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REPRESENTATIONS OF FINITE GROUPS AND CUNTZ-KRIEGER ALGEBRAS

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We investigate the structure of the C^*-algebras $\mathcal{O}_\rho$ constructed by Doplicher and Roberts from the intertwining operators between the tensor powers of a representation ρ of a compact group. We show that each Doplicher-Roberts algebra is isomorphic to a corner in the Cuntz-Krieger algebra $\mathcal{O}_A$ of a $\{0,1\}$-matrix $A = A_\rho$ associated to ρ. When the group is finite, we can then use Cuntz's calculation of the K-theory of $\mathcal{O}_A$ to compute $K_*(\mathcal{O}_\rho)$.

Doplicher and Roberts have recently developed a duality theory for compact subgroups of $SU(n, \mathbf{C})$ in which the dual object consists of a simple C^*-algebra $\mathcal{O}_G$ and an endomorphism of $\mathcal{O}_G$ [**3, 4**]. The construction of $\mathcal{O}_G$ is based on the concrete representation ρ of G in $SU(n, \mathbf{C})$ rather than the abstract group G, so we prefer to call it $\mathcal{O}_\rho$; our work originated in an attempt to find out how the structure of $\mathcal{O}_\rho$ depends on the choice of representation. To this end we have computed the K-theory of $\mathcal{O}_\rho$ for finite G, by embedding it as a corner in a Cuntz-Krieger algebra $\mathcal{O}_A$, and using Cuntz's calculation of $K_*(\mathcal{O}_A)$ [**1**]. One conclusion is that different representations of the same finite group can give algebras which have quite different K-theory, and hence are not even stably isomorphic or Morita equivalent.

The algebra $\mathcal{O}_\rho$ is constructed from the spaces of intertwining operators between the different tensor powers ρ^n of ρ, and its structure is determined by the decompositions of ρ^n into irreducibles, and hence by the decompositions of $\pi \otimes \rho$ for $\pi \in \widehat{G}$. The combinatorics of the situation can be summed up in a bipartite graph with $\widehat{G}$ as vertices, and our main observation is that these combinatorics are similar to those involved in Cuntz and Krieger's construction of a C^*-algebra $\mathcal{O}_A$ from a $\{0,1\}$-matrix A. When G is compact, A is infinite, and there are technical problems in transferring this combinatorial similarity to the C^*-algebra level; indeed, we need to appeal to both [**2**] and [**3**] to do it. For finite groups, we can prove directly that $\mathcal{O}_\rho$ is a corner in $\mathcal{O}_A$, and the simplicity of $\mathcal{O}_\rho$ therefore follows from [**2**] alone. We shall go as far as we can in full generality, since we are optimistic that one can extend the results of [**1**] to cover infinite A, and use them to compute $K_*(\mathcal{O}_\rho)$ for compact G along similar lines.

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We begin with a discussion of the two Doplicher-Roberts algebras ${}^0\mathcal{O}_\rho$, $\mathcal{O}_\rho$ associated to a finite-dimensional representation ρ: ${}^0\mathcal{O}_\rho$ is a *-algebra, and $\mathcal{O}_\rho$ its C^*-enveloping algebra. In Section 2, we associate a $\{0,1\}$-matrix A_ρ to ρ, and show how ${}^0\mathcal{O}_\rho$ can be canonically mapped into the Cuntz-Krieger algebra $\mathcal{O}_{A_\rho}$; in Section 3, we prove that, when G is finite, this mapping induces an isomorphism of $\mathcal{O}_\rho$ onto a corner $P\mathcal{O}_{A_\rho}P$ in $\mathcal{O}_{A_\rho}$. Since $\mathcal{O}_{A_\rho}$ is known to be simple [2], this implies that $\mathcal{O}_\rho$ is Morita equivalent to $\mathcal{O}_{A_\rho}$, and in particular has the same K-theory. In our final section, we compute $K_*(\mathcal{O}_\rho)$ for a few examples of finite groups, using methods which should work whenever we have a character table for G.

One could also hope to investigate the structure of Doplicher-Roberts algebras by realising them as the C^*-algebras of locally compact groupoids whose unit spaces are path spaces associated to the infinite diagram of Section 1, and exploiting general properties of groupoid C^*-algebras, as done for AF-algebras in [7]. At present, though, it is not clear whether the appropriate groupoids for the Cuntz-Krieger algebras of infinite $\{0,1\}$-matrices are locally compact, and hence the present approach may be more easily adapted to compact groups. In [6], we gave a brief discussion of the groupoid approach, and the problems involved in it.

We stress that many of the ideas and results in this paper are either well-known or implicit in the work of Doplicher-Roberts and Cuntz-Krieger. For example, our comments in Section 4 on computing $K_*(\mathcal{O}_A)$ are surely known to all experts. However, we do hope a detailed presentation of this circle of ideas in a technically-straightforward special case will be informative and useful.

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1. Doplicher-Roberts algebras

Let ρ be a finite-dimensional representation of a locally compact group, and for $n \in \mathbf{N}$, let ρ^n be the n-fold tensor power of ρ, acting in $H_\rho \otimes \cdots \otimes H_\rho = H_\rho^n$. For each pair $m, n \in \mathbf{N}$, we denote by (ρ^m, ρ^n) the space of intertwining operators $T : H_\rho^n \to H_\rho^m$; we have chosen this notation so that the composition $S \circ T$ of $S \in (\rho^m, \rho^n), T \in (\rho^n, \rho^p)$ lies in (ρ^m, ρ^p). There is a natural embedding $T \to T \otimes 1$ of (ρ^m, ρ^n) in (ρ^{m+1}, ρ^{n+1}), and we denote the direct limit $\varinjlim(\rho^p, \rho^{p+k})$ by ${}^0\mathcal{O}_\rho^k$. The direct sum ${}^0\mathcal{O}_\rho = \bigoplus_{k\in\mathbf{Z}} {}^0\mathcal{O}_\rho^k$ is a *-algebra in which the product of $S \in (\rho^m, \rho^n)$ and $T \in (\rho^p, \rho^q)$ is

$$\begin{cases} (S \otimes 1_{p-n}) \circ T \in \left(\rho^{m+(p-n)}, \rho^q\right) & \text{if } p \geqslant n \\ S \circ (T \otimes 1_{n-p}) \in \left(\rho^m, \rho^{q+(n-p)}\right) & \text{if } p > n, \end{cases}$$

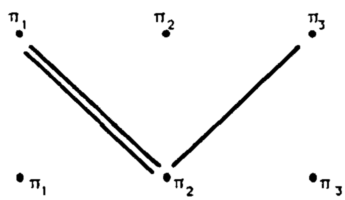
and the adjoint of $S \in (\rho^m, \rho^n)$ is $S^* \in (\rho^n, \rho^m)$.

