




RESEARCH ARTICLE

A combinatorial model for the transition matrix between the Specht and SL_2 -web bases

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Abstract

We introduce a new class of permutations, called web permutations. Using these permutations, we provide a combinatorial interpretation for entries of the transition matrix between the Specht and SL_2 -web bases of the irreducible \mathfrak{S}_{2n} -representation indexed by (n, n) , which answers Rhoades’s question. Furthermore, we study enumerative properties of these permutations.

1. Introduction and the main results

In this article, we study the transition matrix between two famous bases, the Specht basis and the SL_2 -web basis, for the irreducible representation of the symmetric group \mathfrak{S}_{2n} indexed by the partition (n, n) . Motivated by Rhoades’s work [Rho19], we give a combinatorial interpretation for entries of the transition matrix as a certain class of permutations, and we present their interesting properties.

For an integer $n \geq 1$, let \mathfrak{S}_{2n} be the symmetric group on the set $[2n] = \{1, \dots, 2n\}$. It is well-known that each irreducible representation of \mathfrak{S}_{2n} can be indexed by a partition of $2n$. For a partition λ of $2n$, we then denote by \mathcal{S}^λ the irreducible representation indexed by λ , called the *Specht module*. In this article, we focus specifically on the Specht module indexed by the partition (n, n) , and two well-studied bases for $\mathcal{S}^{(n,n)}$.

A *standard Young tableau* of shape (n, n) is a $2 \times n$ array of integers whose entries are $[2n]$, and each row and each column are increasing. See Figure 1 for an example. The set of standard Young tableaux of shape (n, n) , denoted by $\text{SYT}(n, n)$, parametrizes the *Specht basis*

$$\{v_T \in \mathcal{S}^{(n,n)} : T \in \text{SYT}(n, n)\}$$

for $\mathcal{S}^{(n,n)}$. For more details on the Specht basis and related combinatorics, see [Ful97, Sag01].

A (*perfect*) *matching* on $[2n]$ is a set partition of $[2n]$ such that each block has size 2. We also depict a matching on $[2n]$ as a diagram consisting of $2n$ vertices and n arcs where any pair of arcs has no common vertex. A *crossing* is a pair of arcs $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. A matching is called *noncrossing* if the matching has no crossing, and *nonnesting* if there is no pair of arcs $\{a, d\}$ and $\{b, c\}$ with $a < b < c < d$; see Figure 2. For an arc $\{i, j\}$ with $i < j$, i is called an *opener* and j is called

1	3	4	6
2	5	7	8

Figure 1. A standard Young tableau of shape (4, 4).

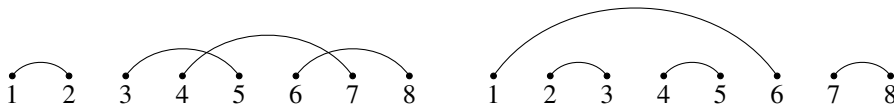


Figure 2. Two matchings on [8]. The first one is nonnesting, while the second one is noncrossing.

a closer. Let Mat_{2n} (NC_{2n} and NN_{2n} , respectively) stand for the set of (noncrossing and nonnesting, respectively) matchings on $[2n]$.

Note that there is a natural bijection between $\text{SYT}(n, n)$ and NN_{2n} . For $T \in \text{SYT}(n, n)$, connect two vertices lying on the same column of T via an arc; then we obtain a nonnesting matching. For instance, the tableau in Figure 1 and the first matching in Figure 2 are under this correspondence. Using this correspondence, we index the Specht basis for $\mathcal{S}^{(n,n)}$ by nonnesting matchings of $[2n]$, instead of standard Young tableaux of shape (n, n) :

$$\{v_M \in \mathcal{S}^{(n,n)} : M \in \text{NN}_{2n}\}.$$

We now consider the $2 \times 2n$ matrix

$$z = \begin{bmatrix} z_{1,1} & z_{1,2} & \dots & z_{1,2n} \\ z_{2,1} & z_{2,2} & \dots & z_{2,2n} \end{bmatrix},$$

where $z_{i,j}$'s are indeterminates. For $1 \leq i < j \leq 2n$, let $\Delta_{ij} := \Delta_{ij}(z)$ be the maximal minor of z with respect to the i th and j th columns (i.e., $\Delta_{ij} = z_{1,i}z_{2,j} - z_{1,j}z_{2,i}$). For a matching $M \in \text{Mat}_{2n}$, let

$$\Delta_M := \Delta_M(z) = \prod_{\{i,j\} \in M} \Delta_{ij} \in \mathbb{C}[z_{1,1}, \dots, z_{2,2n}].$$

