POSTULATES FOR DISTRIBUTIVE LATTICES

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MANY sets of postulates have been given for distributive lattices and for Boolean algebra. For a description of some of the most interesting and for references to others the reader is referred to Birkhoff's "Lattice Theory"[1]. In this paper we give sets of postulates which have some intrinsic interest because of their simplicity. In the first two sections binary operations are used to describe a distributive lattice by 2 identities in 3 variables and a Boolean algebra by 3 identities in 3 variables. In the third section a ternary operation is used to describe distributive lattices with O and I by 2 identities in 5 variables.

1. Distributive lattices. Let $\mathfrak{S}$ be a set of elements $a, b, c, \ldots$ closed under the operations $\cup$ and $\cap$ and satisfying, for all a, b, c in $\mathfrak{S}$, these postulates:

P1. $$a = a \cap (a \cup b),$$

P2. $$a \cap (b \cup c) = (c \cap a) \cup (b \cap a).$$

We wish to prove $\mathfrak{S}$ is a distributive lattice. In identities (1.1), (1.2), and (1.3) below, A denotes $a \cap a$.

(1.1) $$a = a \cap (a \cup a) = A \cup A, \qquad \text{by P1 and P2.}$$

(1.2) $$a = a \cap a.$$

Proof. $$A = A \cap (A \cup A) = A \cap a, \qquad \text{by P1 and (1.1).}$$

Hence $$\begin{aligned} a \cap a &= a \cap (A \cup A) \\ &= (A \cap a) \cup (A \cap a) \\ &= A \cup A = a, \qquad \text{by (1.1), P2, and (1.1).} \end{aligned}$$

(1.3) $$a = A \cup A = a \cup a, \qquad \text{by (1.1) and (1.2).}$$

(1.4) $$a \cap b = b \cap a.$$

Proof. $$\begin{aligned} a \cap b &= (a \cap b) \cup (a \cap b) \qquad \text{by (1.3), P2, and (1.3).} \\ &= b \cap (a \cup a) = b \cap a, \end{aligned}$$

(1.5) $$a = (b \cap a) \cup a.$$

Proof. $$\begin{aligned} a &= a \cap (a \cup b) \\ &= (b \cap a) \cup a, \qquad \text{by P1, P2, and (1.2).} \end{aligned}$$

(1.6) $$a = a \cup [(b \cap a) \cap a].$$

Proof. $$\begin{aligned} a &= a \cap a = a \cap [(b \cap a) \cup a] \\ &= a \cup [(b \cap a) \cap a], \qquad \text{by (1.2), (1.5), P2, and (1.2).} \end{aligned}$$

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(1.7) $$a \cup b = [b \cup (a \cap b)] \cup a.$$

Proof.
$$\begin{aligned} a \cup b &= (a \cup b) \cap (a \cup b) \\ &= [b \cap (a \cup b)] \cup [a \cap (a \cup b)] \\ &= [b \cup (a \cap b)] \cup a, \qquad \text{by (1.2), P2, P2, (1.2) and P1.} \end{aligned}$$

(1.8) $$b = b \cup (a \cap b).$$

Proof.
$$\begin{aligned} b &= (a \cap b) \cup b \\ &= \{b \cup [(a \cap b) \cap b]\} \cup (a \cap b) \\ &= b \cup (a \cap b), \qquad \text{by (1.5), (1.7), and (1.6).} \end{aligned}$$

(1.9) $$a \cup b = b \cup a, \qquad \text{by (1.7) and (1.8).}$$

Since the remainder of the exposition has a pattern common to several previous expositions [1, pp. 135, 136.], we proceed giving somewhat less detail. We have proved the so-called idempotent, commutative, and absorption laws. The associative laws remain to be proved.

We denote $(a \cup b) \cup c$ by P and $a \cup (b \cup c)$ by Q. It is routine to show that $a \cap P = a$, $b \cap P = b$, and $c \cap P = c$. Hence,

$$\begin{aligned} Q &= (a \cap P) \cup [(b \cap P) \cup (c \cap P]) \\ &= (a \cap P) \cup [(b \cup c) \cap P] = Q \cap P. \end{aligned}$$

By left-right symmetry, $Q \cap P = P$. Thus we have $\cup$ associativity and it is now easy to deduce the dual of the distributive law. By duals of proofs previously given, we may prove $\cap$ associativity. We then have

(1.10) $\mathfrak{S}$ is a distributive lattice.

2. Distributive lattices with O and I. In this section we note some immediate extensions of the postulate system P1, P2. Consider the postulates:

P3. $a \cup O = a$, for some O.

P3′. $a \cap I = a$, for some I.

P3″. $O \cup (a \cap I) = a$, for some O and some I.

P3*. To each b there corresponds some b' such that

$$a \cap (b \cup b') = a \cup (b \cap b').$$

Using (1.10) it is easy to prove the following statements. An algebraic system which satisfies P1, P2 and

(i) P3, is a distributive lattice with O,

(ii) P3′, is a distributive lattice with I,

(iii) P3″, is a distributive lattice with O and I,

(iv) P3*, is a Boolean algebra.

In case (i), we have $a \cap O = (a \cup O) \cap O = O$. Moreover O is unique for if an element O' shares the properties of O, then $O = O' \cup O = O'$. Case (ii) is the dual of case (i).

In case (iii), we have $O \cup a = O \cup [O \cup (a \cap I)] = O \cup (a \cap I) = a$ and hence $a = O \cup (a \cap I) = a \cap I$. Thus P3″ implies P3 and P3′.

In case (iv), denote $b \cup b'$ by I and $b \cap b'$ by O. From $a \cap I = a \cup O$, $O \cup (a \cap I) = O \cup (a \cup O) = a \cup O = a \cup (a \cup O) = a \cup (a \cap I) = a$. Hence P3* implies P3″. It is a routine matter to show that the complement, b', of b is unique.

3. Postulates with a ternary operation. The ternary operation used here is the one introduced by Grau [3]. Kiss and Birkhoff [4] have described distributive lattices with O and I in terms of the operation. Croisot [2], using this operation and 5 variables, defines a Boolean algebra by means of 2 identities and a distributive lattice with O and I by means of 3 identities (see Problem 64 in [1]). In the latter case also, it happens that 2 identities are sufficient. We give the result without proof.

Let $\mathfrak{S}$ be an algebraic system with a ternary operation (a, b, c) and with elements O and I such that, identically,

Q1. $$(O, a, (I, b, I)) = a,$$

Q2. $$(a, (b, c, d), e) = ((a, c, e), d, (b, a, e)).$$

If we define $a \cup b = (a, I, b)$ and $a \cap b = (a, O, b)$, then S is a distributive lattice with O and I. Moreover,

$$(a, b, c) = (a \cap b) \cup (b \cap c) \cup (c \cap a).$$

References

[1] Garrett Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications, vol. XXV, 1948.

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