We shall refer to either ${}^0\mathcal{O}_\rho$ or its C^*-enveloping algebra $\mathcal{O}_\rho$ as a *Doplicher-Roberts algebra*; of course, it is not immediately obvious that ${}^0\mathcal{O}_\rho$ has a C^*-enveloping algebra, since a priori

$$\|T\| = \sup\{\|\pi(T)\| : \pi \text{ is a *-representation of } {}^0\mathcal{O}_\rho\}$$

could be infinite. To settle this, we shall describe a natural basis for each (ρ^m, ρ^n), which is parametrised by paths in an infinite graph associated to ρ, and which will be important in our later constructions.

We first let R be the set of (equivalence classes of) irreducible summands of the tensor powers ρ^n, (adding in the trivial representation ι, if necessary), and to each element of R we associate a specific representation $\pi : G \to U(H_\pi)$. We define a bipartite graph with R as the set of vertices, and the number of edges joining π_1 at the top level to π_2 at the lower level equal to the multiplicity of π_2 in $\pi_1 \otimes \rho$. Thus, for example, if π_2 occurs with multiplicity 2 in $\pi_1 \otimes \rho$, and multiplicity 1 in $\pi_3 \otimes \rho$, the graph contains



If x is an edge from π_1 above to π_2 below, we write $s(x) = \pi_1$ and $r(x) = \pi_2$, and we let E denote the set of all edges. We now assign to each edge x an isometric intertwiner $T_x : H_{r(x)} \to H_{s(x)} \otimes H_\rho$, in such a way that, for each π,

$$H_\pi \otimes H_\rho = \bigoplus_{\{x: s(x)=\pi\}} T_x T_x^*(H_\pi \otimes H_\rho)$$

—in other words, such that the edges out of π give a specific decomposition of $H_\pi \otimes H_\rho$ into irreducibles. Next we consider the infinite graph obtained by sticking infinitely many copies of the bipartite graph below the original. We note that a sequence $x_1, x_2, \ldots, x_n$ of edges in the original graph combines to form a vertical path in the infinite graph if and only if $r(x_j) = s(x_{j+1})$ for all j. Each path $x = \{x_1, x_2, \ldots x_n\}$ represents an intertwiner

$$T_x = (T_{x_1} \otimes 1_{n-1}) \circ (T_{x_2} \otimes 1_{n-2}) \circ \cdots \circ T_{x_n} : H_{r(x_n)} \to H_\rho^n,$$

where 1_r denotes the identity operator on H_ρ^r, and the paths x with $s(x_1) = \iota$ provide an explicit decomposition of H_ρ^n into irreducibles:

$$H_\rho^n = \bigoplus_{\{\text{paths } x \text{ with } s(x_1)=\iota\}} T_x T_x^*(H_\rho^n).$$

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Proposition 1.1. *The family*

$$\{T_x T_y^* : |x| = m\, |y| = n, s(x_1) = s(y_1) = \iota, r(x_m) = r(y_n)\}$$

is a basis for (ρ^m, ρ^n)*, and each basis element* $T_x T_y^*$ *is a partial isometry.*

Proof: Each pair of paths x, y with $|x| = m$, $|y| = n$ and $s(x_1) = \iota = s(y_1)$ determines a pair of irreducible summands $T_x\left(H_{r(x_m)}\right)$, $T_y\left(H_{r(y_n)}\right)$ of H_ρ^m, H_ρ^n; the space of intertwiners of these representations is 0 unless $r(x_m) = r(y_n)$, and then is the 1-dimensional space spanned by $T_x T_y^*$. Hence every intertwiner in (ρ^m, ρ^n) can be uniquely expressed as a linear combination of the $T_x T_y^*$, as claimed. Because each T_y is isometric, T_y^* is a partial isometry with range space $H_{r(y_n)}$, and, whenever $r(x_m) = r(y_n)$, $T_x T_y^*$ is also a partial isometry. ∎

Corollary 1.2. *For every* $T \in {}^0\mathcal{O}_\rho$,

$$\|T\| = \sup\{\|\pi(T)\| : \pi \text{ is a *-representation of } {}^0\mathcal{O}_\rho\}$$

is finite.

Proof: As every element of ${}^0\mathcal{O}_\rho$ is a finite sum of elements of ${}^0\mathcal{O}_\rho^k$, and each of these is a direct limit, we may as well suppose that $T \in (\rho^m, \rho^n)$, and hence that T can be uniquely written as a linear combination $\sum \lambda_{x,y} T_x T_y^*$. Now an operator $S \in (\rho^m, \rho^n)$ is a partial isometry if and only if $S = SS^*S$ as operators on H_ρ^n, and hence, by definition of the *-algebra structure on ${}^0\mathcal{O}_\rho$, if and only if $S = SS^*S$ in ${}^0\mathcal{O}_\rho$. Thus $\pi\left(T_x T_y^*\right)$ is a partial isometry for every representation π of ${}^0\mathcal{O}_\rho$, and

$$\|T\| \leqslant \sum |\lambda_{x,y}|\, \left\|\pi\left(T_x T_y^*\right)\right\| \leqslant \sum |\lambda_{x,y}|,$$

which gives the Corollary. ∎

Remark. Although we have not insisted that the group G be compact, as Doplicher and Roberts do, the extra generality is spurious: if ρ is finite-dimensional, the intertwining spaces for the identity representation ι_K of the compact group $K = \overline{\rho(G)} \subset U(H_\rho)$ are exactly the same as those of ρ, and hence $\mathcal{O}_{\iota_K} = \mathcal{O}_\rho$. However, there are non-compact groups with lots of finite-dimensional representations — for example, $SL(2, \mathbf{Z}) \cong \mathbf{Z}_2 * \mathbf{Z}_3$, and the integer Heisenberg group — and there could possibly be interesting interplay between the combinatorics of the representation, the algebra $\mathcal{O}_\rho$, and the underlying non-compact group.

