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Global boundedness and large time behaviour in a higher-dimensional quasilinear chemotaxis system with consumption of chemoattractant

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This paper deals with the following quasilinear chemotaxis system with consumption of chemoattractant

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u\nabla v), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, \quad t > 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N (N = 3, 4, 5)$ with smooth boundary $\partial\Omega$. It is shown that if $m > \max\{1, \frac{3N-2}{2N+2}\}$, for any reasonably smooth nonnegative initial data, the corresponding no-flux type initial-boundary value problem possesses a globally bounded weak solution. Furthermore, we prove that the solution converges to the spatially homogeneous equilibrium $(\bar{u}_0, 0)$ in an appropriate sense as $t \to \infty$, where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0$. This result not only partly extends the previous global boundedness result in Fan and Jin (*J. Math. Phys.* **58** (2017), 011503) and Wang and Xiang (*Z. Angew. Math. Phys.* **66** (2015), 3159–3179) to $m > \frac{3N-2}{2N}$ in the case $N \geqslant 3$, but also partly improves the global existence result in Zheng and Wang (*Discrete Contin. Dyn. Syst. Ser. B* **22** (2017), 669–686) to $m > \frac{3N}{2N+2}$ when $N \geqslant 2$.

Keywords: boundedness; chemotaxis; weak solution; large time behaviour; quasilinear

2020 *Mathematics Subject Classification* Primary: 35B40
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1. Introduction

Chemotaxis is one of the most important components in the process of reproduction and migration, it describes the biased movement of biological species or cells towards chemotaxis substances. In this paper, we study the following quasilinear chemotaxis system with consumption of chemoattractant

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u\nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0, \\ (\nabla u^m - u\nabla v) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, where $m > 1$ is a constant, ν denotes the outer normal derivative on $\partial\Omega$, $u(x, t)$ and $v(x, t)$ denote the density of cells population and the concentration of oxygen, respectively. And the initial data (u_0, v_0) satisfies

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \quad \text{with} \quad u_0 \geqslant 0 \text{ and } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{with } v_0 \geqslant 0 \quad \text{in} \quad \bar{\Omega}. \end{cases} \tag{1.2}$$

To better understand the chemotaxis model (1.1), we recall several previous works. Firstly, we recall the following system that has been studied for more than ten years,

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u\nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0, \end{cases} \tag{1.3}$$

where the function $D(u)$ denotes the diffusive function of cells, and the effect of $D(u)$ on the global solvability of solutions has attracted widespread attention. Note that for the corresponding no-flux type initial-boundary value problem (1.3), for the case of $D(u) = 1$ and $N \geqslant 2$, Tao [**12**] proved that system (1.3) possesses a unique global bounded classical solution under the assumption that $\|v_0\|_{L^\infty(\Omega)}$ is small. Especially, the domain is convex when $N = 3$, Tao and Winkler [**14**] removed the smallness condition of initial data and proved that the system admits at least one global weak solution for arbitrarily large initial data; moreover, they showed that this solution is eventually smooth and converges to the constant equilibria in the large time limit.

Considering the quasilinear diffusion function $D(u) \geqslant D_0(u+1)^{m-1}$ with some constant $D_0 > 0$, if $N \geqslant 2$ and the domain is convex, Wang *et al.* [**20**] established the globally bounded classical solution when $m > 2 - \frac{2}{N}$. Subsequently, Wang *et al.* [**19**] removed the convexity assumption and showed that the global solution is locally bounded if $m > 2 - \frac{6}{N+4}$ with $N \geqslant 3$. In addition, Zheng and Wang [**41**] improved the global existence result for the case $m > \frac{3N}{2N+2}$.

Furthermore, taking account of the degenerate diffusion function $D(u) \geqslant D_0 u^{m-1}$ with some constant $D_0 > 0$, under the assumption $m > \frac{3N-2}{2N}$, $N \geqslant 3$, the globally bounded weak solution was obtained in [**5**, **24**], and the asymptotic behaviour of solution was obtained in [**5**]. And for the convex domain case, the global existence result of the weak solution was further raised to the case $m > \frac{3N}{2N+2}(N \geqslant 2)$ in [**41**]. Since then, no further research was conducted.

Besides, chemotaxis processes with signal absorption coupling to the fluid motion are often considered, such as the following chemotaxis system which describes the

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movement of bacterial cells to oxygen in incompressible fluids [18]

$$\begin{cases} u_t + \mathcal{V}\cdot\nabla u = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(uS\nabla v), & x\in\Omega, t>0,\\ v_t + \mathcal{V}\cdot\nabla v = \Delta v - uv, & x\in\Omega, t>0,\\ \mathcal{V}_t + \kappa(\mathcal{V}\cdot\nabla)\mathcal{V} = \Delta\mathcal{V} - \nabla P + u\nabla\phi, \quad \nabla\cdot\mathcal{V}=0, & x\in\Omega, t>0, \end{cases} \tag{1.4}$$

where S is a given chemotactic sensitivity function, $\kappa\in\mathbb{R}$ is a constant, P is the pressure, ϕ is the gravitational force, and $\mathcal{V}$ is the velocity of the fluid. Note that if the effect of fluids is absent (i.e. $\mathcal{V}=0$), system (1.4) is reduced to system (1.3). Readers who are interested in the study of (1.4) could refer to [**1**–**3**, **6**, **10**, **15**, **16**, **21**–**23**, **26**–**33**, **36**–**40**]. Particularly, we point out that when $D(u)=mu^{m-1}$, there is a long process of how far m can ensure the solvability of system (1.4) with $\kappa=0$ in $\Omega\subset\mathbb{R}^3$. When S is a scalar function, Francesco *et al.* [**3**] proved global bounded weak solutions for $m\in(\frac{7+\sqrt{217}}{12},2]$, Tao and Winkler [**16**] established the existence of global weak solutions for $m>\frac{8}{7}$, Winkler [**31**] and Jin [**6**] enhanced the boundedness result to $m>\frac{9}{8}$ and $m>1$, respectively. When S is a given parameter matrix, Winkler [**28**] presented the boundedness of solutions in convex domains for $m>\frac{7}{6}$. Additionally, this result was extended to the case $m>\frac{10}{9}$ [**37**], $m>\frac{65}{63}$ [**17**], and $m>\frac{11}{4}-\sqrt{3}$ [**39**]. Recently, Winkler [**34**, corollary 1.4] has shown that the system admits a globally bounded weak solution for $m>1$ in the convex domain.

Comparing the results of systems (1.3) and (1.4) in three dimensions, we see that the conditions of parameter $m>\frac{7}{6}$ [**5**, **24**] and $m>\frac{9}{8}$ [**41**] are worse than $m>1$ [**6**, **34**] from the point of ensuring the solvability of the solution. Therefore, it is a natural question whether the range of m of system (1.1) in higher dimensions is wider than that in [**5**, **24**, **41**]. If the range of m can be extended, what is the large time behaviour of the corresponding solution? Indeed, those questions are partially answered by the main results of this paper.

Main idea: As aforementioned, under the assumption $m>\frac{3N-2}{2N}$, the global boundedness weak solution of system (1.1) has been obtained in [**5**, **24**], thus we focus on the case $m\leqslant\frac{3N-2}{2N}$. Without loss of generality, we shall assume that $2>m>1$ in the sequel. Note that the term $\int_\Omega(u_\varepsilon+\varepsilon)^{p-m+1}|\nabla v_\varepsilon|^2$ was decoupled into two parts in [**5**, **24**], namely the integrals containing only u_ε and ∇v_ε separately. In this paper, inspired by [**34**], we choose $p=m+1$ to be the dissipative part of the inequality describing the evolution of $\int_\Omega(u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2$. Unfortunately, there are additional bad terms that must be addressed. Therefore, we construct a new functional

$$y(t)=\tilde{C}_1\int_\Omega(u_\varepsilon+\varepsilon)^{m+1}+\tilde{C}_2\int_\Omega(u_\varepsilon+\varepsilon)^{3-m}+\int_\Omega(u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2+\tilde{C}_3\int_\Omega|\nabla v_\varepsilon|^4$$

to obtain the boundedness of $\|u_\varepsilon+\varepsilon\|_{L^{m+1}(\Omega)}$ for $2>m>1$, which is the most critical step. Next, under the assumption of $2>m>\max\{1,\frac{3N-2}{2N+2}\}$ (where $N=3$, 4, 5), the boundedness of functional $\int_\Omega(u_\varepsilon+\varepsilon)^p+\int_\Omega|\nabla v_\varepsilon|^{2q}$ for large p and large q can be obtained. And then by the iteration procedure in [**13**, theorem A.1], we get the uniform bounds for $\|u_\varepsilon\|_{L^\infty(\Omega)}$ and $\|\nabla v_\varepsilon\|_{L^\infty(\Omega)}$.

At first, we introduce the following definition of weak solutions.

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Definition 1.1. *For a global weak solution of* (1.1)*, we mean a pair of nonnegative functions* (u, v) *satisfying*

$$u \in L^\infty((0,\infty); L^\infty(\Omega)), \quad \nabla u^m \in L^2_{loc}([0,\infty); L^2(\Omega)), \quad v \in L^\infty((0,\infty); L^\infty(\Omega)),$$

and for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$,

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u^m \cdot \nabla\varphi + \int_0^\infty \int_\Omega u\nabla v \cdot \nabla\varphi,$$
$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega vu \cdot \varphi.$$

The main results are stated as follows.

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^N$ $(N = 3, 4, 5)$ *be a bounded domain with smooth boundary, assume that*

$$m > \max\left\{1, \frac{3N-2}{2N+2}\right\},$$

then system (1.1) *with* (1.2) *at least has one globally bounded weak solution.*

Remark 1.3. For the case $3 \leqslant N \leqslant 5$, our theorem 1.2 extends the previous results in [**5**, **24**, **41**]. In addition, our result is consistent with the associated fluid-free system [**6**, **34**] for $N = 3$.

Remark 1.4. In this paper, our method is motivated by Winkler [**34**], but we can only solve the case for $N = 3, 4, 5$, whether it can be further solved for the case $N > 5$ is uncertain. Fortunately, we have removed the convexity assumption on the domain in [**34**].