It is important to note that the polynomials Δ_{ij} satisfy the following relation: For $1 \leq a < b < c < d \leq 2n$,

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc}. \tag{1}$$

We define a vector space W_n to be the \mathbb{C} -span of Δ_M for all $M \in \text{Mat}_{2n}$. In [KR84], it turns out that the set

$$\{\Delta_M \in W_n : M \in \text{NC}_{2n}\} \tag{2}$$

forms a basis for W_n . We call this basis the *web basis*. (The web basis was developed in the SL_2 -invariant theory due to Kuperburg [Kup96], and its original construction slightly differs from the one we describe above. But they are essentially the same; see [Rho19].)

In addition, there is a natural \mathfrak{S}_{2n} -action on W_n as follows: Regarding a permutation $\sigma \in \mathfrak{S}_{2n}$ as a $2n \times 2n$ permutation matrix, define $\sigma \cdot \Delta_M(z) := \Delta_M(z\sigma^{-1})$. Then the space W_n is closed under this action and hence carries an \mathfrak{S}_{2n} -module structure. Furthermore, the \mathfrak{S}_{2n} -module W_n is isomorphic to the Specht module $\mathcal{S}^{(n,n)}$ [PPR09]. Therefore, due to Schur's lemma, there is a unique (up to scalar) isomorphism $\varphi : W_n \rightarrow \mathcal{S}^{(n,n)}$.

Let M_0 be the unique matching which is simultaneously noncrossing and nonnesting (i.e., $M_0 = \{\{1, 2\}, \dots, \{2n - 1, 2n\}\}$). Due to [RT19], the isomorphism φ maps Δ_{M_0} to v_{M_0} up to scalar, and thus

we choose an appropriate scalar so that $\varphi(\Delta_{M_0}) = v_{M_0}$. We also let $w_M := \varphi(\Delta_M)$ for each $M \in \text{NC}_{2n}$. Then the Specht basis can expand into (the image of) the web basis: For $M \in \text{NN}_{2n}$,

$$v_M = \sum_{M' \in \text{NC}_{2n}} a_{MM'} w_{M'}.$$

In [RT19], Russell and Tymoczko initiated the combinatorial study of the transition matrix

$$A = (a_{MM'})_{M \in \text{NN}_{2n}, M' \in \text{NC}_{2n}}.$$

They constructed directed graphs on the standard Young tableaux and noncrossing matchings, and using them, showed the unitriangularity of the matrix. They also gave some open problems related to their results. One of them is the positivity of the entries of A , which was proved by Rhoades soon after.

Theorem 1.1 [Rho19]. *The entries $a_{MM'}$ of the transition matrix A are nonnegative integers.*

Although Rhoades established the positivity phenomenon for entries of A , he did not find an explicit combinatorial interpretation of the nonnegative integer $a_{MM'}$, c.f. [Rho19, Problem 1.3]. Inspired by his work, we introduce a new family of permutations which are enumerated by the integers $a_{MM'}$, and we study their enumerative properties.

Our strategy is based on Rhoades’s observation [Rho19]. He figured out that the entries $a_{MM'}$ are related to resolving crossings of matchings in the following sense: For a matching $M \in \text{Mat}_{2n}$, let $\{a, c\}$ and $\{b, d\}$ be a crossing pair in M (if it exists), where $a < b < c < d$. Let M' and M'' be the matchings identical to M except that $\{a, b\}$ and $\{c, d\}$ in M' , and $\{a, d\}$ and $\{b, c\}$ in M'' . Then, by the relation (1), we have

$$\Delta_M = \Delta_{M'} + \Delta_{M''}. \tag{3}$$

In addition, the number of crossing pairs in M' (respectively, M'') is strictly less than the number of crossing pairs in M . Therefore, iterating the resolving procedure gives the expansion of Δ_M in terms of the basis (2). In other words, when we write

$$\Delta_M = \sum_{M' \in \text{NC}_{2n}} c_{MM'} \Delta_{M'}, \tag{4}$$

the coefficient $c_{MM'}$ is equal to the number of occurrences of the noncrossing matching M' obtained by iteratively resolving crossings in M . Note that the order of the choice of crossing pairs does not affect the expansion of Δ_M . Rhoades showed that for $M \in \text{NN}_{2n}$ and $M' \in \text{NC}_{2n}$, the entry $a_{MM'}$ of the transition matrix equals $c_{MM'}$. Hence, to give a combinatorial interpretation of $a_{MM'}$, we track the resolving process from a nonnesting matching to noncrossing matchings.