2. Representing a Doplicher-Roberts algebra in a Cuntz-Krieger algebra

Again let ρ be a finite-dimensional representation of a locally compact group, and resume the notation of the previous section. Define a (possibly infinite) $\{0, 1\}$-matrix

A_ρ, indexed by the set E of edges in the bipartite graph associated to ρ, as follows:

$$A_\rho(x,y) = \begin{cases} 1 & \text{if } r(x) = s(y) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

THEOREM 2.1. *Let ρ be a finite-dimensional representation of a locally compact group, and use the notation of Section 1. Let $\{S_x : x \in E\}$ be a family of non-zero partial isometries satisfying*

$$S_x^* S_x = \sum_{y \in E} A_\rho(x,y) S_y S_y^*,$$

*let B be the *-algebra generated by $\{S_x\}$, and let*

$$P = \sum_{\{x \in E : s(x) = \iota\}} S_x S_x^*.$$

*Then there is a *-homomorphism of the Doplicher-Roberts algebra ${}^0\mathcal{O}_\rho$ onto the corner PBP.*

The idea is that paths in the infinite diagram of Section 1 have interpretations in the Cuntz-Krieger algebra $C^*(S_x)$, as well as the Doplicher-Roberts algebra ${}^0\mathcal{O}_\rho$. A sequence $x_1, x_2, \dots, x_n$ of edges in the original graph combines to form a vertical path in the infinite graph if and only if $r(x_j) = s(x_{j+1})$ for all j, hence if and only if $A_\rho(x_j, x_{j+1}) = 1$ for all j, and hence exactly when the the product $S_x = S_{x_1} S_{x_2} \dots S_{x_n}$ is non-zero [**2**, p.252]. And, parallel to Lemma 1.1, every element of B is a linear combination of operators $S_x S_y^*$ with $r(x_m) = r(y_n)$.

We now define $\phi_{m,n} : (\rho^m, \rho^n) \to B$ by $\phi_{m,n}(T_x T_y^*) = S_x S_y^*$. Notice that, since $s(x_1) = s(y_1) = \iota$, we have

$$S_x S_y^* = S_{x_1} S_{x_1}^* (S_x S_y^*) S_{y_1} S_{y_1}^* = P(S_{x_1} S_{x_1}^*)(S_x S_y^*)(S_{y_1} S_{y_1}^*)P = P S_x S_y^* P,$$

and hence $\phi_{m,n} : (\rho^m, \rho^n) \to PBP$. We claim that the maps $\phi_{m,n}$ are compatible with the bonding maps $(\rho^m, \rho^n) \to (\rho^{m+1}, \rho^{n+1})$, in the sense that

$$\phi_{m+1,n+1}\big((T_x T_y^*) \otimes 1\big) = \phi_{m,n}(T_x T_y^*). \tag{2.2}$$

To see this, we note that

$$H_{r(x_m)} \otimes H_\rho = \bigoplus_{\{z \in E : s(z) = r(x_m)\}} T_z T_z^* (H_{r(x_m)} \otimes H_\rho),$$

so that

$$\begin{aligned} T_xT_y^* \otimes 1 &= \sum_{\{z:s(z)=r(x_m)=r(y_n)\}} (T_x \otimes 1)(T_zT_z^*)(T_y^* \otimes 1) \\ &= \sum_{\{z:s(z)=r(x_m)=r(y_n)\}} (T_{xz})(T_{yz})^*; \end{aligned}$$

on the other hand,

$$\begin{aligned} S_xS_y^* &= S_x\big(S_{x_m}^*S_{x_m}\big)S_y^* \\ &= S_x\left(\sum_{z\in E} A(x_m,z)S_zS_z^*\right)S_y^* \\ &= \sum_{\{z:s(z)=r(x_m)\}} S_x(S_zS_z^*)S_y^* \\ &= \sum_{\{z:s(z)=r(x_m)\}} (S_{xz})(S_{yz})^*, \end{aligned}$$

and (2.2) follows.

We can now define $\phi = \oplus\phi^k$, at least as a linear map, and we have to verify that ϕ is a *-homomorphism. Well,

$$\phi_{m,n}\big(T_xT_y^*\big)^* = \big(S_xS_y^*\big)^* = S_yS_x^* = \phi_{n,m}(T_yT_x^*),$$

so ϕ is certainly *-preserving. To check that ϕ is multiplicative, consider $T_xT_y^* \in (\rho^m, \rho^n)$, $T_wT_z^* \in (\rho^p, \rho^q)$, and suppose for the sake of argument that $p \geqslant n$. Then

$$\phi\big((T_xT_y^*)(T_wT_z^*)\big) = \phi^{(n-m)+(q-p)}\Big(\big((T_xT_y^*) \otimes 1_{p-n}\big) \circ (T_wT_z^*)\Big).$$

The product $\big(T_y^* \otimes 1_{p-n}\big)T_w$ is by definition the composition

$$\big(T_{y_n}^* \otimes 1_{p-n}\big) \circ \Big(T_{y_{n-1}}^* \otimes 1_{p-n+1}\Big) \circ \cdots \circ \big(T_{y_1}^* \otimes 1_{p-1}\big) \circ \big(T_{w_1} \otimes 1_{p-1}\big) \circ \cdots \circ T_{w_p}.$$

Since

$$T_{y_1}^*T_{w_1} = \begin{cases} 0 & \text{unless } w_1 = y_1 \\ T_{y_1}^*T_{y_1} = 1 & \text{if } w_1 = y_1, \end{cases}$$

and we know y is a path, $T_{y_2}^*\big((T_{y_1}^*T_{y_1}) \otimes 1\big) = T_{y_2}^*$; thus we can omit the two middle terms in $\big(T_y^* \otimes 1\big)T_w$. By induction, we deduce that the composition is 0 unless $y_i = w_i$ for $1 \leqslant i \leqslant n$, and then equals

$$\big(T_{w_{n+1}} \otimes 1_{p-n-1}\big) \circ \cdots \circ T_{w_n} = T_{w'},$$

say. Thus

$$((T_xT_y^*)\otimes 1_{p-n})T_wT_z^* = \begin{cases} (T_x\otimes 1_{p-n})\circ T_{w'}\circ T_z^* & \text{if } y_i = w_i \text{ for } 1\leqslant i\leqslant n \\ 0 & \text{otherwise,}\end{cases}$$
$$= \begin{cases} T_{xw'}T_z^* & \text{if } y_i = w_i \text{ for } 1\leqslant i\leqslant n \\ 0 & \text{otherwise.}\end{cases}$$

But this is precisely the rule for cancelling $S_y^*S_w$:

$$S_x(S_y^*S_w)S_z^* = \begin{cases} S_x(S_{y_n}^*S_{y_n})S_{w'}S_z^* & \text{if } y_i = w_i \text{ for } 1\leqslant i\leqslant n \\ 0 & \text{otherwise}\end{cases}$$
$$= \begin{cases} S_xS_{w'}S_z^* & \text{if } y_i = w_i \text{ for } 1\leqslant i\leqslant n \\ 0 & \text{otherwise,}\end{cases}$$

since $r(x_m) = r(y_n)$. Hence ϕ is multiplicative, as claimed.