Remark 1.5. In the case $N = 3, 4$, we can find that if $m = 1$ the system must be imposed on the smallness condition of the initial data [**12**]. As we all know, the condition $m > 1$ means that the diffusion is stronger than $m = 1$. From these points of view, our result is optimal. But in the case $N = 5$, it is not clear whether the assumptions of $m > \frac{13}{12}$ is optimal to ensure global boundedness of the solution.

As a byproduct of theorem 1.2, large time behaviour of the solution to system (1.1) can be achieved.

Theorem 1.6. *Under the assumptions of theorem* 1.2*, the global weak solution constructed in theorem* 1.2 *satisfies*

$$u(\cdot,t) \overset{*}{\rightharpoonup} \bar{u}_0 \ in\ L^\infty(\Omega), \quad v(\cdot,t) \to 0 \ in\ L^\infty(\Omega) \tag{1.5}$$

as $t \to \infty$, *where* $\bar{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0$.

The rest of this paper is organized as follows. In § 2, we introduce the approximated system (2.1) and provide the local existence of the approximated solution and some crucial properties. In § 3, we present some important estimates, and

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obtain the global bounded classical solution to the approximate system (2.1). In § 4, we deduce some convergence properties and complete the proof of theorem 1.2 by an approximation procedure. Finally, in § 5, we establish the convergence of the solution.

2. Approximate problems and crucial properties

In this section, in order to construct a weak solution of (1.1), we consider the following approximate problems

$$\begin{cases} (u_\varepsilon)_t = \Delta(u_\varepsilon+\varepsilon)^m - \nabla\cdot(u_\varepsilon\nabla v_\varepsilon), & x\in\Omega, t>0,\\ (v_\varepsilon)_t = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x\in\Omega, t>0,\\ \dfrac{\partial u_\varepsilon}{\partial\nu} = \dfrac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u_\varepsilon(x,0)=u_0(x), \quad v_\varepsilon(x,0)=v_0(x), & x\in\Omega \end{cases} \tag{2.1}$$

for $\varepsilon\in(0,\,1)$.

For each $\varepsilon\in(0,\,1)$, the regularized problem (2.1) is locally solvable in the classical sense.

Lemma 2.1. *Suppose that $\Omega\subset\mathbb{R}^N(N\geqslant 3)$ is a bounded domain with smooth boundary. Assume that the initial data $(u_0,\,v_0)$ fulfils (1.2). Then problem (2.1) has a unique classical solution*

$$\begin{cases} u_\varepsilon \in C^0(\bar\Omega\times[0,T_{\max,\varepsilon}))\cap C^{2,1}(\bar\Omega\times(0,T_{\max,\varepsilon}))\cap L^\infty_{loc}([0,T_{\max,\varepsilon});W^{1,\infty}(\Omega)),\\ v_\varepsilon \in C^0(\bar\Omega\times[0,T_{\max,\varepsilon}))\cap C^{2,1}(\bar\Omega\times(0,T_{\max,\varepsilon}))\cap L^\infty_{loc}([0,T_{\max,\varepsilon});W^{1,\infty}(\Omega)), \end{cases} \tag{2.2}$$

where $T_{\max,\varepsilon}$ denotes the maximal existence time. Moreover, if $T_{\max,\varepsilon}<\infty$, then

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \to\infty \quad as \quad t\to T_{\max,\varepsilon}. \tag{2.3}$$

Proof. The local existence, extensibility criterion, and regularity of (2.1) are well-established (see [**20**]), the uniqueness can be achieved by the same procedure as in [**7**, **25**], so we omit the details of the proof for the sake of brevity. □

Lemma 2.2. *The solution $(u_\varepsilon,\,v_\varepsilon)$ of (2.1) satisfies*

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \text{ for all } t\in(0,T_{\max,\varepsilon}), \tag{2.4}$$

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \|v_0\|_{L^\infty(\Omega)} \text{ for all } t\in(0,T_{\max,\varepsilon}). \tag{2.5}$$

Proof. A direct integration in the first equation in (2.1) yields (2.4). Equation (2.5) follows from an application of the maximum principle to the second equation in (2.1). □

In order to prove the main result, we state two basic lemmas which will be used later. The first is Gagliardo–Nirenberg inequality.

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Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ be a bounded smooth domain*

$$1 \leqslant r < q < \frac{N(p+m-1)}{N-2}, \tag{2.6}$$

then there exists $C > 0$ such that

$$\int_\Omega \varphi^q \leqslant C \left\{ \left(\int_\Omega \varphi^{p+m-3} |\nabla \varphi|^2 \right)^{\frac{\frac{q}{2r} - \frac{1}{2}}{\frac{p+m-1}{2r} + \frac{1}{N} - \frac{1}{2}}} \|\varphi\|_{L^r(\Omega)}^{\frac{\frac{p+m-1}{2} + \frac{q}{N} - \frac{q}{2}}{\frac{p+m-1}{2r} + \frac{1}{N} - \frac{1}{2}}} + \|\varphi\|_{L^r(\Omega)}^q \right\} \tag{2.7}$$

for all $\varphi \in C^1(\Omega) \cap L^r(\Omega)$.

Particularly, if $q = p + m - 1 + \frac{2r}{N}$, then there exists $C > 0$ such that

$$\int_\Omega \varphi^q \leqslant C \left\{ \left(\int_\Omega \varphi^{p+m-3} |\nabla \varphi|^2 \right) \|\varphi\|_{L^r(\Omega)}^{\frac{\frac{p+m-1}{2} + \frac{q}{N} - \frac{q}{2}}{\frac{p+m-1}{2r} + \frac{1}{N} - \frac{1}{2}}} + \|\varphi\|_{L^r(\Omega)}^q \right\} \tag{2.8}$$

for all $\varphi \in C^1(\Omega) \cap L^r(\Omega)$.

Proof. Condition (2.6) entails that

$$a = \frac{\frac{p+m-1}{2r} - \frac{p+m-1}{2q}}{\frac{p+m-1}{2r} + \frac{1}{N} - \frac{1}{2}} \in (0,1),$$

and hence the Gagliardo–Nirenberg inequality provides $C_1 > 0$ satisfying

$$\begin{aligned}
\int_\Omega \varphi^q &= \|\varphi^{\frac{p+m-1}{2}}\|_{L^{\frac{2q}{p+m-1}}(\Omega)}^{\frac{2q}{p+m-1}} \\
&\leqslant C_1 \|\nabla \varphi^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2qa}{p+m-1}} \cdot \|\varphi^{\frac{p+m-1}{2}}\|_{L^{\frac{2r}{p+m-1}}(\Omega)}^{\frac{2q(1-a)}{p+m-1}} + C_1 \|\varphi^{\frac{p+m-1}{2}}\|_{L^{\frac{2r}{p+m-1}}(\Omega)}^{\frac{2q}{p+m-1}} \\
&= C_2 \left\{ \left(\int_\Omega \varphi^{p+m-3} |\nabla \varphi|^2 \right)^{\frac{qa}{p+m-1}} \cdot \|\varphi\|_{L^r(\Omega)}^{q(1-a)} + \|\varphi\|_{L^r(\Omega)}^q \right\}
\end{aligned}$$

for all $\varphi \in C^1(\Omega) \cap L^r(\Omega)$ with $C_2 = C_1 \max\{1, \frac{(p+m-1)^2}{4}\}$. Note that

$$\frac{qa}{p+m-1} = \frac{\frac{q}{2r} - \frac{1}{2}}{\frac{p+m-1}{2r} + \frac{1}{N} - \frac{1}{2}} \text{ and } q(1-a) = \frac{\frac{p+m-1}{2} + \frac{q}{N} - \frac{q}{2}}{\frac{p+m-1}{2r} + \frac{1}{N} - \frac{1}{2}}.$$

Thus, (2.7) is proved. Moreover, since $\frac{qa}{p+m-1} = \frac{\frac{q}{2r} - \frac{1}{2}}{\frac{p+m-1}{2r} + \frac{1}{N} - \frac{1}{2}} = 1$ with the additional assumption $q = p + m - 1 + \frac{2r}{N}$, (2.8) results from (2.7). □

The next is interpolation inequality.

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LEMMA 2.4 (Lemma 3.3 of [**24**]). *Suppose that* $q > \max\{1, \frac{N-2}{2}\}$ *and* $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ *is a bounded domain with smooth boundary. Moreover, assume that*

$$\lambda \in \left[2q+2, \frac{N(2q+1)-2(q+1)}{N-2}\right], \tag{2.9}$$

then there exists $C > 0$ *such that for all* $\varphi \in C^2(\bar{\Omega})$ *fulfilling* $\varphi \cdot \frac{\partial \varphi}{\partial \nu} = 0$ *on* $\partial\Omega$, *we have*

$$\|\nabla\varphi\|_{L^\lambda(\Omega)} \leqslant C\||\nabla\varphi|^{q-1}D^2\varphi\|_{L^2(\Omega)}^{\frac{2(\lambda-N)}{(2q-N+2)\lambda}}\|\varphi\|_{L^\infty(\Omega)}^{\frac{2qN-(N-2)\lambda}{(2q-N+2)\lambda}} + C\|\varphi\|_{L^\infty(\Omega)}. \tag{2.10}$$

3. Uniform estimates for $(u_\varepsilon, v_\varepsilon)$ and global boundedness of approximate solutions

In this section, we establish some priori estimates for solutions and get the global boundedness of approximate solutions (2.1). Firstly, we apply standard testing procedures to establish a differential inequality for the first equation in (2.1).