To state our main result, we need some preliminaries. We first note that noncrossing matchings and nonnesting matchings are *Catalan objects*. In other words, they are enumerated by Catalan numbers. Another famous Catalan object is a Dyck path. A *Dyck path* of length $2n$ is a lattice path from $(0, 0)$ to (n, n) consisting of n north steps $(0, 1)$ and n east steps $(1, 0)$ that does not pass below the line $y = x$. We write N and E for the north step and the east step, respectively. We therefore regard a Dyck path as a sequence consisting of n N ’s and n E ’s. Let Dyck_{2n} be the set of Dyck paths of length $2n$. Identifying a Dyck path with the region below the path, we give a natural partial order on Dyck_{2n} by inclusion, denoted by \subseteq . For instance, the Dyck path $N \cdots NE \cdots E$ where n N ’s precede n E ’s is the maximum path in Dyck_{2n} with respect to the partial order, whereas the path $NENE \cdots NE$ is the minimum path. In Section 2, we define a map $D : \text{Mat}_{2n} \rightarrow \text{Dyck}_{2n}$, and by abuse of notation, a map $D : \mathfrak{S}_n \rightarrow \text{Dyck}_{2n}$. We also define a map $M : \mathfrak{S}_n \rightarrow \text{NC}_{2n}$. Finally, we introduce a new family of permutations, called *web permutations*. With these data, we now present one of our main results.

Theorem 1.2. For matchings $M \in \text{NN}_{2n}$ and $M' \in \text{NC}_{2n}$, the entry $a_{MM'}$ is equal to the number of web permutations $\sigma \in \mathfrak{S}_n$ such that $D(\sigma) \subseteq D(M)$ and $M(\sigma) = M'$.

The theorem follows almost immediately from the definition of the novel permutations. However, the definition does not directly tell us whether a given permutation is a web permutation or not. Our second main result, Theorem 3.4, explains how to characterize these permutations in terms of their cycle structures. Using this characterization, we deduce the results in [RT19, IZ22] concerning the unitriangularity of the transition matrix and a necessary and sufficient condition for additional vanishing entries. Furthermore, the characterization leads us to enumerative properties of web permutations. To be more precise, Euler numbers count web permutations (Corollary 4.2), and we present a conjecture related web permutations and the Seidel triangle, which generalizes a result of Nakamigawa [Nak20] (Conjecture 4.9).

The article is organized as follows. In Section 2, we give a new model, called a grid configuration, for representing matchings. Within this model, we resolve crossings in nonnesting matchings until there is no crossing. We then define web permutations from the noncrossing grid configurations, and we prove Theorem 1.2. In the next two sections, we study some properties of web permutations. In Section 3, we give a characterization of web permutations. We show that web permutations are closely related to André permutations. Section 4 provides some interesting enumerative properties of web permutations.

2. Grid configurations and web permutations

In this section, we define grid configurations which represent matchings in a ‘rigid’ setting. We describe the procedure of resolving crossings within this model. We then introduce a new class of permutations, called web permutations. This provides a combinatorial interpretation for the entries $a_{MM'}$ of the transition matrix.

Consider an $n \times n$ grid in the xy -plane with corners $(0, 0)$, $(0, n)$, $(n, 0)$ and (n, n) . We denote each cell by (i, j) , where i and j are the x - and y -coordinates of its upper-right corner. Let $\sigma \in \mathfrak{S}_n$ be a permutation. For each $1 \leq i \leq n$, mark the cell $(i, \sigma(i))$, and draw a horizontal line to the left and a vertical line to the top from the marked cell. We call this the *empty grid configuration* of σ . A cell (i, j) is called a *crossing* of σ if there are both a vertical line and a horizontal line through the cell in the empty grid configuration; that is, $\sigma(i) < j$ and $i < \sigma^{-1}(j)$. We denote by $\text{Cr}(\sigma)$ the set of all crossings of σ . For a subset $E \subseteq \text{Cr}(\sigma)$, the *grid configuration* $G(\sigma, E)$ of a pair (σ, E) is defined to be the empty grid configuration of σ where each crossing in E is replaced by an elbow as shown in Figure 3. In particular, the empty grid configuration of σ is $G(\sigma, \emptyset)$.