The algebra B is spanned by the elements of the form $S_xS_y^*$, which is non-zero only if there exists z with $A(x_m,z) = A(y_n,z) = 1$, that is, only if $r(x_m) = r(y_n)$. Since

$$PS_xS_y^*P = \begin{cases} S_xS_y^* & \text{if } s(x_1) = \iota \text{ and } s(y_1) = \iota \\ 0 & \text{otherwise,}\end{cases}$$

it follows that

$$PS_xS_y^*P = \begin{cases} S_xS_y^* & \text{if } s(x_1) = \iota = s(y_1) \text{ and } r(x_m) = r(y_n) \\ 0 & \text{otherwise,}\end{cases}$$
$$= \begin{cases} \phi_{m,n}(T_xT_y^*) & \text{if } s(x_1) = \iota = s(y_1) \text{ and } r(x_m) = r(y_n) \\ 0 & \text{otherwise.}\end{cases}$$

Thus the non-zero operators of the form $PS_xS_y^*P$ are all in the range of ϕ, and since they span PBP, the homomorphism ϕ maps onto PBP.

This completes the proof of Theorem 2.1. ∎

Corollary 2.2. *There is a surjective homomorphism of the Doplicher-Roberts algebra $\mathcal{O}_\rho$ onto the corner $PC^*(S_x)P$.*

Proof: The algebra $\mathcal{O}_\rho$ is the C^*-enveloping algebra of the $*$-algebra ${}^0\mathcal{O}_\rho$, so the homomorphism $\phi : {}^0\mathcal{O}_\rho \to PBP \subset PC^*(S_x)P$ is by definition continuous, and extends to a homomorphism of $\mathcal{O}_\rho$ into $PC^*(S_x)P$. Since ϕ maps ${}^0\mathcal{O}_\rho$ onto PBP, which is dense in $PC^*(S_x)P$, and homomorphisms between C^*-algebras have closed range, the Corollary follows. ∎

Corollary 2.3. *Suppose ρ is a representation of a compact group G in $SU_n(\mathbf{C})$, for some $n > 1$. If $\{S_x\}, P$ are as in Theorem 2.1, then $\mathcal{O}_\rho$ is isomorphic to $PC^*(S_x)P$.*

Proof: By Theorem 2.12 of [**3**], there is a unique C^*-seminorm on ${}^0\mathcal{O}_\rho$, which is actually a C^*-norm. Since pulling back the operator norm along the homomorphism of ${}^0\mathcal{O}_\rho$ onto PBP induces such a seminorm, we deduce that the homomorphism is isometric, and extends to an isomorphism of $\mathcal{O}_\rho$ onto the closure $PC^*(S_x)P$ of PBP. ∎

Remark 2.4.. When the group is finite, the matrix A_ρ is finite, and it follows from [**2**] that $C^*(S_x) = \mathcal{O}_{A_\rho}$ is simple (see Lemma 3.1 below). As the corner $\mathcal{O}_\rho$ is then necessarily full, we can deduce from [**3**, Corollary 2.3], and [**2**] that $K_*(\mathcal{O}_\rho) \cong K_*(\mathcal{O}_{A_\rho})$ (we shall prove this again in Section 3 without appealing to [**3**] or requiring $\rho(G) \subset SU$). In principle, we can similarly deduce from [**3**] and [**2**] that $K_*(\mathcal{O}_\rho) \cong K_*(\mathcal{O}_{A_\rho})$ when G is compact and $\rho(G) \subset SU$, although some care will be needed in applying [**2**] because A_ρ is infinite if G is. However, since the calculation of $K_*(\mathcal{O}_A)$ in [**1**] does not obviously apply to infinite A, further work is needed before this result can be useful, and we defer it for now.

3. Doplicher-Roberts algebras of finite groups

Our goal here is to prove that, when G is finite, the complete Doplicher-Roberts algebra $\mathcal{O}_\rho$ is isomorphic to a corner in the corresponding Cuntz-Krieger algebra $\mathcal{O}_{A_\rho}$. Before we can state our theorem, we need to check that the $\{0,1\}$-matrix A_ρ is one for which $\mathcal{O}_{A_\rho}$ can be uniquely defined, up to isomorphism, as the C^*-algebra generated by a family of non-zero partial isometries $\{S_x : x \in E\}$ satisfying

$$S_x^* S_x = \sum_{y \in E} A_\rho(x,y) S_y S_y^*. \tag{3.1}$$

Cuntz and Krieger gave a sufficient condition (I) on the $\{0,1\}$-matrix A_ρ [**2**, p.254; Theorem 2.13], and showed that if in addition A_ρ is irreducible, then $\mathcal{O}_{A_\rho}$ is simple [**2**, 2.14]. Both these properties of A_ρ reduce to standard facts about the representation theory of finite groups:

Lemma 3.1. *If ρ is a representation of a finite group and $1 < \dim \rho < \infty$, then A_ρ is irreducible and satisfies the Cuntz-Krieger condition (I).*

Proof: We may as well suppose ρ is faithful: if not, replace the group G by $G/\ker\rho$. Then every irreducible representation of G is contained in some tensor power of ρ [**5**, (4.3) and (2.9)], and hence $R = \widehat{G}$; equivalently, for each $\pi \in \widehat{G}$ there is a path in the infinite diagram starting at ι and finishing at π. If π_c is the contragredient representation $s \to (\pi_{s^{-1}})^t$, then ι is a summand of $\pi \otimes \pi_c$ (since the corresponding characters satisfy $\chi_{\pi_c} = \overline{\chi}_\pi$, this follows from [5, p.48 and (2.9)]), and hence for any $\pi \in \widehat{G}$ there is a path from π to ι. Putting these last two observations together gives a path joining ι to itself passing through any given π, and hence paths joining any given π_1 to any other π_2. Now given $x, y \in E$, we can use a path from $r(x)$ to $s(y)$ to produce a path starting with x and finishing with y, and thus A_ρ is irreducible. To see that A_ρ satisfies (I) we just have to produce two different paths starting and finishing with the same edge x: for then the irreducibility of A_ρ implies that we can connect any other $y \in E$ to x. But if π has maximal dimension, $\dim \rho \geqslant 2$ implies that $\pi \otimes \rho$ must have at least two irreducible summands, and hence that there are at least two edges y, z with $\pi = s(y) = s(z)$. Now we take x to be any edge with $r(x) = \pi$, and joining $r(y)$ and $r(z)$ to $s(x)$ gives two distinct paths starting and ending at x. ∎

REMARK 3.2. The result always fails if $\dim \rho = 1$. For then ρ is an isomorphism of $G/\ker \rho$ onto a finite cyclic subgroup of $\mathbf{T}$, the map $\gamma \to \gamma\rho$ is an automorphism of $(G/\ker\rho)^\frown$, and the matrix A_ρ is a permutation matrix, which never satisfies condition (I). However, since $\rho(G)$ is cyclic, so is $G/\ker\rho$, ρ must generate $(G/\ker\rho)^\frown$, and the permutation matrix is irreducible.