LEMMA 3.1. *Let* $p > 1$ *and* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with smooth boundary. Assume* $(u_\varepsilon, v_\varepsilon)$ *is a classical solution to system* (2.1) *on* $[0, T_{\max,\varepsilon})$. *Then for all* $t \in (0, T_{\max,\varepsilon})$ *and* $\varepsilon \in (0, 1)$, *we can see that*

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_\varepsilon+\varepsilon)^p + \frac{m(p-1)}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{p+m-3}|\nabla u_\varepsilon|^2 \\ \leqslant \frac{p-1}{2m}\int_\Omega (u_\varepsilon+\varepsilon)^{p-m+1}|\nabla v_\varepsilon|^2.\end{aligned} \tag{3.1}$$

Proof. We multiply the first equation of (2.1) by $(u_\varepsilon+\varepsilon)^{p-1}$ integrating by parts and together with Young's inequality to obtain

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon+\varepsilon\|_{L^p(\Omega)}^p + m(p-1)\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}|\nabla u_\varepsilon|^2 \\ &\quad\leqslant (p-1)\int_\Omega (u_\varepsilon+\varepsilon)^{p-2}u_\varepsilon\nabla v_\varepsilon\cdot\nabla u_\varepsilon \\ &\quad\leqslant \frac{m(p-1)}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{m+p-3}|\nabla u_\varepsilon|^2 + \frac{p-1}{2m}\int_\Omega (u_\varepsilon+\varepsilon)^{p-m+1}|\nabla v_\varepsilon|^2\end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max,\varepsilon})$, which results in (3.1). □

LEMMA 3.2. *Let* $\Omega \subset \mathbb{R}^N (N = 3, 4, 5)$ *be a bounded domain with smooth boundary. Assume* $2 > m > 1$, *then for all* $\varepsilon \in (0, 1)$, *there exists* $C > 0$ *independent of* ε *such*

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that the solution of (2.1) *satisfies*

$$\int_\Omega (u_\varepsilon+\varepsilon)^{m+1} \leqslant C \quad \textit{for all } t\in(0,T_{\max,\varepsilon}), \tag{3.3}$$

$$\int_t^{t+\tau}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2 \leqslant C \textit{ for all } t\in(0,T_{\max,\varepsilon}-\tau), \tag{3.4}$$

$$\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^2 \leqslant C \textit{ for all } t\in(0,T_{\max,\varepsilon}-\tau) \tag{3.5}$$

with $\tau:=\min\{1,\frac{T_{\max,\varepsilon}}{2}\}$.

Proof. The proof is divided into five steps.

Step 1. Using (3.1) to u_ε with $p:=m+1$, for all $t\in(0,T_{\max,\varepsilon})$ we get

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_\varepsilon+\varepsilon)^{m+1}+\frac{m^2}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2 \leqslant \frac{1}{2}\int_\Omega (u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2. \tag{3.6}$$

Step 2. For any $\eta_1>0$, one can find four positive constants C_1, $C_2(\eta_1)$, C_3, and $\widetilde{C}$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2+\int_\Omega (u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2+2\int_\Omega (u_\varepsilon+\varepsilon)|D^2v_\varepsilon|^2\\ \leqslant \eta_1\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2+C_1\int_\Omega |\nabla u_\varepsilon|^2\\ +C_2(\eta_1)\int_\Omega |\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2+C_3\int_\Omega|\nabla v_\varepsilon|^2+\widetilde{C} \quad \text{for all } t\in(0,T_{\max,\varepsilon}).\end{aligned} \tag{3.7}$$

Indeed, on the basis of (2.5) and the identities $\nabla v_\varepsilon\cdot\nabla\Delta v_\varepsilon=\frac{1}{2}\Delta|\nabla v_\varepsilon|^2-|D^2v_\varepsilon|^2$ and $\nabla|\nabla v_\varepsilon|^2=2D^2v_\varepsilon\cdot\nabla v_\varepsilon$, upon integrating by parts, for all $t\in(0,T_{\max,\varepsilon})$, we compute

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2\\ &=\int_\Omega (u_\varepsilon)_t\cdot|\nabla v_\varepsilon|^2+2\int_\Omega (u_\varepsilon+\varepsilon)\nabla v_\varepsilon\cdot\nabla(v_\varepsilon)_t\\ &=\int_\Omega |\nabla v|^2\{\Delta(u_\varepsilon+\varepsilon)^m-\nabla\cdot(u_\varepsilon\nabla v_\varepsilon)\}\\ &\quad+2\int_\Omega (u_\varepsilon+\varepsilon)\nabla v_\varepsilon\cdot\{\nabla\Delta v_\varepsilon-\nabla(u_\varepsilon v_\varepsilon)\}\\ &=-2m\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}\nabla u_\varepsilon\cdot D^2v_\varepsilon\cdot\nabla v_\varepsilon+2\int_\Omega u_\varepsilon\nabla v_\varepsilon\cdot D^2v_\varepsilon\cdot\nabla v_\varepsilon\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega (u_\varepsilon+\varepsilon)\Delta|\nabla v_\varepsilon|^2-2\int_\Omega (u_\varepsilon+\varepsilon)|D^2v_\varepsilon|^2\\
&-2\int_\Omega (u_\varepsilon+\varepsilon)v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon-2\int_\Omega (u_\varepsilon+\varepsilon)u_\varepsilon|\nabla v_\varepsilon|^2\\
=&-2m\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}\nabla u_\varepsilon\cdot D^2v_\varepsilon\cdot\nabla v_\varepsilon+2\int_\Omega u_\varepsilon\nabla v_\varepsilon\cdot D^2v_\varepsilon\cdot\nabla v_\varepsilon\\
&-2\int_\Omega \nabla u_\varepsilon(D^2v_\varepsilon\cdot\nabla v_\varepsilon)+\int_{\partial\Omega} u_\varepsilon\cdot\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}-2\int_\Omega (u_\varepsilon+\varepsilon)|D^2v_\varepsilon|^2\\
&-2\int_\Omega (u_\varepsilon+\varepsilon)v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon-2\int_\Omega (u_\varepsilon+\varepsilon)u_\varepsilon|\nabla v_\varepsilon|^2. \qquad (3.8)
\end{aligned}
$$

Now, we will estimate the right-hand side of (3.8). To this end, given any $\eta_1>0$, utilize Young's inequality to see that for all $t\in(0, T_{\max,\varepsilon})$ satisfies

$$
\begin{aligned}
&-2m\int_\Omega (u_\varepsilon+\varepsilon)^{m-1}\nabla u_\varepsilon\cdot D^2v_\varepsilon\cdot\nabla v_\varepsilon\\
&\leqslant \eta_1\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2+\frac{m^2}{\eta_1}\int_\Omega |\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2, \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
&2\int_\Omega u_\varepsilon\nabla v_\varepsilon\cdot D^2v_\varepsilon\cdot\nabla v_\varepsilon\\
&\leqslant \frac12\int_\Omega u_\varepsilon^2|\nabla v_\varepsilon|^2+2\int_\Omega |\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2\\
&\leqslant \frac12\int_\Omega (u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2+2\int_\Omega |\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2, \qquad (3.10)
\end{aligned}
$$

and

$$
-2\int_\Omega \nabla u_\varepsilon(D^2v_\varepsilon\cdot\nabla v_\varepsilon)\leqslant\int_\Omega|\nabla u_\varepsilon|^2+\int_\Omega|\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2. \qquad (3.11)
$$

Next, we will estimate the boundary integral in (3.8). According to [**11**], there exists $c_2>0$ such that $\partial_\nu|\nabla\varphi|^2\leqslant c_2|\nabla\varphi|^2$ on $\partial\Omega$ for all $\varphi\in C^2(\bar\Omega)$ with $\partial_\nu\varphi|_{\partial\Omega}=0$. And notice the equivalent trace inequality [**35**, P.1186]: for all $\epsilon>0$, one has

$$
\|w\|_{L^2(\partial\Omega)}\leqslant\epsilon\|\nabla w\|_{L^2(\Omega)}+C(\epsilon)\|w\|_{L^1(\Omega)},\quad \forall\ w\in H^1(\Omega).
$$

Now, for any $\epsilon>0$, it follows that for all $t\in(0,T_{\max,\varepsilon})$,

$$
\begin{aligned}
\int_{\partial\Omega} u_\varepsilon\cdot\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}&\leqslant c_2\int_{\partial\Omega}u_\varepsilon|\nabla v_\varepsilon|^2\leqslant c_2\int_{\partial\Omega}|\nabla v_\varepsilon|^4+\frac{c_2}{4}\int_{\partial\Omega}u_\varepsilon^2\\
&\leqslant c_2\int_{\partial\Omega}|\nabla v_\varepsilon|^4+\epsilon\int_\Omega|\nabla u_\varepsilon|^2+C(\epsilon)\left(\int_\Omega u_\varepsilon\right)^2. \qquad (3.12)
\end{aligned}
$$

Moreover, let c_3 denote the embedding constant for trace embedding $W^{1,1}(\Omega)\hookrightarrow L^1(\partial\Omega)$ [4]. Using lemma 2.4 for v_ε with $q:=2$, $\lambda:=6$, $3\leqslant N\leqslant 5$, and (2.5), one

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has

$$\int_\Omega |\nabla v_\varepsilon|^6 \leqslant C\int_\Omega |\nabla v_\varepsilon|^2|D^2 v_\varepsilon|^2 + C \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{3.13}$$

This combined with Young's inequality yields

$$\begin{aligned} c_2\int_{\partial\Omega}|\nabla v_\varepsilon|^4 &\leqslant 2c_2c_3\int_\Omega |\nabla v_\varepsilon|^2\nabla|\nabla v_\varepsilon|^2 + c_2c_3\int_\Omega|\nabla v_\varepsilon|^4\\ &\leqslant \frac{1}{32}\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2 + (c_2c_3+32c_2^2c_3^2)\int_\Omega|\nabla v_\varepsilon|^4\\ &= \frac{1}{8}\int_\Omega |\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2 + (c_2c_3+32c_2^2c_3^2)\int_\Omega|\nabla v_\varepsilon|^4\\ &\leqslant \frac{1}{4}\int_\Omega |\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2 + c_4 \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \end{aligned}\tag{3.14}$$

Therefore, combining (3.12), (3.14) and (2.4) by letting $\epsilon = 1$, the boundary integral in (3.8) can be simplified to

$$\int_{\partial\Omega} u_\varepsilon\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \leqslant \frac{1}{4}\int_\Omega|\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2 + \int_\Omega|\nabla u_\varepsilon|^2 + C \quad \text{for all } t\in(0,T_{\max,\varepsilon}).$$

Next, using (2.5) and Young's inequality, it is obvious that for all $t\in(0,\,T_{\max,\varepsilon})$,

$$\begin{aligned} -2\int_\Omega (u_\varepsilon+\varepsilon)v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon &\leqslant \frac{1}{4}\int_\Omega(u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2 + 4\int_\Omega v_\varepsilon^2|\nabla u_\varepsilon|^2\\ &\leqslant \frac{1}{4}\int_\Omega(u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2 + 4\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega|\nabla u_\varepsilon|^2. \end{aligned}\tag{3.15}$$

Since $\varepsilon\in(0,\,1)$, and using Young's inequality, it follows that for all $t\in(0,\,T_{\max,\varepsilon})$,

$$\begin{aligned} -2\int_\Omega(u_\varepsilon+\varepsilon)u_\varepsilon|\nabla v_\varepsilon|^2 &\leqslant -2\int_\Omega(u_\varepsilon+\varepsilon)(u_\varepsilon+\varepsilon-1)|\nabla v_\varepsilon|^2\\ &= -2\int_\Omega(u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2 + 2\int_\Omega(u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2\\ &\leqslant -\frac{7}{4}\int_\Omega(u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2 + 4\int_\Omega|\nabla v_\varepsilon|^2. \end{aligned}\tag{3.16}$$

From (3.9)–(3.16), we obtain (3.7) upon letting $C_1 := 2+4\|v_0\|_{L^\infty(\Omega)}$, $C_2(\eta_1) := \frac{m^2}{\eta_1}+\frac{13}{4}$ and $C_3 = 4$.

