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RINGS WHOSE ADDITIVE ENDOMORPHISMS ARE N-MULTIPLICATIVE

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Sullivan's problem of describing rings, all of whose additive endomorphisms are multiplicative, is generalised to the study of rings R satisfying $\varphi(a_1 \dots a_n) = \varphi(a_1) \dots \varphi(a_n)$ for every additive endomorphism φ of R, and all $a_1, \dots, a_n \in R$, with $n > 1$ a fixed positive integer. It is shown that such rings possess a bounded (finite) ideal A such that $[R/A]^n = 0$ $\left([R/A]^{2n-1} = 0\right)$. More generally, if $f(X_1, \dots, X_t)$ is a homogeneous polynomial with integer coefficients, of degree > 1, and if a ring R satisfies $\varphi[f(a_1, \dots, a_t)] = f[\varphi(a_1), \dots, \varphi(a_t)]$ for all additive endomorphisms φ, and all $a_1, \dots, a_t \in R$, then R possesses a bounded ideal A such that R/A satisfies the polynomial identity f.

Notation.

$Z(n)$	a cyclic additive group of order n.
R	a ring.
$R[n]$	$\{x \in R \mid nx = 0\}$, n a positive integer.
R^+	the additive group of R.
R_t	the torsion part of R^+.
R_p	the p-primary component of R^+, p a prime.
R_P	$\bigoplus_{p \in P} R_p$, P a set of primes.
a_p	the p-primary component of $a \in R_t$, p a prime.
P_n	$\{p \text{ a prime} \mid n \equiv 1 (\text{mod } p-1)\}$, n a positive integer.
End (R^+)	the ring of endomorphims of R^+.
End (R)	the semigroup of ring endomorphisms of R.

Sullivan [4] asked for a description of the rings R satisfying $\text{End}(R) = \text{End}(R^+)$. Kim and Roush [3] classified the finite rings satisfying Sullivan's property. In [1] the torsion rings satisfying $\text{End}(R) = \text{End}(R^+)$ were completely described, and very restrictive necessary conditions were obtained for a general ring to satisfy this property.

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In this note, the following generalisation of Sullivan's problem will be considered. Let $n \geqslant 1$ be a positive integer. Which rings R satisfy $\varphi(a_1 \ldots a_n) = \varphi(a_1) \ldots \varphi(a_n)$ for all $\varphi \in \mathrm{End}(R^+)$ and all $a_1, \ldots, a_n \in R$? More generally, for $f(X_1, \ldots, X_t)$ a polynomial with integer coefficients, which rings R satisfy $\varphi[f(a_1, \ldots, a_t)] = f[\varphi(a_1), \ldots, \varphi(a_t)]$ for all $\varphi \in \mathrm{End}(R^+)$, and all $a_1, \ldots, a_t \in R$? Rings satisfying the polynomial identity f clearly satisfy this property. It will be shown that for f a homogeneous polynomial the converse is "almost" true.

DEFINITION: Let $n > 1$ be a positive integer. A ring R is said to be an AE_n-ring, (additive endomorphisms are n-multiplicative), if $\varphi(a_1 \ldots a_n) = \varphi(a_1) \ldots \varphi(a_n)$ for all $\varphi \in \mathrm{End}(R^+)$, and all $a_1, \ldots, a_n \in R$.

The proof of the following lemma is due to Muskat (through personal communication).

LEMMA 1. *Let $n > 1$ be a positive integer. A prime q satisfies $q \mid p^n - p$ for all primes p if and only if $q \in P_n$.*

PROOF: Suppose that $q \in P_n$, that is $n \equiv 1 (\mathrm{mod}\ q-1)$. For p an arbitrary prime $p^{q-1} - 1 \mid p^{n-1} - 1$. Clearly it may be assumed that $p \neq q$. By the Little Fermat Theorem $p^{q-1} \equiv 1 (\mathrm{mod}\ q)$, that is $q \mid p^{q-1} - 1$, which implies that $q \mid p^{n-1} - 1$, which in turn yields that $q \mid p^n - p$.

