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MODULES WITH FINITE SPANNING DIMENSION

BY
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1. **Introduction.** Modules with finite spanning dimension were defined by P. Fleury [3] in an attempt to dualize the concept of Goldie dimension. In this note we study these modules in some detail, obtain an improved structure theorem for them and also extend the work done in [2] and [3]. Projective modules with finite spanning dimension turn out to be local or artinian.

All the rings considered here are associative with unit and all the modules are unital left modules. We employ the notation and terminology of [3] and [7].

A left module A over a ring R is said to have finite spanning dimension (for short, f.s.d.) if, for every descending chain of submodules $S_1 \supseteq S_2 \supseteq \cdots$, there is an integer k such that $S_i = S_k$ or S_i is small in S_k for all $i > k$.

Fleury [3] indicated how modules with f.s.d. are made up of hollow modules. We first investigate the hollow modules and their generalizations. This throws more light on the nature of the modules with f.s.d. and helps to improve a structure theorem obtained by Fleury for these modules. Quasi-projective modules with f.s.d. are characterized. Modules with f.s.d. over special types of rings like Dedekind domains, left V-rings etc., are described.

2. **Semi-hollow and hollow modules.** Hollow modules are duals of uniform modules. Here we study the hollow and the semi-hollow modules in some detail.

Definition 2.1. Let A be an R-module. (i) A is said to be local if A has a unique proper maximal submodule which contains every other proper submodule of A. (ii) A is called a hollow module if every proper submodule of A is small in A. (iii) We say that A is semi-hollow if every proper finitely generated submodule of A is small in A.

Clearly, a local module is hollow and is further cyclic. Also note that an R-module A is semi-hollow if and only if every proper cyclic submodule of A is small. A better description is given next.

Proposition 2.2. *An R-module A is semi-hollow if and only if A is local or A has no proper maximal submodules.*

Proof. Suppose A is semi-hollow and has a proper maximal submodule M. Let

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$a \in A \setminus M$. Then $A = Ra + M$. Since A is semi-hollow, this will be a contradiction unless $A = Ra$. Thus A is local.

Conversely, suppose A is not semi-hollow. Clearly A is not local. By hypothesis, $A = Rx + T$, for some $x \in A$, where $T \neq A$. Let M be maximal with respect to the property that $Rx + M = A$ and $x \notin M$. M exists by Zorn's lemma. It is clear that M is a maximal proper submodule of A. This completes the proof.

For any module A, let $\operatorname{Rad} A$ be the intersection of all the maximal submodules of A, with the proviso that $\operatorname{Rad} A = A$, if A has no maximal submodules. One can then reformulate 2.2 as: A is semi-hollow if and only if $\operatorname{Rad} A = A$ or $\operatorname{Rad} A$ is maximal and small in A.

Remarks. (i) Since a finitely generated module always has a proper maximal submodule, finitely generated semi-hollow modules are just the local modules.

(ii) The abelian group $Q \oplus Z(p)$, where Q is the additive group of rational numbers and $Z(p)$ is the prime cyclic group of order p, shows that a module may have a unique maximal submodule without being local. It also shows that a direct sum of two semi-hollow modules need not be semi-hollow. Note that $Q \oplus Z(p^\infty)$, where $Z(p^\infty)$ is the Prüfer group, has no proper maximal subgroups so that it is semi-hollow. Since it contains $Q \oplus Z(p)$, we see that a submodule of a semi-hollow module need not be semi-hollow.

We also note in passing that if A is semi-hollow (hollow), then A/K is semi-hollow (hollow) for any submodule K of A. The converse holds if K is small in A.

It is clear that if R is a field, then simple R-modules are the only semi-hollow R-modules. The next proposition describes the semi-hollow modules over Dedekind domains which, to avoid trivial situations, are assumed to be not fields.

Proposition 2.3. *Let R be a Dedekind domain. Then an R-module A is semi-hollow if and only if A is one of the following types*: (i) *A is divisible*; (ii) *A is cyclic, in case R is a discrete valuation ring OR $A \cong R/P^n$ with P a non-zero prime ideal and n a positive integer, if R is otherwise.*

Proof. If A is divisible and $M \neq A$ is a maximal submodule, then A/M is a simple module and is further divisible. This is impossible (since R is not a field). Thus $A = \operatorname{Rad} A$. Copnversely, if $A = \operatorname{Rad} A$, then $A = PA$ for every non-zero prime ideal P of R; For, otherwise, A/PA being a non-zero R/P-module has a proper maximal (R/P-submodule and hence) R-submodule which would give rise to a proper maximal submodule of A, a contradiction. Hence A is divisible.