Step 3. In order to absorb the third term on the right-hand side of (3.7), our goal is to show that the integral $\int_\Omega|\nabla v_\varepsilon|^4$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v_\varepsilon|^4 + \int_\Omega|\nabla v_\varepsilon|^2|D^2v_\varepsilon|^2 \leqslant C_4\int_\Omega|\nabla u_\varepsilon|^2 + C_5 \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{3.17}$$

In fact, using the second equation in (2.1) along with the pointwise identities $\nabla v_\varepsilon\cdot\nabla\Delta v_\varepsilon = \frac{1}{2}\Delta|\nabla v_\varepsilon|^2 - |D^2v_\varepsilon|^2$ and $\nabla|\nabla v_\varepsilon|^2 = 2D^2v_\varepsilon\cdot\nabla v_\varepsilon$, we differentiate

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$\int_\Omega |\nabla v_\varepsilon|^4$ directly and integrate by parts to yield

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v_\varepsilon|^4 &= \int_\Omega |\nabla v_\varepsilon|^2 \nabla v_\varepsilon \cdot \nabla(\Delta v_\varepsilon - u_\varepsilon v_\varepsilon) \\ &= \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 \Delta |\nabla v_\varepsilon|^2 - \int_\Omega |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon|^2 \\ &\quad - \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \nabla v_\varepsilon \nabla u_\varepsilon - \int_\Omega u_\varepsilon |\nabla v_\varepsilon|^4 \\ &= -\frac{1}{2}\int_\Omega \big|\nabla |\nabla v_\varepsilon|^2\big|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla v_\varepsilon|^2 \frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu_\varepsilon} - \int_\Omega |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon|^2 \\ &\quad - \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \nabla v_\varepsilon \nabla u_\varepsilon - \int_\Omega u_\varepsilon |\nabla v_\varepsilon|^4 \\ &\leqslant -\int_\Omega |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla v_\varepsilon|^2 \frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu_\varepsilon} \\ &\quad - \int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \nabla v_\varepsilon \nabla u_\varepsilon - \int_\Omega u_\varepsilon |\nabla v_\varepsilon|^4 \quad \text{for all } t \in (0, T_{\max,\varepsilon}).\end{aligned} \tag{3.18}$$

Applying Young's inequality, for all $\delta > 0$, we obtain

$$-\int_\Omega v_\varepsilon |\nabla v_\varepsilon|^2 \nabla v_\varepsilon \nabla u_\varepsilon \leqslant \frac{\|v_\varepsilon\|^2_{L^\infty(\Omega)}}{4\delta}\int_\Omega |\nabla u_\varepsilon|^2 + \delta \int_\Omega |\nabla v_\varepsilon|^6 \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{3.19}$$

In light of (3.13), we can choose a suitable δ such that exists a constant $c_1 > 0$ satisfying

$$\delta \int_\Omega |\nabla v_\varepsilon|^6 \leqslant \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon|^2 + c_1 \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{3.20}$$

Similar to (3.14), the boundary integral in (3.18) fulfils

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu} |\nabla v_\varepsilon|^2 &\leqslant c_2 \int_{\partial\Omega} |\nabla v_\varepsilon|^4 \\ &\leqslant \frac{1}{4}\int_\Omega |\nabla v_\varepsilon|^2 |D^2 v_\varepsilon|^2 + c_4 \quad \text{for all } t \in (0, T_{\max,\varepsilon}).\end{aligned} \tag{3.21}$$

From (3.18)–(3.21), we obtain (3.17) upon an obvious choice of $C_4 := \frac{\|v_0\|^2_{L^\infty(\Omega)}}{\delta}$ and $C_5 := 4c_1 + 4c_4$.

Step 4. Let the term $\int_\Omega |\nabla u_\varepsilon|^2$ of both (3.7) and (3.17) appears on the left-hand side of an inequality. Namely, for all $t \in (0,\, T_{\max,\varepsilon})$ and for any $\eta_2 > 0$, there exists

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$C_6(\eta_2) := \eta_2^{\frac{m-1}{m-2}} \cdot (\frac{2-m}{2m})^{\frac{1}{2-m}}$ such that

$$\begin{aligned}\frac{1}{3-m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_\varepsilon+\varepsilon)^{3-m} + \frac{(2-m)m}{2}\int_\Omega |\nabla u_\varepsilon|^2 \\ \leqslant \frac{2-m}{2m}\int_\Omega (u_\varepsilon+\varepsilon)^{4-2m}|\nabla v_\varepsilon|^2 \\ \leqslant \eta_2\int_\Omega (u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2 + C_6(\eta_2)\int_\Omega |\nabla v_\varepsilon|^2.\end{aligned} \tag{3.22}$$

Actually, due to $2 > m > 1$, using (3.1) to u_ε with $p := 3-m$, and upon Young's inequality the result is obtained.

Step 5. Subtly combining (3.6), (3.7), (3.17) and (3.22), let

$$\begin{aligned}y(t) := \frac{1}{2m^2(m+1)}\int_\Omega (u_\varepsilon+\varepsilon)^{m+1} + \frac{4C_1+8C_2(\eta_1)C_4}{m(2-m)(3-m)}\int_\Omega (u_\varepsilon+\varepsilon)^{3-m} \\ + \int_\Omega (u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2 + 2C_2(\eta_1)\int_\Omega |\nabla v_\varepsilon|^4 \quad \text{for all } t\in(0,T_{\max,\varepsilon}).\end{aligned} \tag{3.23}$$

Due to $2 > m > 1$, we can see that

$$\begin{aligned}&y'(t) + y(t) + \frac{1}{4}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2 + \Big(C_1 + 2C_2(\eta_1)C_4\Big)\int_\Omega |\nabla u_\varepsilon|^2 \\ &\quad + \frac{3}{4}\int_\Omega (u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2 + C_2(\eta_1)\int_\Omega |\nabla v_\varepsilon|^2|D^2 v_\varepsilon|^2 \\ &\leqslant \eta_2\frac{4C_1+8C_2(\eta_1)C_4}{2-m}\int_\Omega (u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2 \\ &\quad + \left(\frac{\big(4C_1+8C_2(\eta_1)C_4\big)C_6(\eta_2)}{m(2-m)} + C_3\right)\int_\Omega |\nabla v_\varepsilon|^2 \\ &\quad + \eta_1\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2 + 2C_2(\eta_1)C_5 \\ &\quad + \frac{1}{2m^2(m+1)}\int_\Omega (u_\varepsilon+\varepsilon)^{m+1} + \frac{4C_1+8C_2(\eta_1)C_4}{m(2-m)(3-m)}\int_\Omega (u_\varepsilon+\varepsilon)^{3-m} \\ &\quad + \int_\Omega (u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2 + 2C_2(\eta_1)\int_\Omega |\nabla v_\varepsilon|^4 + \widetilde{C} \quad \text{for all } t\in(0,T_{\max,\varepsilon}).\end{aligned} \tag{3.24}$$

Choose suitable η_1 and η_2 such that $\eta_1 = \frac{1}{16}$ and $\eta_2\frac{4C_1+8C_2(\eta_1)C_4}{2-m} = \frac{1}{2}$. In view of lemma 2.3, Young's inequality, and (2.4), there exists $c_1 > 0$ such that

$$\begin{aligned}\frac{1}{2m^2(m+1)}\int_\Omega (u_\varepsilon+\varepsilon)^{m+1} + \frac{4C_1+8C_2(\eta_1)C_4}{m(2-m)(3-m)}\int_\Omega (u_\varepsilon+\varepsilon)^{3-m} \\ \leqslant \frac{1}{8}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2 + c_1 \quad \text{for all } t\in(0,T_{\max,\varepsilon}).\end{aligned} \tag{3.25}$$

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Using Young's inequality, we have

$$\int_\Omega (u_\varepsilon+\varepsilon)|\nabla v_\varepsilon|^2 \leqslant \frac{1}{4}\int_\Omega (u_\varepsilon+\varepsilon)^2|\nabla v_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2 \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{3.26}$$

Upon Young's inequality and (3.20), there exist c_2 and $c_3>0$ such that

$$\begin{aligned}
&\int_\Omega |\nabla v_\varepsilon|^2 + \left(\frac{(4C_1+8C_2(\eta_1)C_4)C_6(\eta_2)}{m(2-m)} + C_3\right)\int_\Omega |\nabla v_\varepsilon|^2 + 2C_4\int_\Omega |\nabla v_\varepsilon|^4 \\
&\quad\leqslant 2\delta C_2(\eta_1)\int_\Omega |\nabla v_\varepsilon|^6 + c_2 \\
&\quad\leqslant C_2(\eta_1)\int_\Omega |\nabla v_\varepsilon|^2|D^2 v_\varepsilon|^2 + c_3 \quad \text{for all } t\in(0,T_{\max,\varepsilon}).
\end{aligned} \tag{3.27}$$

It follows from (3.24)–(3.27) that

$$y'(t)+y(t)+h(t)\leqslant \widehat{C} \quad \text{for all } t\in(0,T_{\max,\varepsilon}), \tag{3.28}$$

where $h(t):=\frac{1}{16}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2 + (C_1+2C_2(\eta_1)C_4)\int_\Omega |\nabla u_\varepsilon|^2$ and $\widehat{C}:= 2C_2(\eta_1)C_5+c_1+c_3+\widetilde{C}$. By an ordinary differential equations (ODE) comparison argument, it yields

$$y(t)\leqslant \max\{\widehat{C}, y(0)\} \quad \text{for all } t\in(0,T_{\max,\varepsilon}),$$

which implies (3.3). Finally, integration of (3.28) shows that (3.4)–(3.5) hold. □

According to lemma 3.2, the boundedness of $\|u_\varepsilon+\varepsilon\|_{L^{m+1}(\Omega)}$ can be obtained without any restriction on m except for the condition $2>m>1$. Next, we use the boundedness of $\|u_\varepsilon+\varepsilon\|_{L^{m+1}(\Omega)}$ to establish a further estimate for solutions to the approximated system (2.1). As in [**13**], we provide an estimate on ∇v_ε. The proof of the following lemma is the same as that in lemma 3.2 of [**24**], so we omit it.

