1 < p \le q < \infty$, if and only if, $A_H = \max_{i=0,1,2,3}(A_i, F_0, F_1) < \infty$, where A_i, $i = 0, 1, 2, 3$ are the constants of Theorem 3.3 and

$$F_0 = \sup_{t>0}\Big\{\int_0^t w\Big\}^{1/q}\Big\{\int_t^\infty \ln^{p'}(s/t)\mathcal{V}(s)\,ds\Big\}^{1/p'},$$

$$F_1 = \sup_{t>0}\Big\{\int_0^t \ln^q(t/s) w(s)\,ds\Big\}^{1/q}\Big\{\int_t^\infty \mathcal{V}\Big\}^{1/p'}.$$

Moreover $C \approx A_H$.

(b) For $1 < q < p < \infty$, $1/r = 1/q - 1/p$, the inequality holds, if and only if, $B_H = \max_{i=0,1,2,3}(B_i, F_2, F_3) < \infty$, where B_i, $i = 0, 1, 2, 3$ are given in Theorem 3.3 and

$$F_2 = \Big\{\int_0^\infty \Big(\int_0^t w\Big)^{r/p}\Big(\int_t^\infty \ln^{p'}(s/t)\mathcal{V}(s)\,ds\Big)^{r/p'} w(t)\,dt\Big\}^{1/r}$$

$$F_3 = \Big\{\int_0^\infty \Big(\int_0^t \ln^q(t/s)w(s)\,ds\Big)^{r/q}\Big(\int_t^\infty \mathcal{V}(s)\,ds\Big)^{r/q'} \mathcal{V}(t)\,dt\Big\}^{1/r}.$$

Moreover $C \approx B_H$.

(c) For $0 < p \le 1 < q < \infty$, the inequality holds, if and only if, $D_H = \max_{i=0,1,2,3}(D_i) < \infty$, where D_i, $i = 0, 1, 2$ are given in Theorem 3.3 and

$$D_3 = \sup_{t>0}\Big\{\int_0^t \ln^q(s/t)w(s)\,ds\Big\}^{1/q} V^{1/p}(t).$$

Moreover $C \approx D_H$.

PROOF. By [3; Theorem 4.8, p. 138] and [17] the rearrangement inequality $(\mathrm{Hf})^*(t) \ll (P^*f)(t) + (\tilde{P}f^*)(t) \ll (\mathrm{Hf}^*)(t)$ is satisfied. But since $Pf(t) + \tilde{P}f(t) = P(\tilde{P}f)(t)$ the boundedness of $H: \Gamma_p(v) \longrightarrow \Gamma_q(w)$ is equivalent to the inequality

$$\left\{\int_0^\infty [P^2(\tilde{P}f)]^q w\right\}^{1/q} \leq C\left\{\int_0^\infty [Pf]^p v\right\}^{1/p}$$

for $0 \leq f \downarrow$. By the reverse Hölder inequality and the fact that the adjoint of $P^2\tilde{P}$ is $P\tilde{P}^2$, the inequality is equivalent to

$$\sup_{0\leq f\downarrow} \frac{\int_0^\infty fP(\tilde{P}^2 g)}{\{\int_0^\infty [Pf]^p v\}^{1/p}} \leq C\left\{\int_0^\infty g^{q'} w^{1-q'}\right\}^{1/q'}, \tag{3.7}$$

where $g \geq 0$ is arbitrary. But since $P(\tilde{P}^2 g) \downarrow$ we can apply Theorem 2.2(a) and (b), so that (3.7) is equivalent to

$$\left\{\int_0^\infty \left[\int_0^x P(\tilde{P}^2 g)\right]^{p'} \mathcal{V}(x)\,dx\right\}^{1/p'} \leq C\left\{\int_0^\infty g^{q'} w^{1-q'}\right\}^{1/q'} \tag{3.8}$$

if $1 < p < \infty$, and

$$\sup_{x>0}\left\{\int_0^x P(\tilde{P}^2 g)\right\} V^{-1/p}(x) \leq C\left\{\int_0^\infty g^{q'} w^{1-q'}\right\}^{1/q'}, \tag{3.9}$$

if $0 < p \leq 1$, where $\mathcal{V}$ and V are defined by (3.1). Since

$$\int_0^x P\tilde{P}^2 g = P^2 g(x) + 2\big(\tilde{P}g(x) + Pg(x)\big) + \tilde{P}^2 g(x)$$

we proceed as in the proof of Theorem 3.3: In case $1 < p < \infty$, (3.8) shows that we must characterize the weights for which each of the integrals

i) $\{\int_0^\infty (Pg)^{p'} \mathcal{V}\}^{1/p'}$,

ii) $\{\int_0^\infty (\tilde{P}g)^{p'} \mathcal{V}\}^{1/p'}$

iii) $\{\int_0^\infty [\int_0^x \ln(x/s) g(s)\,ds]^{p'} x^{-p'} \mathcal{V}(x)\,dx\}^{1/p'}$

and

iv) $\{\int_0^\infty [\int_x^\infty \ln(s/x) g(s) \frac{ds}{s}]^{p'} \mathcal{V}(x)\,dx\}^{1/p'}$

is dominated by the right side of (3.8). In case $0 < p \leq 1 < q$ (3.9) applies and we must characterize the weights for which

v) $\sup_{x>0}\{\int_0^x (Pg)\} V^{-1/p}(x)$,

vi) $\sup_{x>0}\{\int_0^x \tilde{P}g\} V^{-1/p}(x)$

vii) $\sup_{x>0}\{\int_0^x [\frac{1}{t} \int_0^t \ln(t/s) g(s)\,ds]\,dt\} V^{-1/p}(x)$

and

viii) $\sup_{x>0}\{\int_0^x [\int_t^\infty \ln(s/t) g(s) \frac{ds}{s}]\,dt\} V^{-1/p}(x)$

are dominated by the right side of (3.9). But since these weight characterizations are known and follow as in the proof of Theorem 3.3, we omit the details.

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