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Suppose A is of type (ii). If R is a discrete valuation ring, then R is a local R-module and hence the cyclic R-module A is local. If R is not a discrete valuation ring, then $A \cong R/P^nR$ and has PR/P^nR as the unique maximal submodule, whence A is local. Thus, in any case, A is semi-hollow.

It is known ([5]) that a divisible module over a Dedekind domain R is a direct sum of (i) copies of K, the quotient field of R and (ii) copies of $R(P^\infty)$ for various non-zero prime ideals P, where $R(P^\infty)$ denotes the P-primary component of K/R. This helps us to deduce next that very few modules over a Dedekind domain are hollow.

COROLLARY 2.4. *Let R be a Dedekind domain. Then an R-module A is hollow if and only if* (i) *A is a submodule of K or K/R, in case R is a discrete valuation ring and* (ii) *A is a submodule of $R(P^\infty)$, for some non-zero prime ideal P, if R is otherwise.*

Proof. Since a hollow module is indecomposable, a divisible hollow module over R is isomorphic to K or $R(P^\infty)$, for some prime ideal P. But K is hollow if and only if R is a discrete valuation ring. To see this, note that if R is not a discrete valuation ring, then one can define R-submodules R^P and R_p by $R^P/R = R(P^\infty)$ and $K/R = R^P/R \oplus R_p/R$, so that $K = R^P + R_p$ with $R^P \neq K \neq R_p$. On the other hand, if R is a discrete valuation ring, then the proper R-submodules of K are cyclic, being of the form p^nR, where p is the generator of the unique non-zero prime ideal of R and n is an integer (negative, zero, or positive). The corollary then follows from 2.3.

REMARK. (i) In particular, an abelian group A is hollow if and only if A is a subgroup of the Prüfer group $Z(p^\infty)$ for some prime p. This shows that the claim (and its proof) made in [2] that the abelian group of p-adic integers is hollow is incorrect.

(ii) It is clear from 2.3 and 2.4 that a semi-hollow module need not be hollow.

Since $\operatorname{Rad} A \neq A$ for any non-zero projective module A ([1]), we have.

PROPOSITION 2.5. *A projective R-module P is semi-hollow $\Leftrightarrow P$ is hollow $\Leftrightarrow$ P is local.*

The next proposition describes the quasi-projective hollow modules.

PROPOSITION 2.6. *Let A be a quasi-projective R-module. Then A is hollow if and only if $S = \operatorname{End}_R(A)$ is a local ring.*

Proof. Sufficiency: A result of Sandomierski (see [9]) states that the Jacobson radical $J(S) = \{\beta \in S \mid \operatorname{im} \beta \text{ is small in } A\}$. Let S be local. Suppose $B \subseteq A$ with $B + C = A$ for some $C \neq A$. Then $A/(B \cap C) = B/(B \cap C) \oplus C/(B \cap C)$. Let

$\mu : A/(B \cap C) \to B/(B \cap C)$ be the corresponding projection and $\lambda : A \to A/(B \cap C)$ be the natural map. Then the quasi-projectivity of A gives an α making the diagram

$$\begin{tikzcd} & A \arrow[dl, dashed, "\alpha"'] \arrow[d, "\lambda\mu"] \\ A \arrow[r, "\lambda"] & A/B \cap C \end{tikzcd}$$

commutative. Since $A\alpha\lambda = A\lambda\mu = B/(B \cap C)$, $A\alpha \subseteq B$. Also $B = A\alpha + (B \cap C)$, so that $A\alpha + C = A\alpha + (B \cap C) + C = B + C = A$. This shows that $A\alpha$ is not small in A and thus $\alpha \notin J(S)$. Since S is a local ring. α is an automorphism. Then $B = A$, proving that A is hollow.