LEMMA 3.3. *Let $q>1$, then for all $\varepsilon\in(0,1)$, we have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v_\varepsilon|^{2q} + \frac{2(q-1)}{q}\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + q\int_\Omega |\nabla v_\varepsilon|^{2(q-1)}|D^2 v_\varepsilon|^2 \\
&\quad\leqslant q(2q-2+\sqrt{N})^2\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^2|\nabla v_\varepsilon|^{2q-2} + C_7
\end{aligned} \tag{3.29}$$

for all $t\in(0,\,T_{\max,\varepsilon})$ with a positive constant C_7.

Next, we will estimate the combination of $\int_\Omega (u_\varepsilon+\varepsilon)^p + \int_\Omega |\nabla v_\varepsilon|^{2q}$.

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LEMMA 3.4. *Assume that $2 > m > 1$, then for all $p > 2$ and any $q > 1$, one can find three constants C_8, C_9, $C_{10} > 0$ such that*

$$\begin{aligned} &F_\varepsilon'(t) + F_\varepsilon(t) + \frac{p(p-1)}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{p+m-3}|\nabla u_\varepsilon|^2 \\ &\leqslant C_8\int_\Omega (u_\varepsilon+\varepsilon)^{\frac{(p-m+1)(2q+2)}{2q}} + C_9\int_\Omega (u_\varepsilon+\varepsilon)^{q+1} \\ &+\int_\Omega (u_\varepsilon+\varepsilon)^p + C_{10} \quad \textit{for all } t\in(0,T_{\max,\varepsilon}), \end{aligned} \tag{3.30}$$

where the function $F_\varepsilon(t)$ is defined as

$$F_\varepsilon(t) := \int_\Omega (u_\varepsilon+\varepsilon)^p + \int_\Omega |\nabla v_\varepsilon|^{2q} \quad \textit{for all } t\in(0,T_{\max,\varepsilon}) \textit{ and } \varepsilon\in(0,1). \tag{3.31}$$

Proof. Combining lemma 3.1 with lemma 3.3 and using $m > 1$, we obtain

$$\begin{aligned} &F_\varepsilon'(t) + F_\varepsilon(t) + \frac{p(p-1)}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{p+m-3}|\nabla u_\varepsilon|^2 \\ &\quad + \frac{2(q-1)}{q}\int_\Omega |\nabla|\nabla v_\varepsilon|^q|^2 + q\int_\Omega |\nabla v_\varepsilon|^{2(q-1)}|D^2 v_\varepsilon|^2 \\ &\leqslant \frac{p(p-1)}{2m}\int_\Omega (u_\varepsilon+\varepsilon)^{p-m+1}|\nabla v_\varepsilon|^2 + q(2q-2+\sqrt{N})^2\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^2|\nabla v_\varepsilon|^{2q-2} \\ &\quad + C_7 + \int_\Omega (u_\varepsilon+\varepsilon)^p + \int_\Omega |\nabla v_\varepsilon|^{2q} \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \end{aligned} \tag{3.32}$$

Next, by Young's inequality for any $\eta_3 > 0$, we have

$$\frac{p(p-1)}{2m}\int_\Omega (u_\varepsilon+\varepsilon)^{p-m+1}|\nabla v_\varepsilon|^2 \leqslant \eta_3\int_\Omega |\nabla v_\varepsilon|^{2q+2} + C(\eta_3)\int_\Omega (u_\varepsilon+\varepsilon)^{\frac{(p-m+1)(2q+2)}{2q}},$$

and

$$q(2q-2+\sqrt{N})^2\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^2|\nabla v_\varepsilon|^{2q-2} \leqslant \eta_3\int_\Omega |\nabla v_\varepsilon|^{2q+2} + C(\eta_3)\int_\Omega (u_\varepsilon+\varepsilon)^{q+1}.$$

Using interpolation inequality (2.10) and Young's inequality, we can choose a suitably η_3 such that

$$2\eta_3\int_\Omega |\nabla c_\varepsilon|^{2q+2} + \int_\Omega |\nabla v_\varepsilon|^{2q} \leqslant q\int_\Omega |\nabla v_\varepsilon|^{2(q-1)}|D^2 v_\varepsilon|^2 + c_1 \quad \text{for all } t\in(0,T_{\max,\varepsilon}).$$

It follows from (3.32) that (3.30) holds. □

Therefore, we obtain the boundedness of $\int_\Omega (u_\varepsilon+\varepsilon)^p + \int_\Omega |\nabla v_\varepsilon|^{2q}$ with $p > 2$ and $q > 1$.

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LEMMA 3.5. *Let $\Omega \subset \mathbb{R}^N (N = 3,\, 4,\, 5)$ be a bounded domain with smooth boundary. Assume $2 > m > \max\{1,\, \frac{3N-2}{2N+2}\}$, whenever $p > 2$ and $q > 1$ are such that*

$$\frac{p-m+1}{m-1+\frac{m+1}{N}} < 2q < 2p + 2m - 4 + \frac{4(m+1)}{N}, \tag{3.33}$$

then for all $\varepsilon \in (0,\, 1)$, we can find a constant $C = C(p,\, q) > 0$ satisfying

$$\int_\Omega (u_\varepsilon + \varepsilon)^p + \int_\Omega |\nabla v_\varepsilon|^{2q} \leqslant C \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{3.34}$$

Proof. Using the boundedness of $\|u_\varepsilon + \varepsilon\|_{L^{m+1}(\Omega)}$ to (2.8) with $r := m + 1$, there exists $C_{11} > 0$ such that for all $\varepsilon \in (0,\, 1)$ satisfies

$$\int_\Omega (u_\varepsilon + \varepsilon)^{p+m-1+\frac{2(m+1)}{N}} \leqslant C_{11} \int_\Omega (u_\varepsilon + \varepsilon)^{p+m-3} |\nabla u_\varepsilon|^2 + C_{11} \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{3.35}$$

Moreover, $m > \max\{1,\, \frac{3N-2}{2N+2}\}$ ensures that hypothesis (3.33) holds. (3.33) asserts that $q + 1 < p + m - 1 + \frac{2(m+1)}{N}$ and

$$\begin{aligned} 0 < \frac{(p-m+1)(2q+2)}{2q} &= p - m + 1 + \frac{2(p-m+1)}{2q} \\ &< p - m + 1 + 2\left(m - 1 + \frac{m+1}{N}\right) = p + m - 1 + \frac{2(m+1)}{N}. \end{aligned}$$

Consequently, utilizing Young's inequality, we obtain

$$\int_\Omega (u_\varepsilon + \varepsilon)^{\frac{(p-m+1)(2q+2)}{2q}} < \frac{p(p-1)}{6C_8 C_{11}} \int_\Omega (u_\varepsilon + \varepsilon)^{p+m-1+\frac{2(m+1)}{N}} + c_1 \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{3.36}$$

and

$$\int_\Omega (u_\varepsilon + \varepsilon)^{q+1} < \frac{p(p-1)}{6C_9 C_{11}} \int_\Omega (u_\varepsilon + \varepsilon)^{p+m-1+\frac{2(m+1)}{N}} + c_2 \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{3.37}$$

Since $p < p + m - 1 + \frac{2(m+1)}{N}$, it is similar to deduce

$$\int_\Omega (u_\varepsilon + \varepsilon)^p < \frac{p(p-1)}{6C_{11}} \int_\Omega (u_\varepsilon + \varepsilon)^{p+m-1+\frac{2(m+1)}{N}} + c_3 \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{3.38}$$

Collecting (3.35)–(3.38) and (3.30), it follows that

$$F'_\varepsilon(t) + F_\varepsilon(t) \leqslant c_4 \quad \text{for all } t \in (0, T_{\max,\varepsilon})$$

with $c_4 > 0$. It is obvious to obtain (3.34) by a comparison argument. □

By means of Moser–Alikakos iteration procedure, an application of the above to suitably large but fixed p and q yields bounds in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$.

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Lemma 3.6. *Let* $\Omega \subset \mathbb{R}^N (N = 3, 4, 5)$ *be a bounded domain with smooth boundary. Assume that* $2 > m > \max\{1, \frac{3N-2}{2N+2}\}$. *Then there exists* $C > 0$ *such that for all* $\varepsilon \in (0, 1)$ *the solution of* (2.1) *satisfies*

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant C \quad \text{for all } t > 0, \tag{3.39}$$

and

$$\|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \leqslant C \quad \text{for all } t > 0. \tag{3.40}$$

Proof. In light of using the result of lemma 3.5, lemma A.1 in [**13**], and (2.5), we derive

$$\|u_\varepsilon\|_{L^\infty(\Omega)} + \|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \leqslant C \quad \text{for all } t \in (0, T_{\max,\varepsilon})$$

where $C > 0$ is independent of $\varepsilon \in (0, 1)$. From this and the extensibility criterion (2.3), it is evident that $T_{\max,\varepsilon} = \infty$, which finishes the proof. □

4. Global bounded weak solutions to system (1.1)

In order to finish the proof of theorem 1.2, we will give some regularity properties with $(u_\varepsilon, v_\varepsilon)$ in this section.

