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ASYMPTOTIC EQUIVALENCE OF A LINEAR AND NONLINEAR SYSTEM WITH IMPULSE EFFECT

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1. Introduction

The present paper deals with the problem of asymptotic equivalence of the system with impulse effect

$$\frac{dx}{dt}=Ax+g(t), \quad t\neq t_k; \qquad \Delta x\big|_{t=t_k}=Bx(t_k) \tag{1}$$

and

$$\frac{dy}{dt}=Ay+g(t)+f(t,y), \quad t\neq t_k; \qquad \Delta y\big|_{t=t_k}=By(t_k)+b_k(y(t_k)), \tag{2}$$

where $x,y:I\to R^n$; $g:I\to R^n$; $f:I\times R^n\to R^n$; $b_k:R^n\to R^n$; $I=[0,\infty)$; R^n is the n-dimensional Euclidean space with a norm $|\cdot|$; A and B are constant matrices; the moments $\{t_k\}$ constitute an increasing sequence $0<t_1<\cdots<t_k<\cdots$, $\lim_{k\to\infty}t_k=\infty$.

The systems with impulse effect of type (1) are characterized by the fact that at the moments $\{t_k\}$ under the action of instant effect (impulse), the mapping point (t,x) jumps from the position $(t_k,x(t_k))$ into the position $(t_k,x(t_k)+\Delta x(t_k))$. It is also supposed that at the moments of impulse effect $\{t_k\}$ the solutions of systems (1) and (2) are left continuous, i.e. $x(t_k-0)=x(t_k)$, $\Delta x|_{t=t_k}=x(t_k+0)-x(t_k)$.

We shall make use of the following definition for asymptotic equivalence.

Definition 1. The systems (1) and (2) are said to be *asymptotically equivalent* if there is a one-to-one correspondence between their solutions such that

$$\lim_{t\to\infty}|x(t)-y(t)|=0, \tag{3}$$

for the corresponding solutions $x(t)$ and $y(t)$.

The main theorem of this paper is an analogue of the theorem of Brauer [1] for asymptotic equivalence of systems without impulse effect.

2. Preliminary remarks

Further on the following notation is used: $i(t,s)$—the number of the points t_k inside the interval (t,s); $\|A\|=\sup_{|x|=1}|Ax|$—the norm of the matrix $A=(a_{ij})_1^n$; E – the unit $n\times n$

matrix; 0_m—the zero $m\times m$ matrix; $\operatorname{diag}(A_1, A_2)$—the quasidiagonal $n\times n$ matrix with blocks A_1 and A_2.

In the proof of our main result we shall use the following lemma.

Lemma 1 [2]. *Let the following conditions be fulfilled*:

1. *The function $u:I\to I$ is piecewise continuous on I being left continuous at the points of discontinuity $\{t_k\}$, and $0<t_1<\cdots<t_k<\cdots$, $\lim_{k\to\infty} t_k=\infty$.*

2. *The function $\lambda:I\to R$ is continuous on I and the numbers d_k, $k=0,1,\dots$, are non-negative.*

3. *For $t\in I$ the inequality*

$$u(t)\leqq d_0+\int_0^t \lambda(s)u(s)\,ds+\sum_{0<t_k<t} d_k u(t_k)$$

holds.

Then for $t\in I$ the following inequality holds

$$u(t)\leqq d_0 \prod_{t_0<t_k<t}(1+d_k)\exp\left(\int_0^t \lambda(s)\,ds\right).$$

Denote by (A) the following set of conditions:

A1. All solutions of system (1) are bounded on I.

A2. Constants $Q>0$ and $p>0$ exist such that

$$|i(t_0,t)-p(t-t_0)|\leqq Q, \quad \text{for} \quad 0\leqq t_0\leqq t<\infty.$$

A3. The function $g:I\to R^n$ is continuous on I.

A4. $\det(E+B)\neq 0$ and the matrices A and B commute.

A5. The inverse functions h_k^{-1} of the functions $h_k:R^n\to R^n$, $h_k(y)=y+By+\mathscr{C}_k(y)$, $k=1,2,\dots$, exist.

A6. The functions $f:I\times R^n\to R^n$ and $b_k:R^n\to R^n$, $k=1,2,\dots$ are continuous on their domains and a non-negative continuous function $\lambda:I\to I$ and non-negative constants β_k, $k=1,2,\dots$ exist, such that

$$|f(t,y)|\leqq \lambda(t)|y|, \quad \text{for} \quad t\in I \quad \text{and} \quad y\in R^n, \tag{4}$$

$$|f(t,y)-f(t,z)|\leqq \lambda(t)|y-z| \quad \text{for} \quad t\in I \quad \text{and} \quad y,z\in R^n, \tag{5}$$

$$|b_k(y)|\leqq \beta_k|y|, \quad \text{for} \quad y\in R^n, \quad k=1,2,\dots, \tag{6}$$

$$|b_k(y)-b_k(z)|\leqq \beta_k|y-z|, \quad \text{for} \quad y,z\in R^n, \quad k=1,2,\dots, \tag{7}$$

$$\int_0^\infty \lambda(s)\,ds+\sum_{k=1}^\infty \beta_k\leqq L<\infty. \tag{8}$$