We now fix a family $\{S_x : x \in E\}$ of non-zero partial isometries on a Hilbert space H satisfying (3.1), view $\mathcal{O}_{A_\rho}$ as $C^*(S_x : x \in E)$, and let

$$P = \sum_{\{x \in E : s(x) = \iota\}} S_x S_x^*.$$

Our main result is:

THEOREM 3.3. *Let ρ be a representation of a finite group with $1 < \dim\rho < \infty$. Then $\mathcal{O}_\rho$ is isomorphic to the corner $P\mathcal{O}_{A_\rho}P$.*

We first have to establish the algebraic version. For it, we resume the notation of Sections 1 and 2.

LEMMA 3.4. *Suppose G is finite and $1 < \dim\rho < \infty$. Then the homomorphism ϕ of Theorem 2.1 is an isomorphism of ${}^0\mathcal{O}_\rho$ onto PBP.*

PROOF: We begin by letting

$$B_{m,n} = \mathrm{sp}\,\{S_x S_y^* : |x| = m, |y| = n, r(x_m) = r(y_n)\},$$

so that by definition $\phi_{m,n}$ maps (ρ^m, ρ^n) onto $PB_{m,n}P$ (recall that $PS_xS_y^*P = S_xS_y^*$ or 0, so $PB_{m,n}P$ is spanned by those $S_xS_y^*$ where $s(x_1) = s(y_1) = \iota$). In fact we claim that the generators $S_xS_y^*$ for $B_{m,n}$ are linearly independent, so that $\phi_{m,n}$ is a linear isomorphism. To see why, suppose $\sum_{|x|=m,|y|=n} \lambda_{x,y} S_xS_y^* = 0$ in B. If $|w| = m, |z| = n$

then $S_w^* S_z = \delta_{z,w} S_{z_m}^* S_{z_m}$ [2, 2.1], and hence

$$S_w^* \Big(\sum_{|x|=m, |y|=n} \lambda_{x,y} S_x S_y^* \Big) S_z = \lambda_{w,z} S_{w_m}^* S_{w_m} S_{z_n}^* S_{z_n}.$$

Thus $\lambda_{w,z} = 0$ whenever $|w| = m, |z| = n$ and $r(w_m) = r(z_n)$, and the $S_x S_y^*$ in $B_{m,n}$ are independent, as claimed.

The direct limit of the isomorphisms $\{\phi_{m,n}\}$ is an isomorphism

$$\phi^k : {}^0\mathcal{O}_\rho^k = \varinjlim \left(\rho^p, \rho^{p+k}\right) \to PB^k P = \varinjlim PB_{p,p+k}P = \bigcup_p PB_{p,p+k}P,$$

and to show the direct sum $\phi = \oplus \phi^k$ is an isomorphism, it is enough to show that the range PBP is the (algebraic) direct sum of the subspaces PB^kP. This is a highly nontrivial property of the algebra $\mathcal{O}_{A_\rho} = C^*(S_x)$, essentially established by Cuntz and Krieger in [2, 2.8, 2.9], and is only true because the matrix A_ρ satisfies condition (I) by Lemma 3.1.

As shown in [2, bottom of p.255], every X in B can be written in the form

$$\sum_{k=-M}^{-1} \Big(\sum_{|x|=|k|} S_x X_x \Big) + X_0 + \sum_{k=1}^{N} \Big(\sum_{|y|=k} X_y S_y^* \Big),$$

where X_0, X_x, X_y are all linear combinations of elements $S_w S_z^*$ with $|w| = |z|$. Since, for example, $\sum_{|y|=k} X_y S_y^* \in B^k$, and the recipe given in [2] shows that $X_y S_y^*$ lies in PB^kP when $X \in PBP$, our problem is to show that this expression is unique. So suppose we have written 0 as a sum

$$\sum_{k=-M}^{N} Z_k = \sum_{k=-M}^{-1} \Big(\sum_{|x|=|k|} S_x Z_x \Big) + Z_0 + \sum_{k=1}^{N} \Big(\sum_{|y|=k} Z_y S_y^* \Big).$$

If Z denotes the formal sum on the right-hand side, then $Z = 0$ implies $Z^*Z = 0$, and hence, by [2, 2.8], that the homogeneous term $(Z^*Z)_0 \in B^0$ vanishes. But this term is

$$\begin{aligned} &\sum_{k=-M}^{-1} \Big(\sum_{|x|=|k|=|x'|} Z_x^* S_x^* S_{x'} Z_{x'} \Big) + Z_0^* Z_0 + \sum_{k=1}^{N} \Big(\sum_{|y|=k} Z_y S_y^* \Big)^* \Big(\sum_{|y'|=k} Z_{y'} S_{y'}^* \Big) \\ &\quad = \sum_{k=-M}^{-1} \Big(\sum_{|x|=|k|} Z_x^* S_x^* S_x Z_x \Big) + Z_0^* Z_0 + \sum_{k=1}^{N} \Big(\sum_{|y|=k} Z_y S_y^* \Big)^* \Big(\sum_{|y|=k} Z_y S_y^* \Big). \end{aligned}$$

Because the sum of positive operators can be 0 only if each term is 0, we can deduce from this that $Z_0 = 0$ and $S_x Z_x = 0$ for each x, and hence that $Z_k = 0$ for $k < 0$. The same argument using $ZZ^* = 0$ gives $Z_y S_y^* = 0$ for each y, so that $Z_k = 0$ for $k > 0$. We have shown that, algebraically at least, $B = \bigoplus_{k\in\mathbf{Z}} B^k$ and $PBP = \bigoplus_{k\in\mathbf{Z}} PB^kP$, and it follows that $\phi = \oplus\phi^k$ is an isomorphism, as required. $\square$

PROOF OF THEOREM 3.3: Cuntz and Krieger prove the uniqueness of $\mathcal{O}_A$ by showing that the *-algebra B generated by the partial isometries has a unique C^*-norm $\|\cdot\|_B$, namely that coming from its action on H. Since we know from the Lemma that ${}^0\mathcal{O}_\rho$ is *-isomorphic to PBP, our problem is to show that the enveloping C^*-norm $\|\cdot\|_{C^*}$ on PBP coincides with $\|\cdot\|_B$ on PBP. We certainly have $\|\cdot\|_B \leqslant \|\cdot\|_{C^*}$, so it will be enough to show that, for any *-representation π of PBP, there is a *-representation τ of B such that $\|\pi(Y)\| \leqslant \|\tau(Y)\|$ for $Y \in PBP$; if so, then

$$\|Y\|_B = \|\mathrm{id} \oplus \tau(Y)\| = \sup\{\|Y\|_B, \|\tau(Y)\|\}$$

forces

$$\begin{aligned} \|Y\|_{C^*} &= \sup\{\|\pi(Y)\| : \pi \text{ is a *-representation of } PBP\} \\ &\leqslant \sup\{\|\tau(Y)\| : \tau \text{ is a *-representation of } B\} \\ &\leqslant \|Y\|_B . \end{aligned}$$

Given π, we intend to write down a formula for such a τ, but we need to do some background work first.