Conversely, suppose that the prime $q \notin P_n$, that is $n \not\equiv 1 (\mathrm{mod}\ q-1)$. Let g be a primitive root of the congruence $X^{q-1} \equiv 1 (\mathrm{mod}\ q)$. By Dirichlet's Theorem the sequence $\{q + kg \mid k = 1, 2, \ldots\}$ contains a prime $p \neq q$. Since $q - 1 \nmid n - 1$, it follows that $p^{n-1} \not\equiv 1 (\mathrm{mod}\ q)$, and so $q \nmid p(p^{n-1} - 1)$. ∎

THEOREM 2. *Let R be an AE_n-ring. Then $R^n \subseteq \bigoplus_{p \in P_n} R[p]$.*

PROOF: Let p be a prime. The map $R^+ \to R^+$ via $x \mapsto px$ belongs to $\mathrm{End}(R^+)$, so for all $a_1, \ldots, a_n \in R$, the equation $pa_1 \ldots a_n = p^n a_1 \ldots a_n$ is satisfied, that is $(p^n - p)R^n = 0$. It follows from Lemma 1 that $R^n \subseteq R_{P_n}$. Let $a \in R_p^n$, p a prime. Then $p(p^{n-1} - 1)a = 0$. Since $p \nmid p^{n-1} - 1$, it follows that $pa = 0$, and so $R^n \subseteq \bigoplus_{p \in P_n} R[p]$. ∎

LEMMA 3. *Let R be an AE_n-ring, and let H be a direct summand of R^+. Then $R^k H R^{n-k-1} \subseteq H$ for all $0 \leqslant k \leqslant n-1$.*

PROOF: Suppose that $R^+ = H \oplus K$. Let π_K be the natural projection of R^+ onto K along H. For $a_1, \ldots, a_{n-1} \in R$, and $h \in H$, the fact that $\pi_K \in \mathrm{End}(R^+)$ yields that

$$\pi_K(a_1 \ldots a_k h a_{k+1} \ldots a_{n-1}) = \pi_K(a_1) \ldots \pi_K(a_K) \pi_K(h) \pi_K(_{k+1}) \ldots \pi_K(a_{n-1}).$$

Since $\pi_K(h) = 0$, it follows that $\pi_K(R^k H R^{n-k-1}) = 0$, that is $R^k H R^{n-k-1} \subseteq H$. ∎

Since P_n is a finite set of primes, Theorem 2 implies that an AE_n-ring R is nilpotent modulo a bounded ideal in R. Actually, if R is AE_n, then R is nilpotent modulo a finite ideal in R.

THEOREM 4. *Let R be an AE_n-ring, and let $P = \{p \in P_n \mid R_p^+ = Z(p)\}$. Then $R^{2n-1} \subseteq R_P$.*