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3. Main results

Theorem 1. *Let the conditions* (A) *be fulfilled. Then the systems* (1) *and* (2) *with impulse effect are asymptotically equivalent.*

Proof. Let $x(t)=x(t;t_0,x_0)$ be a solution of (1) such that $x(t_0+0)=x_0$, and $y(t)=y(t;t_0,x_0)$—a solution of (2) such that $y(t_0+0)=x_0$.

We are going to show that for t_0 sufficiently large, there exists a one-to-one correspondence between the initial states x_0 and y_0 which in turn generates a one-to-one correspondence between the solutions $x(t)$ and $y(t)$ of systems (1) and (2). Further on we shall demonstrate that for each two corresponding solutions the relation (3) holds.

The general solution X of the linear non-homogeneous system (1) has the form

$$X=\eta+Z, \tag{9}$$

where η is any solution of (1) and Z is the general solution of the linear homogeneous system

$$\frac{dz}{dt}=Az, \quad t\neq t_k; \quad \Delta z|_{t=t_k}=Bz(t_k). \tag{10}$$

It follows from A1 and (9) that all solutions of (10) are bounded. Since the matrices A and B commute, then according to [2] the solution $z(t)=z(t;t_0,x_0)$ of (10) has the form

$$z(t)=(E+B)^{i(t_0,t)}e^{A(t-t_0)}, \quad \text{for} \quad t>t_0,$$

or

$$z(t)=(E+B)^{i(t_0,t)-p(t-t_0)}e^{\Lambda(t-t_0)}z_0, \quad \text{for} \quad t>t_0, \tag{11}$$

where $\Lambda=A+p\ln(E+B)$.

Bearing in mind the boundedness of the solution $z(t)$ and condition A2, it follows from (11) that the matrix $e^{\Lambda(t-t_0)}$ is bounded for $0\leqq t_0\leqq t<\infty$. Hence the matrix Λ has the structure $\Lambda=S^{-1}\operatorname{diag}(\Lambda_-,\Lambda_0)S$, where Λ_- is a $q\times q$ Jordan matrix whose eigenvalues λ_i have negative real parts $\operatorname{Re}\lambda_i<-\alpha<0$, $i=1,\dots,q$; Λ_0 is a $r\times r$ Jordan matrix whose eigenvalues μ_i have zero real parts and simple elementary divisors, $\operatorname{Re}\mu_j=0$, $j=1,\dots,r$; $q+r=n$; $\det S\neq 0$.

Introduce the matrix functions

$$G(t,s)=\begin{cases}(E+B)^{i(s,t)-p(t-s)}S^{-1}\operatorname{diag}(e^{\Lambda_-(t-s)},e^{\Lambda_0(t-s)})S, & t>s;\\ (E+B)^{-i(t,s)+p(s-t)}S^{-1}\operatorname{diag}(e^{\Lambda_-(t-s)},e^{\Lambda_0(t-s)})S, & t\leqq s;\end{cases}$$

$$G_-(t,s)=\begin{cases}(E+B)^{i(s,t)-p(t-s)}S^{-1}\operatorname{diag}(e^{\Lambda_-(t-s)},O_r)S, & t>s;\\ (E+B)^{-i(t,s)+p(s-t)}S^{-1}\operatorname{diag}(e^{\Lambda_-(t-s)},O_r)S, & t\leqq s;\end{cases} \tag{12}$$

$$G_0(t,s)=\begin{cases}(E+B)^{i(s,t)-p(t-s)}S^{-1}\operatorname{diag}(O_q,e^{\Lambda_0(t-s)})S, & t>s;\\ (E+B)^{-i(t,s)+p(s-t)}S^{-1}\operatorname{diag}O_q,e^{\Lambda_0(t-s)})S, & t\leqq s.\end{cases} \tag{13}$$

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An immediate verification shows that

$$G(t,s)=G_-(t,s)+G_0(t,s), \tag{14}$$

$$G(t,t)=E, \quad \text{for} \quad t\in I, \tag{15}$$

$$G(t_k+0,t_k)=E, \tag{16}$$

$$G(t_k+0,s)-G(t_k,s)=BG(t_k,s), \quad \text{for} \quad s<t_k, \tag{17}$$

$$\frac{\partial U}{\partial t}=AU \quad \text{for} \quad t\neq t_k, \tag{18}$$

where U is any of the matrices G, G_- or G_0.

