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PARTITIONING INTERVALS, SPHERES AND BALLS INTO CONGRUENT PIECES

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ABSTRACT. We survey results on partitioning some common sets into m congruent pieces, and prove that a ball in $\mathbf{R}^n$ cannot be so partitioned if $2 \le m \le n$.

A recent problem in this journal [1] discussed the question of partitioning an interval (or ball) into two congruent pieces. In fact, this problem has an interesting history which we survey in this note, together with a proof of a conjecture in [1] and some further questions.

If a group G acts on X, let us say that X is **m-divisible** w.r.t. G if X may be partitioned into m pieces which are pairwise congruent via G. For any finite m, it is clear that a half-open interval, or the circumference of a circle, is m-divisible w.r.t. translations, resp. rotations. The first nontrivial result is Vitali's classical construction of a non-Lebesgue measurable set, which shows that a circle is $\aleph_0$-divisible w.r.t. rotations. This result does not easily transfer to a half-open interval because of the necessity of using addition modulo one (in the case of $[0, 1)$). Von Neumann, responding to a question of Steinhaus (see [6, p. 8]) proved in [13] that, nevertheless, all half-open intervals are $\aleph_0$-divisible w.r.t. translations; see [9] for a simplified proof. The chart below summarizes further results related to divisibility of intervals, spheres and balls in $\mathbf{R}^n$ (S^n denotes (the surface of) the unit sphere in $\mathbf{R}^{n+1}$). The result of [1] regarding 2-divisibility of intervals is a special case of work done by Gustin [5]; he not only proved the negative results for open and closed intervals mentioned in the chart, but also showed that each piece in a partition witnessing the finite divisibility of a half-open interval w.r.t. isometries must be a finite union of intervals. Special cases of Gustin's negative results ($m = 2, 3$) were rediscovered by Sierpiński [12, p. 63] and Schinzel (see [9]). Generalizations of these results to the case where m is an infinite cardinal have been provided by Mycielski [9] and Ruziewicz [11] (see [12, p. 64]).

The results on spheres are interesting because they are derived from work (Robinson [10]) aimed at minimizing the number of pieces in a Banach-Tarski

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X	$[a, b)$	$[a, b]$ or (a, b)	S^1	$S^n (n \geq 2)$	closed or open ball in $\mathbf{R}^n$
G	isometries of $\mathbf{R}$	isometries of $\mathbf{R}$	rotations	rotations	isometries of $\mathbf{R}^n$
$2 \leq m < \infty$	YES	NO (Gustin)	YES	$m = 2$: YES if n is odd NO if n is even $m \geq 3$: YES (Robinson)	$2 \leq m \leq n$: NO (see Theorem below) $m > n$: ?
$m = \aleph_0$	YES (von Neumann)		YES (Vitali)	YES (Dekker and de Groot; Mycielski)	?
$\aleph_0 < m < 2^{\aleph_0}$	YES (Mycielski)		YES (Ruziewicz)		?

A summary of results on m-divisibility of intervals, spheres and balls.

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duplication of the sphere. Robinson proved that S^2 is m-divisible using rotations for any finite $m \geq 3$ (2-divisibility of any S^{2k} is impossible because all rotations of such spheres have fixed points), and this result can be combined with the obvious splittings of S^1 to yield the m-divisibility of all higher dimensional spheres when $m \geq 3$. For, given n, we may write $n+1$ as $2j+3k$ and represent a point $P \in S^n$ as $(u_1, v_1, \ldots, u_j, v_j, x_1, y_1, z_1, \ldots, x_k, y_k, z_k)$. Choose partitions of S^1 and S^2 witnessing m-divisibility and consider the first of the pairs (u, v) or triples (x, y, z) that has at least one nonzero coordinate, normalizing it by dividing by its length. Then place P in the ith piece of a partition of S^n if this (normalized) pair or triple lies in the ith piece of the partition of S^1 or S^2. The resulting subsets of S^n are congruent by rotations obtained by pasting together the rotations used to witness the congruences in S^1 or S^2; precisely, use the embedding of $SO_2^j \times SO_3^k$ into SO_{n+1} given by the formation of a direct sum, i.e., use a block diagonal matrix in SO_{n+1}. Note that if n is odd then the antipodal lies in S^n's rotation group, SO_{n+1}, and so S^n is 2-divisible. Extensions of the results on S^2 (and hence also higher dimensional spheres) to the case of infinitely many pieces were given, independently, by Dekker and de Groot [2, 3] and Mycielski [7].

Robinson's result, which is based on the techniques of the Banach-Tarski paradox, uses the Axiom of Choice. Mycielski [8, 9] asked whether the 3-divisibility of S^3 using rotations could be proved without appealing to that axiom. By modern consistency results of Solovay, the necessity of Choice would follow if it could be shown that S^2 cannot be split into 3 rotationally congruent pieces, each of which is Lebesgue measurable, but it is even unknown whether a splitting into three congruent Borel sets is possible.

Finally, we consider the situation for balls in $\mathbf{R}^2$ and beyond. The results of the theorem below for $m=2$ was conjectured by Cater [1]. After I mentioned to R. M. Robinson that this conjecture could be proved by a method which he had used [10, p. 257], he pointed out the proof below, which extends the result to all m satisfying $2 \leq m \leq n$.

THEOREM. *A closed or open ball in* $\mathbf{R}^n$ *is not* m*-divisible w.r.t. isometries if* $2 \leq m \leq n$.

Proof. Consider first the case of a closed ball B, assumed to be centered at the origin. Suppose that B is partitioned into $A_1, \ldots, A_m$ and σ_j is an isometry of $\mathbf{R}^n$ mapping A_1 onto A_j. Let S denote the surface of B. We may assume that $\mathbf{0}$ lies in A_1. Then for $j>1$, $\sigma_j(\mathbf{0}) \neq \mathbf{0}$, so $\sigma_j(B) \neq B$, and it follows that there is a closed hemisphere of S disjoint from $\sigma_j(B)$. Since $A_j \subseteq B \cap \sigma_j(B)$, $A_j \cap S$ lies within an open hemisphere of S. Similarly, $A_j \cap \sigma_j(S)$ lies in an open hemisphere of $\sigma_j(S)$, which implies that $A_1 \cap S$ lies in an open hemisphere of S. Since $A_1 \cup \cdots \cup A_m \supseteq S$, this yields a covering of the surface S by m open hemispheres, which, since $m \leq n$, is impossible. For if one passes a hyperplane through the poles of all the hemispheres then one end of the perpendicular diameter will be at a distance at least 90° from all the poles. Alternatively, that is easily proved by induction, since the boundary of a hemisphere in $\mathbf{R}^n$ is a sphere in $\mathbf{R}^{n-1}$. This same proof works for an open ball provided we let S be, instead of B's surface, the surface of a sphere centered at $\mathbf{0}$ with a radius strictly between $\sqrt{(1-\delta^2)}$ and 1, where δ is the minimum distance from $\mathbf{0}$ to $\sigma_j(\mathbf{0})$, $j=2,\ldots,m$.

The question of m-divisibility of a ball in $\mathbf{R}^n$ when $m>n\geq 2$ remains unresolved. In particular, is a disc in the plane 3-divisible w.r.t. isometries?

M. Edelstein [4] has recently investigated the question of 2-divisibility of the unit ball in Banach spaces, showing that the ball is not 2-divisible provided the Banach space is strictly convex and reflexive.

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