For each edge x, we choose a path $\alpha(x)$ starting at the vertex ι and ending at x: if $s(x) = \iota$, we insist that $\alpha(x)$ consists of the single edge x. We then define $R_x = S_x S_{\alpha(x)}^*$, so that if $s(x) = \iota$, we have $R_x = S_x S_x^*$, and in general, R_x is a partial isometry with initial projection $R_x^* R_x \leqslant P$. For single edges w, z we have $S_w^* S_z = 0$ unless $w = z$, and therefore

$$\begin{aligned} S_z^* S_y^* S_y S_z &= S_z^*(\textstyle\sum_w A(y,w) S_w S_w^*) S_z \\ &= A(y,z) S_z S_z^*, \end{aligned}$$

which is 0 or $S_z S_z^*$; since we know $\alpha(x)$ is a path, $S_{\alpha(x)} \neq 0$ and cancellation from the centre out shows

$$\begin{aligned} R_x R_x^* &= S_x \left(S_x^* \cdots S_{\alpha(x)_j}^* \cdots S_{\alpha(x)_1}^*\right) \left(S_{\alpha(x)_1} \cdots S_{\alpha(x)_j} \cdots S_x\right) S_x^* \\ &= S_x (S_x^* S_x) S_x^* \\ &= S_x S_x^*. \end{aligned}$$

Thus we have

$$1 = \sum_{x\in E} S_x S_x^* = \sum_{x\in E} R_x R_x^*. \tag{3.1}$$

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We now define $\tau : B \to B(H^E) = M_E(B(H))$ by letting $\tau(Y)$ be the $E\times E$ matrix with (x,y)-entry $\tau(Y)_{x,y} = \pi(R_x^* Y R_y)$; because both $R_x^*R_x$ and $R_y^*R_y$ are dominated by P, $R_x^*YR_y$ lies in PBP, and we can legitimately apply π to it. We claim τ is a *-homomorphism: it is clearly linear, equation (3.1) implies that it is multiplicative:

$$\begin{aligned}(\tau(Y)\tau(Z))_{x,z} &= \textstyle\sum_y \pi(R_x^*YR_y)\pi(R_y^*ZR_z)\\ &= \pi\Big(R_x^*Y\Big(\textstyle\sum_y R_yR_y^*\Big)ZR_z\Big)\\ &= \pi(R_x^*(YZ)R_z)\\ &= \tau(YZ)_{x,z},\end{aligned}$$

and it is easily seen to preserve adjoints:

$$\big(\tau(Y)^*\big)_{x,z} = \big(\tau(Y)_{z,x}\big)^* = \pi(R_z^*YR_x)^* = \pi(R_x^*Y^*R_z) = \tau(Y^*)_{x,z}.$$

Finally, note that because $R_x = S_xS_x^*$ when $x \in I = \{x \in E : s(x) = \iota\}$, we have $P = \sum\limits_{x\in I} R_x = \sum\limits_{x\in I} R_x^*$, and hence for $Y \in PBP$

$$\pi(Y) = \sum_{x,y\in I} \pi(R_x^*YR_y).$$

Since the ranges of the partial isometries R_y are mutually orthogonal, the norm of this sum is equal to the norm of the $I \times I$ matrix

$$(\pi(R_x^*YR_y))_{x,y\in I} \in M_I(B(H));$$

but this is a submatrix of the $E \times E$ matrix $\tau(Y)$, and hence

$$\|\pi(Y)\| = \Big\|(\pi(R_x^*YR_y))_{x,y\in I}\Big\| \leqslant \|\tau(Y)\|,$$

as required. ∎

Corollary 3.5. *For any representation ρ of a finite group satisfying $1 < \dim\rho < \infty$, $\mathcal{O}_\rho$ is a simple C^*-algebra which is Morita equivalent to the corresponding $\mathcal{O}_{A_\rho}$.*

Proof: We have already shown that $A = A_\rho$ is irreducible and satisfies condition (I), so $\mathcal{O}_A$ is simple by [**2**, Theorem 2.14]. Thus the corner $P\mathcal{O}_AP$ is full — there is no nontrivial ideal which can contain it. This implies that the $\mathcal{O}_A$–$P\mathcal{O}_AP$ bimodule $\mathcal{O}_AP$ is an imprimitivity bimodule with the inner products

$$\begin{aligned}\langle XP, YP\rangle_{P\mathcal{O}_AP} &= PX^*YP,\\ {}_{\mathcal{O}_A}\langle XP, YP\rangle &= XPY^*;\end{aligned}$$

the fullness of $P\mathcal{O}_AP$ says precisely that the span of the range of the $\mathcal{O}_A$-valued inner product is dense in $\mathcal{O}_A$. Thus the result follows from the Theorem. ∎

4. The K-theory of Doplicher-Roberts algebras

We want to compute the K-theory of a Doplicher-Roberts algebra $\mathcal{O}_\rho$ using Cuntz's computation of $K_*(\mathcal{O}_{A_\rho})$, which is isomorphic to $K_*(\mathcal{O}_\rho)$ because the C^*-algebras are Morita equivalent. The key result is [1, Proposition 3.1], which asserts that $K_0(\mathcal{O}_A)$ and $K_1(\mathcal{O}_A)$ are, respectively, the cokernel and kernel of the map $1 - A^t : \mathbf{Z}^E \to \mathbf{Z}^E$. Now when we constructed A_ρ from the bipartite graph, we chose to use the set E of edges rather than the set R of vertices as our index set. This has the advantage that A_ρ is always a $\{0,1\}$-matrix, as opposed to an integer matrix, but the disadvantage that E is usually a lot bigger than R, which makes calculations messier. So we want to first show that either matrix can be used in our calculation of K-theory. In fact this is quite generally true: if A, B are the two matrices associated to any bipartite graph, then $1 - A^t$, $1 - B^t$ have the same kernel and cokernel, and if both are $\{0,1\}$-matrices, they give isomorphic Cuntz-Krieger algebras. These facts are surely well-known — for example, they are implicit in the way Cuntz and Krieger handle general integer matrices [2, 2.16] — but we do not know where the details have been written down.