Then the solution $x(t)$ of (1) is of the form

$$x(t)=G(t,t_0)x_0+\int_{t_0}^{t} G(t,s)g(s)\,ds. \tag{19}$$

Using (15)–(18) we see that the solution $y(t)$ of (2) satisfies, for $t>t_0$, the equation

$$y(t)=G(t,t_0)y_0+\int_{t_0}^{t} G(t,s)g(s)\,ds+\int_{t_0}^{t} G(t,s)f(s,y(s))\,ds+\sum_{t_0<t_k<t} G(t,t_k)b_k(y(t_k)). \tag{20}$$

Bearing in mind the structure of the matrix Λ, A2, (12) and (13), the following estimates for the matrices G_-, G_0 and G can be obtained

$$\|G_-(t,s)\|\leqq a\exp(-\alpha(t-s)), \quad \text{for} \quad 0\leqq s\leqq t<\infty, \tag{21}$$

$$\|G_0(t,s)\|\leqq a, \quad \text{for} \quad t\in I, s\in I, \tag{22}$$

$$\|G(t,s)\|\leqq a, \quad \text{for} \quad 0\leqq s\leqq t<\infty, \tag{23}$$

where the constant $a>0$ does not depend on s and t.

The inequalities (4), (6), (21) and (23) together with (19), (20) yield the estimate

$$|y(t)-x(t)|\leqq a|y_0-x_0|+\int_{t_0}^{t} a\lambda(s)|y(s)|\,ds+\sum_{t_0<t_k<t} a\beta_k|y(t_k)|. \tag{24}$$

Let $c=\sup_{t\in I}|x(t)|$. Then it follows from (23)

$$|y(t)|\leqq c+a|y_0-x_0|+\int_{t_0}^{t} a\lambda(s)|y(s)|\,ds+\sum_{t_0<t_k<t} a\beta_k|y(t_k)|. \tag{25}$$

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Applying Lemma 1 to (25) one obtains

$$
\begin{aligned}
|y(t)| &\leqq (c+a|y_0-x_0|)\exp\left(\int_{t_0}^{t} a\lambda(s)\,ds\right)\prod_{t_0<t_k<t}(1+a\beta_k)\\
&=(c+a|y_0-x_0|)\exp\left(\int_{t_0}^{t} a\lambda(s)\,ds+\sum_{t_0<t_k<t}\ln(1+a\beta_k)\right)\\
&\leqq (c+a|y_0-x_0|)\exp\left(a\left(\int_{t_0}^{t}\lambda(s)\,ds+\sum_{t_0<t_k<t}\beta_k\right)\right)\\
&\leqq (c+a|y_0-x_0|)\exp(aL)<\infty.
\end{aligned}
$$

Therefore, each solution of (2) is bounded and, in view of A3, A5 and A6, is defined on I.

If $\bar{y}(t)=y(t;t_0,\bar{y}_0)$ is a solution of (2), then similar arguments yield the estimate

$$|y(t)-\bar{y}(t)|\leqq M|y_0-\bar{y}_0| \qquad (M=a\exp(aL)). \tag{26}$$

It is easy to see that the following relations hold:

$$G_0(t,s)=G(t,t_0)G_0(t_0,s), \quad \text{for} \quad t>t_0, s>t_0, t\neq t_k, s\neq t_k, \tag{27}$$

$$G_0(t,t_k)=G(t,t_0)(E+B)^{\omega}G_0(t_0,t_k), \quad \text{for} \quad t_0\in I \quad \text{and} \quad k=1,2,\dots, \tag{28}$$

where ω equals -1, 1 or 0 depending on the mutual deployment of t_0, t_k and t.

It follows from (22) that the matrix

$$F(t_0,t_k)=(E+B)^{\omega}G_0(t_0,t_k) \tag{29}$$

can be estimated as

$$\|F(t_0,t_k)\|\leqq N, \tag{30}$$

where the constant $N>0$ does not depend on t_0 and t_k.

Introduce the mapping

$$y_0\mapsto x_0, \quad x_0=y_0+S_{t_0}(y_0), \tag{31}$$

where

$$S_{t_0}(y_0)=\int_{t_0}^{\infty} G_0(t_0,s)f(s,y(s))\,ds+\sum_{t_0<t_k} F(t_0,t_k)b_k(y(t_k)). \tag{32}$$

Now from the boundedness of $y(t)$, the estimates (23), (30) and conditions (4), (6) and (8) one is able to conclude that for each fixed $t_0 \in I$ the mapping $S_{t_0}(y_0)$ is defined for all $y_0 \in R^n$.