Lemma 4.1. *Let* $\Omega \subset \mathbb{R}^N (N = 3, 4, 5)$ *be a bounded domain with smooth boundary. Assume that* $m > \max\{1, \frac{3N-2}{2N+2}\}$, *then we get*

$$\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \leqslant \int_\Omega v_0 \quad \text{for all } \varepsilon \in (0, 1), \tag{4.1}$$

and

$$\int_0^\infty \int_\Omega |\nabla v_\varepsilon|^2 \leqslant \frac{1}{2} \int_\Omega v_0^2 \quad \text{for all } \varepsilon \in (0, 1). \tag{4.2}$$

Proof. Multiplying the second equation of (2.1) by 1 and v_ε separately, upon integrating by parts, and integrating with respect to t, we obtain

$$\int_\Omega v_\varepsilon(\cdot, t) + \int_0^t \int_\Omega u_\varepsilon v_\varepsilon = \int_\Omega v_0$$

and

$$\frac{1}{2} \int_\Omega v_\varepsilon^2 + \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 + \int_0^t \int_\Omega u_\varepsilon v_\varepsilon^2 = \frac{1}{2} \int_\Omega v_0^2$$

for all $\varepsilon \in (0, 1)$. As a result, we immediately obtain (4.1) and (4.2). □

Lemma 4.2. *Let* $\Omega \subset \mathbb{R}^N (N = 3, 4, 5)$ *be a bounded domain with smooth boundary. Assume that* $2 > m > \max\{1, \frac{3N-2}{2N+2}\}$, *then for any* $T > 0$ *there exists* $C(T) > 0$

such that

$$\int_0^T \|\partial_t(u_\varepsilon+\varepsilon)^m(\cdot,t)\|_{(W_0^{N,2}(\Omega))^*}\mathrm{d}t \leqslant C(T) \quad \textit{for all } \varepsilon\in(0,1), \tag{4.3}$$

and

$$\int_0^T\int_\Omega |\nabla(u_\varepsilon+\varepsilon)^m|^2 \leqslant C(T) \quad \textit{for all } \varepsilon\in(0,1). \tag{4.4}$$

Proof. Due to lemma 3.2, for any $T>0$, we have

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \leqslant C(T), \tag{4.5}$$

and

$$\int_0^T\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}|\nabla u_\varepsilon|^2 \leqslant C(T), \tag{4.6}$$

which indicates (4.4) is valid. Due to lemma 3.6, there exists a positive constant $c_1>0$ such that $0<u_\varepsilon<c_1$ in $\Omega\times(0,\,\infty)$ for all $\varepsilon\in(0,\,1)$. Multiplying the first equation of (2.1) by $(u_\varepsilon+\varepsilon)^{m-1}$, one has

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_\varepsilon+\varepsilon)^m + \frac{m(m-1)}{2}\int_\Omega (u_\varepsilon+\varepsilon)^{2m-3}|\nabla u_\varepsilon|^2 \leqslant \frac{m-1}{2m}(c_1+1)\int_\Omega |\nabla v_\varepsilon|^2,$$

integrating with respect to t and using (4.2), which indicates

$$\int_0^t\int_\Omega (u_\varepsilon+\varepsilon)^{2m-3}|\nabla u_\varepsilon|^2 \leqslant C. \tag{4.7}$$

Next, multiplying the first equation of (2.1) by $(u_\varepsilon+\varepsilon)^m\psi$ with $\psi\in C_0^\infty(\Omega)$, and integrating over Ω, for all $\varepsilon\in(0,\,1)$, we obtain

$$\begin{aligned}\frac{1}{m}\int_\Omega \partial_t(u_\varepsilon+\varepsilon)^m\cdot\psi = &-m\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}\nabla u_\varepsilon\cdot\nabla\psi\\ &- m(m-1)\int_\Omega (u_\varepsilon+\varepsilon)^{2m-3}|\nabla u_\varepsilon|^2\psi\\ &+\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}\nabla v_\varepsilon\cdot\nabla\psi\\ &+(m-1)\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\psi.\end{aligned} \tag{4.8}$$

Using the boundedness of $\|u_\varepsilon\|_{L^\infty(\Omega)}$ and Young's inequality, it yields

$$\begin{aligned}-m\int_\Omega (u_\varepsilon+\varepsilon)^{2m-2}\nabla u_\varepsilon\cdot\nabla\psi &\leqslant m(c_1+1)^{2m-2}\|\nabla\psi\|_{L^\infty(\Omega)}\cdot\int_\Omega|\nabla u_\varepsilon|\\ &\leqslant C\left(\int_\Omega|\nabla u_\varepsilon|^2+4|\Omega|\right)\cdot\|\nabla\psi\|_{L^\infty(\Omega)},\end{aligned} \tag{4.9}$$

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and

$$-m\int_\Omega (u_\varepsilon+\varepsilon)^{2m-3}|\nabla u_\varepsilon|^2\psi \leqslant m\|\psi\|_{L^\infty(\Omega)}\cdot\left(\int_\Omega (u_\varepsilon+\varepsilon)^{2m-3}|\nabla u_\varepsilon|^2\right). \tag{4.10}$$

Moreover, by the boundedness of $\|u_\varepsilon\|_{L^\infty(\Omega)}$ and $\|\nabla v_\varepsilon\|_{L^\infty(\Omega)}$, we get

$$\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-1}\nabla v_\varepsilon\cdot\nabla\psi \leqslant C\|\nabla\psi\|_{L^\infty(\Omega)}, \tag{4.11}$$

and

$$m\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{m-2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\psi \leqslant C\|\psi\|_{L^\infty(\Omega)}\left(\int_\Omega (u_\varepsilon+\varepsilon)^{2m-3}|\nabla u_\varepsilon|^2\right). \tag{4.12}$$

According to (4.6)–(4.12) and the continuity of the embedding $W_0^{N,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$, there exists $C(T)>0$ such that

$$\int_0^T \|\partial_t u_\varepsilon(\cdot,t)^m\|_{(W_0^{N,2}(\Omega))^*} \leqslant C(T) \quad \text{for all } \varepsilon\in(0,1).$$

The proof of this lemma is completed. □

LEMMA 4.3. *Let* $2>m>\max\{1,\frac{3N-2}{2N+2}\}$, *then for any* $T>0$ *there exists* $C(T)>0$ *such that*

$$\|\frac{\partial v_\varepsilon}{\partial t}\|_{L^2(0,T;(W^{1,2}(\Omega))^*)} \leqslant C(T) \quad \textit{for all } \varepsilon\in(0,1). \tag{4.13}$$

Proof. Multiplying $\psi(x)$ on both sides of the second equation with $\psi(x)\in W^{1,2}(\Omega)$, and integrating over Ω and using the Hölder inequality, we have

$$\int_\Omega \frac{\partial v_\varepsilon}{\partial t}\psi = -\int_\Omega \nabla v_\varepsilon\cdot\nabla\psi - \int_\Omega u_\varepsilon v_\varepsilon\psi \leqslant \|\nabla v_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} + \|u_\varepsilon v_\varepsilon\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}.$$

By the boundedness of $\|u_\varepsilon\|_{L^\infty(\Omega)}$ and $\|v_\varepsilon\|_{L^\infty(\Omega)}$, it shows that

$$\|\frac{\partial v_\varepsilon}{\partial t}\|_{(W^{1,2}(\Omega))^*} \leqslant \|\nabla v_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon v_\varepsilon\|_{L^2(\Omega)} \leqslant C + \|\nabla v_\varepsilon\|_{L^2(\Omega)}. \tag{4.14}$$

Combining (4.2) and (4.14), it follows that

$$\|\frac{\partial v_\varepsilon}{\partial t}\|^2_{L^2(0,T;(W^{1,2}(\Omega))^*)} \leqslant \int_0^T\int_\Omega |\nabla v_\varepsilon|^2 + \int_0^T\int_\Omega |u_\varepsilon v_\varepsilon|^2 \leqslant C(T).$$

□

The proof of throrem 1.2. Lemma 3.6 shows that there exists $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\to 0$ as $j\to\infty$ and that $u_{\varepsilon_j}\overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega\times(0,\infty))$ and $\nabla v_{\varepsilon_j}\overset{*}{\rightharpoonup}\nabla v$ in $L^\infty(\Omega\times(0,\infty))$ hold. Lemma 4.2 implies that $(u_\varepsilon^m)_{\varepsilon\in(0,1)}$ is bounded in $L^2([0,T];W^{1,2}(\Omega))$. Hence, $\nabla u_{\varepsilon_j}^m\rightharpoonup\nabla u^m$ in $L^2_{loc}([0,\infty);L^2(\Omega))$. Furthermore, using the Aubin–Lions lemma and $(\partial_t u_\varepsilon^m)_{\varepsilon\in(0,1)}$ is bounded in

$L^1([0,T];(W_0^{N,2}(\Omega))^*)$, it yields $u_{\varepsilon_j}^m \to u^m$ in $L^2([0,T];L^2(\Omega))$. By the Riesz lemma and $m>1$, we have $u_{\varepsilon_j} \to u$ a.e. in $\Omega \times (0,\infty)$. Likewise, by lemmas 3.6 and 4.3, it follows that $v_{\varepsilon_j} \to v$ in $L^2([0,T];L^2(\Omega))$ and a.e. in $\Omega \times (0,\infty)$. Because of these convergence properties, one may readily prove that (u,v) is a global weak solution of (1.1) in the sense of definition 1.1. Consequently, (u,v) is a global bounded weak solution of (1.1) by lemma 3.6. The proof is completed. □

5. Large time behaviour

This section discusses the asymptotic behaviour of the system for large time. Motivated by [**5**, **28**], the required properties of the solutions are first presented.