Suppose, then, that we have a bipartite graph with vertices V, edges E and range, source maps $r, s : E \to R$. We define

$$B(i,j) = \#\{x \in E : s(x) = i, r(x) = j\}$$
$$A(x,y) = \begin{cases} 1 & \text{if } r(x) = s(y) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.1. *(1) If B is a $\{0,1\}$-matrix satisfying (I), then A satisfies (I) and $\mathcal{O}_B \cong \mathcal{O}_A$.*

(2) There are isomorphisms

$$\ker\left((1 - B^t) : \mathbf{Z}^V \to \mathbf{Z}^V\right) \cong \ker\left((1 - A^t) : \mathbf{Z}^E \to \mathbf{Z}^E\right)$$
$$\mathbf{Z}^V/(1 - B^t)(\mathbf{Z}^V) \cong \mathbf{Z}^E/(1 - A^t)(\mathbf{Z}^E).$$

Proof: If B has entries in $\{0,1\}$, paths of vertices are essentially the same as paths of edges, and the first assertion is essentially clear. For the second, suppose S_i are partial isometries satisfying

$$S_i^* S_i = \sum_{j \in V} B(i,j) S_j S_j^*,$$

and define $T_x = S_{s(x)} S_{r(x)} S_{r(x)}^*$. Then certainly each T_x is a partial isometry in $C^*(S_i)$, and

$$\begin{aligned} S_i = S_i S_i^* S_i &= \sum_{j \in V} B(i,j) S_i S_j S_j^* \\ &= \sum_{\{j : B(i,j)=1\}} S_i S_j S_j^* \\ &= \sum_{\{x : s(x) = i\}} S_{s(x)} S_{r(x)} S_{r(x)}^*, \end{aligned}$$

since $B(i,j) = 1$ if and only if there is is an edge x from i to j. Thus $C^*(S_i) = C^*(T_x)$. We now verify that the T_x generate $\mathcal{O}_A$. On the one hand,

$$\begin{aligned} \sum_{y \in E} A(x,y) T_y T_y^* &= \sum_{\{y : s(y) = r(x)\}} S_{s(y)} \left(S_{r(y)} S_{r(y)}^* \right)^2 S_{s(y)}^* \\ &= S_{r(x)} \Big(\sum_{\{y : s(y) = r(x)\}} S_{r(y)} S_{r(y)}^* \Big) S_{r(x)}^* \\ &= S_{r(x)} \Big(\sum_{\{j : B(r(x), j) = 1\}} S_j S_j^* \Big) S_{r(x)}^* \\ &= S_{r(x)} \left(S_{r(x)}^* S_{r(x)} \right) S_{r(x)}^* \\ &= S_{r(x)} S_{r(x)}^*; \end{aligned}$$

on the other, since the S_i have mutually orthogonal ranges, we also have

$$\begin{aligned} T_x^* T_x &= S_{r(x)} S_{r(x)}^* \left(S_{s(x)}^* S_{s(x)} \right) S_{r(x)} S_{r(x)}^* \\ &= S_{r(x)} S_{r(x)}^* \Big(\sum_{\{j : B(s(x), j) = 1\}} S_j S_j^* \Big) S_{r(x)} S_{r(x)}^* \\ &= S_{r(x)} S_{r(x)}^*, \end{aligned}$$

so the T_x do satisfy the Cuntz-Krieger relations for A. Thus by the Cuntz-Krieger uniqueness theorem we have

$$\mathcal{O}_B \cong C^*(S_i) = C^*(T_x) \cong \mathcal{O}_A,$$

giving (1).

To establish (2), we use the source and range maps to define $V \times E$ and $E \times V$ matrices:

$$S(i,x) = \begin{cases} 1 & \text{if } s(x) = i \\ 0 & \text{otherwise} \end{cases}$$

$$R(x,i) = \begin{cases} 1 & \text{if } r(x) = i \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$(RS)(x,y) = \sum_i R(x,i)S(i,y),$$

and since each summand is 0 or 1,

$$\begin{aligned} R(x,i)S(i,y) = 1 &\Leftrightarrow R(x,i) = 1 = S(i,y) \\ &\Leftrightarrow r(x) = i = s(y). \end{aligned}$$

For each fixed pair (x,y), this can happen for exactly one i, and hence we can deduce that $RS = A$. Similarly,

$$\begin{aligned} (SR)(i,j) &= \sum_x S(i,x)R(x,j) \\ &= \#\{x \in E : S(i,x) = 1 = R(x,j)\} \\ &= \#\{x \in E : s(x) = i, r(x) = j\}, \end{aligned}$$

and $SR = B$. Of course, we also have $R^tS^t = B^t$, $S^tR^t = A^t$, and hence the following standard lemma gives what we need:

LEMMA 4.2. *Suppose R, S are $V \times E, E \times V$ matrices with entries in $\{0,1\}$, and $B = RS \in M_V(\mathbf{Z})$, $A = SR \in M_E(\mathbf{Z})$. Then the transformation $S : \mathbf{Z}^V \to \mathbf{Z}^E$ induces isomorphisms of* $\ker\big((1-B) : \mathbf{Z}^V \to \mathbf{Z}^V\big)$ *onto* $\ker(1-A)$, *and* $\operatorname{coker}(1-B) = \mathbf{Z}^V/(1-B)(\mathbf{Z}^V)$ *onto* $\operatorname{coker}(1-A)$.

PROOF: We first observe that, for each $\lambda \neq 0$, $S : \mathbf{R}^V \to \mathbf{R}^E$ is an isomorphism of the eigenspace

$$E_\lambda^B = \{v \in \mathbf{R}^V : Bv = \lambda v\}$$

onto $E_\lambda^A \subset \mathbf{R}^E$, with inverse given by $\lambda^{-1}R$. Since both R, S have integer entries, it follows that S restricts to an isomorphism of $\ker(1-B) = E_1^B \cap \mathbf{Z}^V$ onto $\ker(1-A) = E_1^A \cap \mathbf{Z}^E$ with inverse R. Next, we note that if $z \in \operatorname{im}(1-B)$, say $z = (1-B)v$, then

$$Sz = S(1-RS)v = (1-SR)Sv = (1-A)(Sv),$$

so S does map $\operatorname{im}(1-B)$ into $\operatorname{im}(1-A)$, and induces a homomorphism ϕ of $\operatorname{coker}(1-B)$ into $\operatorname{coker}(1-A)$. In the same way, R induces a homomorphism ψ of $\operatorname{coker}(1-A)$ into $\operatorname{coker}(1-B)$, which we claim is an inverse for ϕ. For

$$\begin{aligned} \psi \circ \phi(v + \operatorname{im}(1-B)) &= SRv + \operatorname{im}(1-B) \\ &= v - (v - SRv) + \operatorname{im}(1-B) \\ &= v + \operatorname{im}(1-B), \end{aligned}$$

and similarly $\phi \circ \psi$ is the identity on $\operatorname{coker}(1-A)$. ∎