Let $x_0 \in R^n$ be fixed and consider the mapping

$$U_{t_0}: R^n \to R^n, \quad U_{t_0}(y_0) = y_0 - S_{t_0}(y_0).$$

Using conditions (5), (7) and the estimates (23), (30) and (26) we obtain

$$|U_{t_0}(y_0) - U_{t_0}(\bar{y}_0)| \leqq \left(\int_{t_0}^{t} aM\lambda(s)\,ds + NM \sum_{t_0 < t_k} \beta_k \right) |y_0 - \bar{y}_0|. \tag{33}$$

It follows from (33) and (8) that for t_0 sufficiently large the mapping U_{t_0} is contractive and has a unique fixed point $y_0 \in R^n$, $U_{t_0}(y_0) = y_0$, i.e. the mapping (31) is one-to-one. Let us fix such a t_0.

Since the solutions of (1) and (2) are uniquely determined by the initial conditions x_0 and y_0, then the mapping (31) generates a one-to-one correspondence between the solutions $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; t_0, y_0)$ of these systems.

Now (19), (20), (31), (14), (27)–(29) and (32) yield the following relation between two corresponding solutions

$$\begin{aligned} y(t) - x(t) = {} & \int_{t_0}^{t} G_-(t,s) f(s, y(s))\,ds + \sum_{t_0 < t_k < t} G_-(t, t_k) b_k(y(t_k)) \\ & - \int_{t}^{\infty} G_0(t,s) f(s, y(s))\,ds - \sum_{t \leqq t_k} G_0(t, t_k) b_k(y(t_k)). \end{aligned} \tag{34}$$

Let $K = \sup_{t \in I} |y(t)|$. Then, using (34), (21), (22), (4) and (6) one obtains the estimate

$$\begin{aligned} |y(t) - x(t)| \leqq Ka \Bigg(& \int_{t_0}^{t} \exp(-\alpha(t-s))\lambda(s)\,ds + \sum_{t_0 < t_k < t} \exp(-\alpha(t-t_k))\beta_k \\ & + \int_{t}^{\infty} \lambda(s)\,ds + \sum_{t \leqq t_k} \beta_k \Bigg). \end{aligned} \tag{35}$$

It follows from (8) that

$$\lim_{t \to \infty} \left(\int_{t}^{\infty} \lambda(s)\,ds + \sum_{t \leqq t_k} \beta_k \right) = 0. \tag{36}$$

Let $t > 2t_0$. Then

$$\begin{aligned} & \int_{t_0}^{t} \exp(-\alpha(t-s))\lambda(s)\,ds + \sum_{t_0 < t_k < t} \exp(-\alpha(t-t_k))\beta_k \\ & \leqq \exp\left(-\frac{\alpha t}{2}\right) \left(\int_{0}^{\infty} \lambda(s)\,ds + \sum_{k=1}^{\infty} \beta_k \right) + \int_{t/2}^{\infty} \lambda(s)\,ds + \sum_{t/2 \leqq t_k} \beta_k. \end{aligned} \tag{37}$$

With (36) in mind the estimates (35) and (37) yield $\lim_{t\to\infty}|y(t)-x(t)|=0$.
Thus, Theorem 1 is proved.

Remark 1. In the case when the system (2) is linear ($f(t,y)=P(t)y, b_k(y)=P_k y$; $P(t)$ and $P_k, k=1,2,\dots$ are $n\times n$ matrices) Theorem 1 remains valid if A1–A4 hold and A5, A6 are replaced by

A5′. $\det(E+B+P_k)\neq 0, \quad k=1,2,\dots$.
A6′. The matrix function $P(t)$ is continuous on I and

$$\int_0^\infty \|P(t)\|\,dt+\sum_{k=1}^\infty \|P_k\|<\infty.$$

Remark 2. The analysis of the proof of Theorem 1 shows that the Lipschitz conditions (5) and (7) are used only when establishing the uniqueness and continuity of the solutions of (2) as well as the existence of the inverse mapping of (31). These conditions together with (8) restrict the application of Theorem 1. If one omits the invertibility of the mapping (31), then the following result takes place.

Theorem 2. *Let the following conditions be fulfilled*:

1. *The conditions* A1–A5 *hold.*

2. *The functions $f(t,y)$ and $b_k(y)$, $k=1,2,\dots$ are continuous on their domains and there exist a non-negative continuous function $\lambda: I\to I$ and constants $\beta_k\geqq 0$, $k=1,2,\dots$, such that*

$$|f(t,y)|\leqq\lambda(t)|y|, \quad \textit{for} \quad t\in I \quad \textit{and} \quad y\in R^n,$$

$$|b_k(y)|\leqq\beta_k|y|, \quad \textit{for} \quad y\in R^n,$$

$$\int_0^\infty \lambda(t)\,dt+\sum_{k=1}^\infty \beta_k<\infty.$$

3. *The function $f(t,x)$ is locally Lipschitzian in y in the domain $I\times R^n$.*

Then for each solution $y(t)$ of (2) *there exists a solution $x(t)$ of* (1) *such that* $\lim_{t\to\infty}|y(t)-x(t)|=0$.

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