Necessity. Observe that the kernel of any epi-endomorphism of a quasi-projective module A is a summand (see Lemma 4.3, [8]). If A is further hollow, we get that every epi-endomorphism of A is an automorphism. But, for any endomorphism α of a hollow module, α or $1-\alpha$ is always an epi-endomorphism. Thus we conclude that $\mathrm{End}_R(A)$ is a local ring.

Since, by 2.5, a hollow projective module is local, we obtain (perhaps in a simpler way) the following theorem of R. Ware ([9]).

COROLLARY 2.7. ([9]). *Let P be a projective module over a ring R. Then* $\mathrm{End}_R(P)$ *is a local ring if and only if P is local.*

Since, for any two sided ideal I of R, R/I is quasi-projective (see [8]), we get

COROLLARY 2.8. ([2]). *Let I be a two sided ideal of R. If R/I is hollow then (and only then)* $\mathrm{End}_R(R/I)$ *is local.*

PROPOSITION. 2.9. *If A is quasi-injective and hollow, then* $\mathrm{End}_R(A)$ *is local.*

Proof. Since A is fully invariant in its injective hull A^*, every endomorphism α' of A arises as a restriction to A of an endomorphism α of A^*. Thus $\alpha \mapsto \alpha'$ is a ring epimorphism from $\mathrm{End}_R(A^*)$ to $\mathrm{End}_R(A)$. Now A, being hollow, is indecomposable and its full invariance in A^* makes A^* indecomposable. Then $\mathrm{End}_R(A^*)$ is local and hence $\mathrm{End}_R(A)$ is local.

Note that a quasi-injective module A with $\mathrm{End}_R(A)$ local need not be hollow, as is clear from considering the Z-module Q of all rational numbers.

2.10. **Hollow modules over special types of rings**

(i) Let R be left perfect. Then an R-module A is hollow $\Leftrightarrow A$ is semi-hollow $\Leftrightarrow A$ is local.

This is immediate from the fact that every non-zero R-module has a proper maximal submodule.

(ii) Let R be a left V-ring (that is, every simple R-module is injective). Then an R-module A is hollow $\Leftrightarrow A$ is semi-hollow $\Leftrightarrow A$ is simple.

This is obvious from the fact that $\operatorname{Rad} M = 0$ for any R-module M.

3. **Modules with finite spanning dimension.** In this section, we clarify some of the properties of modules with f.s.d. and this results in an improved structure theorem (Theorem 3.4) for these modules.

We first observe that a module A has f.s.d. exactly when the non-small submodules of A satisfy the minimum condition.

If B is a submodule of A, then by a supplement of B in A we mean a submodule S with the property that $B+S=A$, but $B+T\neq A$ for any proper submodule T of S. It is worth noting that if S is a supplement of B in A, then a submodule T of S is small in A if and only if it is small in S. For, if T is small in A and $S=T+U$, then $A=B+S=B+T+U=B+U$ and the minimality of S implies that $U=S$, showing that T is small in S. Thus, in particular, $\operatorname{Rad} S = S \cap \operatorname{Rad} A$.

We now collect a few simple properties of modules with f.s.d. some of which have been proved in [3].

LEMMA 3.1. (i) *If A has* f.s.d., *then any homomorphic image also has* f.s.d. *Every submodule of A has a supplement in A.*

(ii) *If A has* f.s.d. *and S is not small in A, then A/S is artinian.*

(iii) *Let A be an R-module and B a supplement of a submodule in A. Then* $\operatorname{Rad} B = B \cap \operatorname{Rad} A$. *$B$ has* f.s.d., *if A has.*

(iv) *If A has* f.s.d., *then* $A/\operatorname{Rad} A$ *is a direct sum of finitely many simple modules. Moreover, if* $\operatorname{Rad} A$ *is not essential in A, then A is artinian.*

Proof. Assertions (i), (ii), and (iii) are implicit in [3]. We prove (iv). Now, by (i), $A^*=A/\operatorname{Rad} A$ has f.s.d. Since $\operatorname{Rad} A^*=0$, A^* is artinian. For the same reason, A^* is a direct sum of finitely many simple modules. If $\operatorname{Rad} A$ is not essential in A, then since $A/\operatorname{Rad} A$ is a direct sum of simple modules, there exists a simple submodule S with $S \cap \operatorname{Rad} A = 0$. Clearly S is not small so that there exists a submodule $M\neq A$ with $S+M=A$. Then $S\cap M=0$ and so $A=S\oplus M$. By (ii), we conclude that A is artinian.