PROOF: It may be assumed that $R^{2n-1} \neq 0$. Let $a_1, \ldots, a_{2n-1} \in R$ such that $a = \prod_{i=1}^{2n-1} a_i \neq 0$. Let $b = \prod_{n=1}^{n} a_i$, and $c = \prod_{i=n+1}^{2n-1} a_i$. It follows from Theorem 2 that $b = \sum_{p \in P_n} b_p$ with $|b_p| = p$ or $|b_p| = 0$ for all $p \in P_n$. Let $p \in P_n$ such that $a_p \neq 0$. Suppose that b_p has nonzero p-height, that is $b = pb'$ for some $b' \in R$ and $a = pb'c$. Since $b'c \in R^n$, it follows that $(b'c)_p \in R[p]$ by Theorem 2. Hence $a_p = p(b'c)_p = 0$, a contradiction. Therefore $R^+ = (b_p) \oplus H$ with (b_p) the cyclic group of order p generated by b_p, [**2**, Proposition 27.1]. Let $d \in R^+$, with $|d| = p$, and let $\varphi\colon R^+ \to R^+$ be the endomorphism induced by the maps $b_p \mapsto d$, and $h \mapsto 0$ for all $h \in H$. Then $d = \varphi(b) = \prod_{i=1}^{n} \varphi(a_i)$. Since $\varphi(a_i) \neq 0$, it follows that $\varphi(a_i) = k_i d$, with $1 \leqslant k_i \leqslant p-1$ for all $1 \leqslant i \leqslant n$. Hence $d = kd^n$ with $k = \prod_{i=1}^{n} k_i$. Since $p \nmid k$ it follows that $(d^n) = (d)$, that is, $d^n = md$, with $1 \leqslant m \leqslant p-1$. If d has nonzero p-height, then $d = pd'$ for some $d' \in R_p$, and $d^n = p^n (d')^n = 0$ by Theorem 2, a contradiction. Therefore every element d of order p in R^+ generates a cyclic direct summand of R^+, and $d^n = md$, with $1 \leqslant m \leqslant p-1$. This implies that $R^+ = \bigoplus_{i \in I} (a_i) \oplus K$ with $|a_i| = p$, $a_i^n = m_i a_i$ with $1 \leqslant m_i \leqslant p-1$ for all $i \in I$, and $K_p = 0$. If $|I| > 1$, then $R^+ = (a_1) \oplus (a_2) \oplus L$ with $1, 2 \in I$. Let $\Psi\colon R^+ \to R^+$ be the endomorphism induced by the maps $a_i \mapsto a_1$ for $i = 1, 2$, and $x \mapsto 0$ for $x \in L$. Then $\Psi(a_1^{n-1} a_2) = a_1^n = m_1 a_1 \neq 0$. However, $a_1^{n-1} a_2 \in [(a_1) R^{n-1}] \cup [R^{n-1}(a_2)]$. Lemma 3 yields that $(a_1) R^{n-1} \subseteq (a_1)$, and $R^{n-1}(a_2) \subseteq (a_2)$, that is, $a_1^{n-1} a_2 = 0$, and so $\Psi(a_a^{n-1} a_2) = 0$, a contradiction. ∎

A slight modification of the proof of Theorem 2 yields:

THEOREM 5. *Let $f(X_1, \ldots, X_t)$ be a homogeneous polynomial of degree $n > 1$ with integer coefficients, and let m be the greatest common divisor of the coefficients of f. If $\varphi[f(a_1, \ldots, a_t)] = f[\varphi(a_1), \ldots, \varphi(a_t)]$ for all $\varphi \in \operatorname{End}(R^+)$ and all $a_1, \ldots, a_t \in R$, then*

$$R/\{\bigoplus_{\substack{p \in P_n \\ p \nmid m}} R[p] \oplus \bigoplus_{\substack{p \notin P_n \\ p \mid m}} R_p[p^{k_p}] \oplus \bigoplus_{\substack{p \in P_n \\ p \mid m}} R_p[p^{k_p+1}]\}$$

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satisfies the polynomial identity f, where each p is a prime, and p^{k_p} is the greatest power of p dividing m.

If S is a set of homogeneous polynomials satisfying the conditions of Theorem 5, then there exists a torsion ideal $A \trianglelefteq R$ such that R/A satisfies all the polynomial identities $f \in S$. If S is finite, then the ideal A obtained is bounded.

The following example shows that the homogeneity condition in Theorem 5 cannot be eliminated.

Example 6. Let G be a non-torsion additive group, and let R be the zeroring with $R^+ = G$, that is, $R^2 = 0$. Then $\varphi(a^2 - a) = [\varphi(a)]^2 - \varphi(a) = -\varphi(a)$ for all $\varphi \in \operatorname{End}(R^+)$, and all $a \in R$. However R/R_t clearly does not satisfy the polynomial identity $x^2 - x$.

Any polynomial with integer coefficients and possessing a nonzero linear summand provides a counterexample to Theorem 5, similar to Example 6. If $f(X_1, \ldots, X_t)$ is a sum of monomials each with integer coefficient and degree > 1, and $\varphi[f(a_1, \ldots, a_t)] = f[\varphi(a_1), \ldots, \varphi(a_t)]$ for all $\varphi \in \operatorname{End}(R^+)$ and all $a_1, \ldots, a_t \in R$, does R/R_t satisfy the polynomial identity f?

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