LEMMA 5.1. *Let $m > \max\{1, \frac{3N-2}{2N+2}\}$, then there exists $\theta \in (0,1)$ such that for some $C>0$, we obtain*

$$\|v_\varepsilon(\cdot,t)\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \leqslant C \quad \text{for all } t \geqslant 0, \tag{5.1}$$

and for all $\tau>0$, we can find $C(\tau)>0$ such that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \leqslant C \quad \text{for all } t \geqslant \tau. \tag{5.2}$$

Proof. In view of lemma 3.6, $-u_\varepsilon v_\varepsilon$ is bounded in $L^\infty(\Omega\times(0,\infty))$ for all $\varepsilon\in(0,1)$. Therefore, applying the standard parabolic regularity theory [**8**, Chapter III], both estimates (5.1) and (5.2) are obtained. □

LEMMA 5.2. *Let $\Omega \subset \mathbb{R}^N (N=3,4,5)$ be a bounded domain with smooth boundary. Assume that $m > \max\{1, \frac{3N-2}{2N+2}\}$ and $p > \max\{1, m-1\}$, then there exists $C>0$ such that*

$$\int_0^\infty \int_\Omega |\nabla (u_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}| \leqslant C \quad \text{for all } \varepsilon \in (0,1). \tag{5.3}$$

Proof. By virtue of lemma 3.6, there exists a positive constant $c_1>0$ such that $0<u_\varepsilon<c_1$ in $\Omega\times(0,\infty)$ for all $\varepsilon\in(0,1)$. From lemma 3.1 and $p>\max\{1,m-1\}$, it follows that

$$\int_\Omega (u_\varepsilon+\varepsilon)^{p-m+1}|\nabla v_\varepsilon|^2 \leqslant (c_1+1)^{p-m+1}\int_\Omega |\nabla v_\varepsilon|^2 \text{ for all } t>0 \text{ and } \varepsilon\in(0,1).$$

Therefore, an integration of (3.1) shows that

$$\begin{aligned} &\frac{2m(p-1)}{(p+m-1)^2}\int_0^t\int_\Omega |\nabla(u_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}| \\ &\quad \leqslant \frac{1}{p}\int_\Omega (u_0+\varepsilon)^p + \frac{p-1}{2m}(c_1+1)^{p-m+1}\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 \end{aligned}$$

for all $\varepsilon\in(0,1)$, which together with (4.2) indicate that (5.3) is valid. □

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LEMMA 5.3. *Let $\Omega \subset \mathbb{R}^N (N = 3,\, 4,\, 5)$ be a bounded domain with smooth boundary. Assume that $m > \max\{1,\, \frac{3N-2}{2N+2}\}$, then there exist $C > 0$ such that*

$$\|\partial_t u_\varepsilon(\cdot, t)\|_{(W_0^{2,2}(\Omega))^*} \leqslant C, \quad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0, 1). \tag{5.4}$$

Particularly,

$$\|u_\varepsilon(\cdot, t) - u_\varepsilon(\cdot, s)\|_{(W_0^{2,2}(\Omega))^*} \leqslant C|t - s|, \quad \textit{for all } t \geqslant 0, s \geqslant 0 \textit{ and } \varepsilon \in (0, 1). \tag{5.5}$$

Proof. Multiplying the first equation of (2.1) by ψ with $\psi \in C_0^\infty(\Omega)$, and integrating over Ω, one has

$$\begin{aligned} \int_\Omega \partial_t u_\varepsilon \cdot \psi &= -\int_\Omega \nabla(u_\varepsilon + \varepsilon)^m \cdot \nabla\psi + \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\psi \\ &= \int_\Omega (u_\varepsilon + \varepsilon)^m \Delta\psi + \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\psi \quad \text{for all } t > 0. \end{aligned} \tag{5.6}$$

According to lemma 3.6, there exist two positive constants $c_1,\, c_2 > 0$ such that $0 < u_\varepsilon < c_1$ and $|\nabla v_\varepsilon| \leqslant c_2$ in $\Omega \times (0,\, \infty)$ for all $\varepsilon \in (0,\, 1)$. Then (5.6) implies

$$\left| \int_\Omega \partial_t u_\varepsilon(\cdot, t) \cdot \psi \right| \leqslant (c_1 + 1)^m \int_\Omega |\Delta\psi| + c_1 c_2 \int_\Omega |\nabla\psi| \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

and

$$\|\partial_t u_\varepsilon(\cdot, t)\|^2_{(W_0^{2,2}(\Omega))^*} = \sup_{\psi \in W_0^{2,2}(\Omega), \|\psi\|_{W_0^{2,2}(\Omega)} = 1} \left| \int_\Omega \partial_t u_\varepsilon(\cdot, t) \cdot \psi \right|^2 \leqslant C.$$

This indicates that (5.4) and (5.5) are valid. □

With the above information on solutions, the convergence of u is shown as follows.

LEMMA 5.4. *Let $m > \max\{1,\, \frac{3N-2}{2N+2}\}$ (where $N = 3,\, 4,\, 5$) and $(u,\, v)$ as given by theorem 1.2, we can see that*

$$u(\cdot, t) \overset{*}{\rightharpoonup} \bar{u}_0 \textit{ in } L^\infty(\Omega) \textit{ as } t \to \infty. \tag{5.7}$$

Proof. Similar to lemma 5.1 of [**28**] and lemma 3.16 of [**5**], the proof of this lemma can be completed. In fact, assuming the lemma is false, then there exists a sequence $(t_j)_{j \subset \mathbb{N}} \subset (0,\, \infty)$ such that $t_j \to \infty$ as $j \to \infty$, and such that for some $\tilde{\psi} \in L^1(\Omega)$ one has

$$\int_\Omega u(x, t_j)\tilde{\psi}(x)\mathrm{d}x - \int_\Omega \bar{u}_0 \tilde{\psi}(x)\mathrm{d}x \geqslant c_1 \quad \text{for all } j \subset \mathbb{N} \tag{5.8}$$

with $c_1 > 0$. Note that theorem 1.2 implies that there exists a positive constant $c_2 > 0$ such that $0 < u < c_2$ for a.e. $(x,\, t) \in \Omega \times (0,\, \infty)$. And then using the density

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of $C_0^\infty(\Omega)$ in $L^1(\Omega)$ in choosing $\psi \in C_0^\infty(\Omega)$ such that $\|\psi - \tilde{\psi}\|_{L^1(\Omega)} \leqslant \frac{c_1}{4c_2}$, (5.8) yields

$$\begin{aligned}\int_\Omega u(x,t_j)\psi(x)\mathrm{d}x - \int_\Omega \bar{u}_0\psi(x)\mathrm{d}x \geqslant & \int_\Omega u(x,t_j)\tilde{\psi}(x)\mathrm{d}x - \int_\Omega \bar{u}_0\tilde{\psi}(x)\mathrm{d}x \\ & - \{\|u(x,t_j)\|_{L^\infty(\Omega)} + \bar{u}_0\}\|\psi - \tilde{\psi}\|_{L^1(\Omega)} \\ \geqslant & \frac{c_1}{2} \quad \text{for all } j \subset \mathbb{N}.\end{aligned} \tag{5.9}$$

Due to $L^\infty(\Omega) \hookrightarrow (W_0^{2,2}(\Omega))^*$ is compact, using Arzelà–Ascoli theorem, the equicontinuity properties (5.5) and the boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $C^0([0,\infty); L^\infty(\Omega))$ ensure that $u_\varepsilon \to u$ in $C^0_{loc}([0,\,\infty); (W_0^{2,2}(\Omega))^*)$ holds. According to (5.5), there exists a positive constant $c_3 > 0$ such that

$$\|u_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,s)\|_{(W_0^{2,2}(\Omega))^*} \leqslant c_3|t-s|, \quad \text{for all } t \geqslant 0, s \geqslant 0 \text{ and } \varepsilon \in (0,1).$$

Then, taking limits to get

$$\|u(\cdot,t) - u(\cdot,s)\|_{(W_0^{2,2}(\Omega))^*} \leqslant c_3|t-s|, \quad \text{for all } t \geqslant 0, s \geqslant 0.$$

If let $\tau \in (0,\,1)$ such that $\tau \leqslant \frac{c_1}{4c_3\|\psi\|_{W_0^{2,2}(\Omega)}}$, then for all $j \subset \mathbb{N}$ and each $t \in (t_j,\, t_j + \tau)$ one has

$$\begin{aligned}\left|\int_\Omega u(x,t_j)\psi(x)\mathrm{d}x - \int_\Omega u(x,t)\psi(x)\mathrm{d}x\right| &\leqslant \|u(x,t_j) - u(x,t)\|_{(W_0^{2,2}(\Omega))^*}\|\psi\|_{W_0^{2,2}(\Omega)} \\ &\leqslant c_3|t_j - t| \cdot \|\psi\|_{W_0^{2,2}(\Omega)} \\ &\leqslant \frac{c_1}{4},\end{aligned}$$

which together with (5.9) implies that

$$\int_\Omega u(x,t)\psi(x)\mathrm{d}x - \int_\Omega \bar{u}_0\psi(x)\mathrm{d}x \geqslant \frac{c_1}{4} \quad \text{for all } t \in (t_j, t_j+\tau) \text{ and each } j \subset \mathbb{N}. \tag{5.10}$$