Lemma 3.1. (iv) helps us to get, with less effort, structure theorems for modules with f.s.d. which are slightly more satisfactory than the ones given by Fleury ([3]).

PROPOSITION 3.2. *Any module A with* f.s.d. *can be written as an irredundant sum $A=L_1+\cdots+L_n+B$, where the L_i are local (and not small) and B (if not zero) is a semi-hollow non-local module with* f.s.d. *If $A=L_1'+\cdots+L_k'+B'$ is another irredundant sum with the L_i' local and B' semi-hollow non-local, then $n=k$. Moreover, $B=0$ if and only if $B'=0$.*

Proof. By Lemma 3.1 (iv), $A/\text{Rad } A = S_1 \oplus \cdots \oplus S_n$, S_i simple. Let, for each i, L_i be cyclic satisfying $(L_i + \text{Rad } A)/\text{Rad } A = S_i$. Since A has f.s.d., we may assume that $L_i + \text{Rad } A \neq T + \text{Rad } A$ for any proper submodule T of L_i. Then L_i is (hollow and hence is) local and $L_i + \cdots + L_n + \text{Rad } A = A$. Let $B \subseteq \text{Rad } A$ be a supplement of $L_1 + \cdots + L_n$ in A. Then $L_1 + \cdots + L_n + B = A$ and, by 3.1(iii), B is semi-hollow and has f.s.d. Since n is the length of $A/\text{Rad } A$ and a semi-hollow non-local submodule of A is contained in Rad A, we conclude that any other irredundant decomposition of A as a sum of local modules and a semi-hollow non-local module would involve exactly n local modules. Finally, suppose $B = 0$. Then $A = L_1 + \cdots + L_n$ is finitely generated and so Rad A is small. If $B' \neq 0$, then $B' \subseteq \text{Rad } A$ is small and this will contradict the irredundancy of the sum $L'_1 + \cdots + L'_R + B'$. Hence $B' = 0$.

Since a semi-hollow module over an artinian ring is local (see 2.10(i)), we have the following

COROLLARY 3.3. *A module over a left artinian ring has* f.s.d. *if and only if it is an artinian module.*

THEOREM 3.4. *If an R-module A has* f.s.d., *then $A = L_1 \oplus \cdots \oplus L_n \oplus P$, where the L_i are simple and P is s^3-free (that is, has no simple summands). If we also have $A = L'_1 \oplus \cdots \oplus L'_k \oplus P'$, with L'_i simple and P' s^3-free. then $P \cong P'$, $n = k$ and there exists a bijection γ of $\{1, \ldots, n\}$ to itself such that $L_i \cong L_{\gamma(i)}$, for $i = 1, \ldots, n$.*

Proof. Let P be an essential closure (that is, a maximal essential extension) of Rad A in A. Let S be a submodule maximal with respect to the property that $S \cap \text{Rad } A = 0$. Then $S \cap P = 0$ and $(S+P)/P$ is an essential submodule of A/P. Since $A/\text{Rad } A$ is a direct sum of simple modules, so is A/P. This implies that $(S+P)/P = A/P$. Thus $A = S \oplus P$, where S, being isomorphic to $(S + \text{Rad } A)/\text{Rad } A$, is a direct sum $L_1 \oplus \cdots \oplus L_n$ of simple modules and P is s^3-free, since Rad A is essential in P. Suppose $A = L'_1 \oplus \cdots \oplus L'_k \oplus P$, with each L'_i simple and P' s^3-free, then clearly $\text{Rad } A \subseteq P'$ and is essential in P', since $P'/\text{Rad } A$ is a direct sum of simple modules and P' is s^3-free. Thus P' is an essential closure of Rad A in A and, by the above argument, $S \oplus P' = A$. Then $P \cong P'$ and $S \cong L'_1 \oplus \cdots \oplus L'_k$ showing that $n = k$ and that a bijection γ of $\{1, \ldots, n\}$ to itself exists such that $L_i \cong L_{\gamma(i)}$, for $i = 1, \ldots, n$.