This lemma completes the proof of Proposition 4.1. ∎

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EXAMPLE 4.3. $G = S_3$. The character table of S_3 is

	e	(12)	(123)
ι:	1	1	1
σ:	1	-1	1
π:	2	0	-1

The obvious representation to take for ρ is the 2-dimensional representation π: it is faithful because

$$\ker \pi = \{s \in G : \chi_\pi(s) = \chi_\pi(e) = 2\} = \{e\}$$

[5, (2.19)]. We trivially have $\iota^2 = \iota$, $\iota \otimes \sigma = \sigma$, $\iota \otimes \pi \sim \pi$, and $\sigma^2 = \iota$; the characters of the other tensor products are given by

$$\begin{gathered} \chi_{\sigma\otimes\pi} = \chi_\sigma \chi_\pi = \chi_\pi, \quad \text{and} \\ \chi_{\pi\otimes\pi} = (\chi_\pi)^2 = \chi_\iota + \chi_\sigma + \chi_\pi, \end{gathered}$$

and since the decomposition of the character determines the decomposition of the representation [5, (2.9)], we have $\sigma \otimes \pi \sim \pi$ and $\pi^2 \sim \iota \oplus \pi \oplus \sigma$. We therefore have

$$B_\pi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad 1 - B_\pi^t = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Since $\det(1 - B_\pi^t) = 2$, $\ker(1 - B_\pi^t) = 0$ and $K_1(\mathcal{O}_\pi) \cong K_1(\mathcal{O}_{B_\pi}) = 0$. However, for $(m, n, p) \in \mathbf{Z}^3$, the unique solution v of $(1 - B^t)v = (m, n, p)$ in $\mathbf{R}^3$ is

$$v = \left(\frac{m - n - p}{2}, \frac{-m - n - p}{2}, \frac{-m - n + p}{2}\right),$$

which lies in $\mathbf{Z}^3$ if and only if $m + n + p \in 2\mathbf{Z}$. Thus

$$(m, n, p) \to (m + n + p) + 2\mathbf{Z}$$

induces an isomorphism of $K_0(\mathcal{O}_\pi) \cong K_0(\mathcal{O}_{B_\pi}) \cong \mathbf{Z}^3/(1 - B_\pi^t)(\mathbf{Z}^3)$ onto $\mathbf{Z}_2$.

If we take for ρ the faithful representation $\pi \oplus \iota$, we have instead

$$B_\rho = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad 1 - B_\rho^t = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Thus for this choice of ρ,

$$K_1(\mathcal{O}_\rho) \cong K_1(\mathcal{O}_{B_\rho}) \cong \ker(1 - B_\rho^t) \cong \mathbf{Z},$$

and the map $(m,n,p) \to m - p$ induces an isomorphism

$$K_0(\mathcal{O}_\rho) \cong K_0(\mathcal{O}_{B_\rho}) \cong \mathbf{Z}^3/(1 - B_\rho^t)(\mathbf{Z}^3) \cong \mathbf{Z}.$$

Alternatively, if $\rho = \pi \oplus \sigma$, we have

$$1 - B_\rho^t = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

Here $\det(1 - B_\rho^t) = -4$, so $K_1(\mathcal{O}_\rho) = 0$, but $(1 - B_\rho^t)v = (m,n,p)$ has solution

$$v = \left(\frac{m-n}{2}, \frac{-m-p}{2}, p - n\right),$$

and $(m,n,p) \to (m-n, -m-p)$ induces an isomorphism of $\operatorname{coker}(1 - B_\rho^t) \cong K_0(\mathcal{O}_\rho)$ onto $\mathbf{Z}_2 \times \mathbf{Z}_2$.

EXAMPLE 4.4. $G = A_5 \cong PSL(2,5) \cong SL(2,4)$. It is important in the work of Doplicher and Roberts that the representation ρ is faithful and special unitary, and we shall now discuss an example where there are several irreducible representations of this kind — indeed, since this group has only the trivial one-dimensional representation, $s \to \det \pi(s)$ is always identically 1, and any representation is special unitary. We write π_i $(1 \leqslant i \leqslant 5)$ for the irreducible representations, with $\pi_1 = \iota$, and χ_i for the corresponding characters. Then the character table for A_5 is:

	1	2	3	5_1	5_2
$\chi_1 = \iota$:	1	1	1	1	1
χ_2 :	4	0	1	-1	-1
χ_3 :	5	1	-1	0	0
χ_4 :	3	-1	0	α_1	α_2
χ_5 :	3	-1	0	α_2	α_1

where $\alpha_1 = (1+\sqrt{5})/2$, $\alpha_2 = (1-\sqrt{5})/2$. Calculating as in the previous example with $\rho = \pi_2$ gives

$$B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad 1 - B_2^t = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 1 & -1 \\ 0 & -1 & -1 & -1 & 1 \end{pmatrix}$$

The rank of $1 - B_2^t$ is 4, with

$$K_1(\mathcal{O}_{\pi_2}) \cong \ker\left(1 - B_2^t\right) = \{(n, n, -n, 0, 0)\} \cong \mathbf{Z}.$$

Given $\mathbf{m} = (m, n, p, q, r) \in \mathbf{Z}^5$, the equation $(1 - B_2^t)v = \mathbf{m}$ has a solution in $\mathbf{R}^5$ only if $p = n + m$, and then the solution space in $\mathbf{R}^5$ is

$$\{t(1, 1, -1, 0, 0) + \left(m, 0, \frac{-q-r}{2}, \frac{q-p}{2}, \frac{r-p}{2}\right)\};$$

it follows that

$$(m, n, p, q, r) \to (m + n - p, q - p \bmod 2, r - p \bmod 2)$$

induces an isomorphism of $K_0(\mathcal{O}_{\pi_2}) \cong \mathbf{Z}^5/(1 - B_2^t)(\mathbf{Z}^5)$ onto $\mathbf{Z} \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

Next we take $\rho = \pi_4$. This time

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 1 - B_4^t = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

We have $\det(1 - B_4^t) = 4$, so $\ker(1 - B_4^t) = 0 = K_1(\mathcal{O}_{\pi_4})$, and if $\mathbf{m} = (m, n, p, q, r)$, then $(1 - B_4^t)v = \mathbf{m}$ has unique solution

$$\left(\frac{r-p-3q+m+2n}{4}, \frac{-r-p+q+m}{2}, \frac{-r+p-q-m-2n}{4}, \right.$$
$$\left. \frac{r-p-3q-3m+2n}{4}, \frac{r-p+q+m-2n}{4}\right)$$

which lies in $\mathbf{Z}^5$ if and only if $r - p - 3q + m + 2n \in 4\mathbf{Z}$; thus

$$K_0(\mathcal{O}_{\pi_4}) \cong K_0(\mathcal{O}_{B_4}) \cong \mathbf{Z}^5/\left(1 - B_4^t\right)\left(\mathbf{Z}^5\right) \cong \mathbf{Z}/4\mathbf{Z}.$$

In particular, the K-groups of $\mathcal{O}_{\pi_4}$ and $\mathcal{O}_{\pi_2}$ are quite different, even though both π_4 and π_2 are faithful, irreducible, special unitary representations of A_5.

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