Next, we will prove (5.10) contradicts lemma 5.2. Taking the Poincaré constant $c_4 > 0$ such that

$$\int_\Omega \left|\varphi(x) - \frac{1}{|\Omega|}\int_\Omega \varphi(x)\right|^2 \mathrm{d}x \leqslant c_4 \int_\Omega |\nabla\varphi|^2 \quad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Fix any $p > 1$ such that $p \geqslant \max\{1,\, m-1,\, 3-m\}$, and using lemma 5.2, it is obvious that

$$\begin{aligned}\int_0^\infty \int_\Omega \left|u_\varepsilon^{\frac{p+m-1}{2}}(x,t) - a_\varepsilon^{\frac{p+m-1}{2}}(t)\right|^2 \mathrm{d}x\mathrm{d}t &\leqslant c_4 \int_\Omega \left|\nabla u_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \\ &\leqslant c_4 \int_\Omega \left|\nabla(u_\varepsilon + \varepsilon)^{\frac{p+m-1}{2}}\right|^2 \\ &\leqslant c_5\end{aligned} \tag{5.11}$$

for all $\varepsilon \in (0,\, 1)$, where $a_\varepsilon(t) := (\frac{1}{|\Omega|}\int_\Omega u_\varepsilon^{\frac{p+m-1}{2}})^{\frac{2}{p+m-1}}$ and $c_5 > 0$. Using the convergence property $u_\varepsilon \to u$ a.e. in $\Omega \times (0,\, \infty)$ as $\varepsilon \to 0$, the boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega \times (0,\, \infty))$ and the dominated convergence theorem, one obtains

$$a_\varepsilon(t) \to a(t) \quad \text{for a.e. } t > 0, \text{ as } \varepsilon \to 0, \tag{5.12}$$

where $a(t) := (\frac{1}{|\Omega|}\int_\Omega u^{\frac{p+m-1}{2}})^{\frac{2}{p+m-1}}$. Again using $u_\varepsilon \to u$ a.e. in $\Omega \times (0,\, \infty)$ as $\varepsilon \to 0$ and Fatou's lemma, (5.11) and (5.12) imply that

$$\int_0^\infty \int_\Omega |u_\varepsilon^{\frac{p+m-1}{2}}(x,t) - a^{\frac{p+m-1}{2}}(t)|^2 \mathrm{d}x\mathrm{d}t \leqslant c_5. \tag{5.13}$$

Review the following inequality: If $\mu > 1$, then $\frac{\xi^\mu - \eta^\mu}{\xi - \eta} \geqslant \eta^{\mu-1}$ for all $\xi,\, \eta \leqslant 0$ with $\xi \neq \eta$. And since by the Hölder inequality, and the $L^1(\Omega)$ conservation of u means that

$$\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u \leqslant \frac{1}{|\Omega|}\left(\int_\Omega u^{\frac{p+m-1}{2}}\right)^{\frac{2}{p+m-1}} \cdot |\Omega|^{\frac{p+m-3}{p+m-1}} = a(t).$$

Thus, on the left-hand side of (5.13) indicates

$$\begin{aligned}\int_\Omega |u^{\frac{p+m-1}{2}}(x,t) - a^{\frac{p+m-1}{2}}(t)|^2 \mathrm{d}x &\geqslant a^{p+m-3}(t)\int_\Omega |u(x,t) - a(t)|^2 \mathrm{d}x \\ &\geqslant \bar{u}_0^{p+m-3} \cdot \int_\Omega |u(x,t) - a(t)|^2 \mathrm{d}x,\end{aligned}$$

and

$$\int_0^\infty \int_\Omega |u(x,t) - a(t)|^2 \mathrm{d}x\mathrm{d}t \leqslant c_6 := \frac{c_5}{\bar{u}_0^{p+m-3}}. \tag{5.14}$$

Now, we introduce

$$u_j(x,s) := u(x, t_j + s), \quad (x,s) \in \Omega \times (0,\tau) \text{ for all } j \subset \mathbb{N},$$

and

$$a_j(s) := a(t_j + s), \quad s \in (0,\tau), \text{ for all } j \subset \mathbb{N}.$$

Therefore, (5.14) implies that

$$\int_0^\tau \int_\Omega |u_j(x,s) - a_j(s)|^2 \mathrm{d}x\mathrm{d}t = \int_{t_j}^{t_j+\tau} \int_\Omega |u(x,t) - a(t)|^2 \mathrm{d}x\mathrm{d}t \to 0 \text{ as } j \to \infty,$$

which means

$$u_j(x,s) - a_j(s) \to 0 \text{ in } L^2(\Omega \times (0,\tau)) \text{ as } j \to \infty. \tag{5.15}$$

By the definition of $a(t)$ and the boundedness of $u(x,t)$, $(a_j)_{j\subset\mathbb{N}}$ is bounded in $L^2((0,\, \tau))$. Then, for some nonnegative $a_\infty \in L^2((0,\, \tau))$ satisfying

$$a_j \rightharpoonup a_\infty \text{ in } L^2((0,\tau)) \text{ as } j \to \infty. \tag{5.16}$$

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By utilizing the $L^1(\Omega)$ conservation of u, we have

$$\int_0^\tau \int_\Omega (u_j(x,s) - a_j(s))\mathrm{d}x\mathrm{d}s = \tau\bar{u}_0|\Omega| - |\Omega| \int_0^\tau a_j(s)\mathrm{d}s \\ \to \tau\bar{u}_0|\Omega| - |\Omega| \int_0^\tau a_\infty(s)\mathrm{d}s \text{ as } j \to \infty,$$

which combined with (5.15), one has

$$\int_0^\tau a_\infty(s)\mathrm{d}s = \tau\bar{u}_0. \tag{5.17}$$

On the other hand, (5.10) and (5.17) show that

$$\begin{aligned}\frac{c_1}{4} &\leqslant \int_0^\tau \int_\Omega u_j(x,s)\psi(x)\mathrm{d}x\mathrm{d}s - \int_0^\tau \int_\Omega \bar{u}_0\psi(x)\mathrm{d}x\mathrm{d}s \\ &= \int_0^\tau \int_\Omega (u_j(x,s) - a_j(s))\mathrm{d}x\mathrm{d}s + \int_0^\tau \int_\Omega a_j(s)\psi(x)\mathrm{d}x\mathrm{d}s - \tau\bar{u}_0 \int_\Omega \psi(x)\mathrm{d}x \\ &= \int_0^\tau \int_\Omega (u_j(x,s) - a_j(s))\mathrm{d}x\mathrm{d}s + \int_0^\tau a_j(s)\mathrm{d}s \cdot \int_\Omega \psi(x)\mathrm{d}x \\ &\to \int_0^\tau a_\infty(s)\mathrm{d}s \cdot \int_\Omega \psi(x)\mathrm{d}x - \tau\bar{u}_0 \int_\Omega \psi(x)\mathrm{d}x = 0 \text{ as } j \to \infty.\end{aligned}$$

This is a contradiction and then the proof of this lemma is completed. □

Finally, the convergence of v can be obtained.

LEMMA 5.5. *Let $m > \max\{1, \frac{3N-2}{2N+2}\}$ (where $N = 3, 4, 5$) and (u, v) as given by theorem 1.2, we obtain*

$$v(\cdot,t) \to 0 \textit{ in } L^\infty(\Omega) \textit{ as } t \to \infty. \tag{5.18}$$

Proof. Similar to lemma 5.2 of [**28**] and lemma 3.17 of [**5**], the proof of this lemma can be completed. Similarly, assume the lemma is false, then there exist two sequences $(x_j)_{j\subset\mathbb{N}} \subset \Omega$ and $(t_j)_{j\subset\mathbb{N}} \subset (0, \infty)$ such that $t_j \to \infty$ as $j \to \infty$ satisfies

$$v(x_j, t_j) \geqslant c_1 \quad \text{for all } j \subset \mathbb{N} \tag{5.19}$$

with $c_1 > 0$, where passing to subsequences we may assume that there exists $x_0 \in \bar{\Omega}$ such that $x_j \to x_0$ as $j \to \infty$. Due to lemma 5.1, v is uniformly continuous in $\cup_{j\in\mathbb{N}}(\bar{\Omega} \times [t_j, t_j + 1])$, which entails that there exist $\delta > 0$, $\tau \in (0, 1)$ and $B :=$

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$B_\delta(x_0) \cap \Omega$ such that

$$v(x,t) \geqslant \frac{c_1}{2} \quad \text{for all } x \in B, t \in (t_j, t_j+\tau) \ \text{ and } j \in \mathbb{N}. \tag{5.20}$$

Now, let $u_j(x,\, s) := u(x,\, t_j + s)$ and $v_j(x,\, s) := v(x,\, t_j + s)$ for $x \in \Omega$, $s \in (0,\, \tau)$ and $j \in \mathbb{N}$, then from (4.1), we noticed that

$$\begin{aligned} \int_0^\tau \int_B u_j(x,s)v_j(x,s)\mathrm{d}x\mathrm{d}s &= \int_{t_j}^{t_j+\tau} \int_B u(x,t)v(x,t)\mathrm{d}x\mathrm{d}t \\ &\leqslant \int_{t_j}^{\infty} \int_B u(x,t)v(x,t)\mathrm{d}x\mathrm{d}t \\ &\to 0 \text{ as } j \to \infty. \end{aligned} \tag{5.21}$$

On the other hand, let $\psi(x) := \chi_B(x)$ for $x \in \Omega$, then in light of lemma 5.4, it follows that

$$\begin{aligned} &\left| \int_{t_j}^{t_j+\tau} \int_\Omega u(x,t)\chi_B(x)\mathrm{d}x\mathrm{d}t - \bar{u}_0\tau|B| \right| \\ &= \left| \int_{t_j}^{t_j+\tau} \left\{ \int_\Omega u(x,t)\chi_B(x)\mathrm{d}x - \int_\Omega \bar{u}_0\chi_B(x)\mathrm{d}x \right\}\mathrm{d}t \right| \\ &\leqslant \tau \sup_{t\in(t_j,t_j+\tau)} \left| \left\{ \int_\Omega u(x,t)\chi_B(x)\mathrm{d}x\mathrm{d}t - \int_\Omega \bar{u}_0\chi_B(x)\mathrm{d}x \right\} \right| \\ &\to 0 \text{ as } j \to \infty, \end{aligned}$$

and that hence

$$\int_0^\tau \int_\Omega u_j(x,s)\mathrm{d}x\mathrm{d}s \to \bar{u}_0\tau|B| \text{ as } j \to \infty. \tag{5.22}$$

In summary, the combination of (5.20) and (5.21) indicates that

$$\begin{aligned} \frac{c_1}{2}\bar{u}_0\tau|B| &= \liminf_{j\to\infty} \left\{ \frac{c_1}{2} \int_0^\tau \int_\Omega u(x,s)\mathrm{d}x\mathrm{d}s \right\} \\ &\leqslant \int_0^\tau \int_B u_j(x,s)v_j(x,s)\mathrm{d}x\mathrm{d}s, \end{aligned}$$

which contradicts (5.21), then the proof of this lemma is completed. □

The proof of theorem 1.6. The claimed convergence properties are precisely asserted by lemmas 5.4 and 5.5. □

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