We next describe projective modules with f.s.d.

PROPOSITION 3.5. *A projective (quasi-projective) module P has* f.s.d. *if and only if P is local (hollow) or artinian.*

Proof. We need only to prove the necessity. If P were hollow, then, by 2.5, P is local. Suppose P is a non-hollow quasi-projective module. Then $P = A + B$, where $A \neq P \neq B$ and where, by 3.1(i), A and B can be taken to be

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supplements of each other. An adaptation of standard arguments (see, for e.g., the Satz in [6]) shows that $P = A \oplus B$. We indicate the proof for the sake of completeness. By quasi-projectivity, the diagram

$$\begin{array}{ccc} & & P \\ & & \big\downarrow{\scriptstyle \eta} \\ A & \xrightarrow{\eta'} & (A+B)/B \end{array}$$

where η is the natural map and $\eta' = \eta \mid A$, gives an $\alpha : P \to A$ satisfying $\eta'\alpha = \eta$. Since $\ker \eta' = A \cap B$ is small in A, α is epic so that $P = A + \ker \alpha$. But $\ker \alpha \subseteq B$ and B a supplement of A, whence $B = \ker \alpha$. Then $0 = \alpha(B) = A \cap B$. Thus $P = A \oplus B$. By 3.1(ii), P is artinian.

REMARK 3.6. Thus, in particular, a ring R has f.s.d. as a left R-module exactly when R is local or left artinian. This remark shows that, contrary to the claim made in [3], a semiprimary ring S need not have f.s.d. as an S-module, as is clear by taking S to be the ring direct sum, $S = R \oplus R$, where R is the ring of all rational numbers with denominators not divisible by 2.

The direct factors A and B in the proof of 3.5 are again quasi-projective with f.s.d. and, if they are not hollow, the same arguments can be used to decompose each of them. Proceeding like this and using the f.s.d. of P, Proposition 2.6 and the Krull–Schmidt–Azumaya theorem, we obtain the following

COROLLARY 3.7. *A quasi-projective module P with* f.s.d. *is a direct sum of a finite number of hollow (quasi-projective) modules, $P = H_1 \oplus \cdots \oplus H_n$, H_i hollow. If $P = H'_1 \oplus \cdots \oplus H'_k$, with each H'_i hollow, then $k = n$ and there is a bijection σ of $\{1, \ldots, n\}$ to itself such that $H_i \cong H_{\sigma(i)}$, $i = 1, \ldots, n$.*

Harada [4] has proved that if P is an artinian projective module, then $\mathrm{End}_R(P)$ is left artinian. He first proves that $\mathrm{End}_R(P)$ is semi-primary, uses this to show that P is finitely generated and then deduces from his Lemma 2.6 that $\mathrm{End}_R(P)$ is artinian. Since it is clear from 3.7 that an artinian projective module is a direct sum of finitely many local modules, one can directly appeal to Lemma 2.6 of Harada [4] to get that $\mathrm{End}_R(P)$ is left artinian. We do not know whether a hollow artinian quasi-projective R-module P with $\mathrm{End}_R(P)$ a division ring must be a local module. If this is true, then Harada's theorem can be generalized to quasi-projective modules.

As a consequence of Harada's theorem and 3.5, we have

COROLLARY 3.8. *If P is a projective R-module with* f.s.d., *then $S = \mathrm{End}_R(P)$ has* f.s.d. *as a left S-module.*

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Since a module with f.s.d. is the sum of finitely many hollow modules [3], we get from 2.4, the assertions (i) and (ii) of the following proposition.

PROPOSITION 3.9. (i) *Let R be a Dedekind domain which is not a discrete valuation ring. Then the R-modules with* f.s.d. *are just the artinian R-modules.*

(ii) *Let R be a discrete valuation ring. Then an R-module A has* f.s.d. *exactly when A is artinian or is a direct sum of finitely many R-submodules of K, the quotient field of R.*

(iii) *Let R be a left V-ring. Then the R-modules with* f.s.d. *are just the artinian R-modules and are direct sums of finitely many simple R-modules.*

(iv) *Let R be left perfect. Then an R-module with* f.s.d. *is finitely generated and is a sum of finitely many local modules.*

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