For the $n \times n$ grid, we label leftmost vertical intervals from bottom to top with 1 through n and uppermost horizontal intervals from left to right with $n + 1$ through $2n$. With this label of boundary intervals, a grid configuration can be considered as a matching on $[2n]$ as follows: Each strand joining i th and j th boundary intervals represents an arc connecting i and j ; see Figure 4. We denote by $M(\sigma, E)$ the matching associated to the grid configuration $G(\sigma, E)$. For short, we write $M(\sigma) = M(\sigma, \text{Cr}(\sigma))$.

We define a partial order on cells of the $n \times n$ grid by $(x, y) \geq (x', y')$ if $x \leq x'$ and $y \geq y'$. In other words, $(x, y) \geq (x', y')$ if and only if the cell (x, y) lies on the upper-left quadrant at (x', y') .

In the relation (3), the crossing pair $(\{a, c\}$ and $\{b, d\})$ in M is resolved in two different ways. Similarly, we introduce two operations to resolve a crossing in a grid configuration. For a permutation σ and $E \subseteq \text{Cr}(\sigma)$, let $c = (i, j)$ be a crossing in the grid configuration $G(\sigma, E)$, i.e., $c \in \text{Cr}(\sigma) \setminus E$. Then one can resolve the crossing c by replacing it with an elbow, which gives the grid configuration

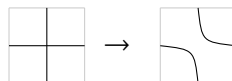


Figure 3. A crossing to an elbow.

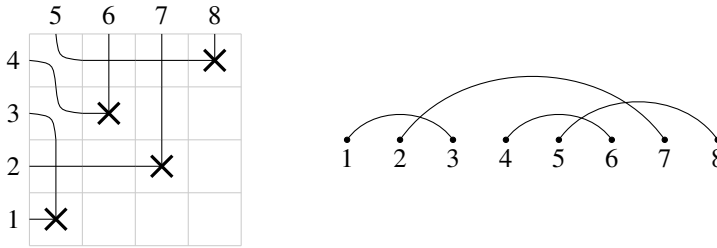


Figure 4. The grid configuration $G(1324, \{(1, 3), (1, 4)\})$ and the corresponding matching.

$G(\sigma, E \cup \{c\})$. We call this operation *smoothing*. Note that the smoothing operation coincides with the process giving M' from M in the relation (3). To obtain a grid configuration corresponding to M'' , we need an additional assumption on the grid configuration $G(\sigma, E)$ and the crossing c . We assume that

$$\text{there is no cell in } E \text{ which is less than the crossing } c. \tag{5}$$

Let $c = (i, j)$, $k = \sigma(i)$ and $\ell = \sigma^{-1}(j)$. Due to the assumption, the crossing is made by the vertical line from the marked cell (i, k) and the horizontal line from the marked cell (ℓ, j) . Let σ' be a permutation given by

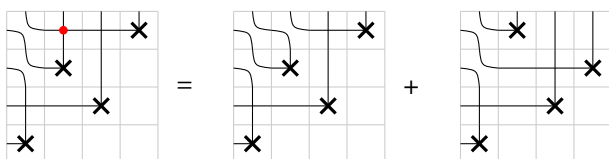
$$\sigma'(m) = \begin{cases} j, & \text{if } m = i, \\ k, & \text{if } m = \ell, \\ \sigma(m), & \text{otherwise.} \end{cases}$$

Equivalently, we let $\sigma' = \sigma \cdot t_{i,\ell}$, where $t_{i,\ell}$ is the transposition which swaps i and ℓ . Then $c \notin \text{Cr}(\sigma')$, and every crossing in $\text{Cr}(\sigma)$ which is not less than c belongs to $\text{Cr}(\sigma')$, and vice versa. Hence, we have $E \subseteq \text{Cr}(\sigma')$. Moreover, at the level of matchings, the only two arcs going through the cell c in $G(\sigma, E)$ are changed in $M(\sigma', E)$. In other words, the crossing c is resolved in $G(\sigma', E)$, and the others leave unchanged. In addition, $M(\sigma, E \cup \{c\}) \neq M(\sigma', E)$. Therefore, we conclude $M(\sigma', E) = M''$ in the relation (3). We call this operation, giving $G(\sigma', E)$ from $G(\sigma, E)$, *switching*. Roughly speaking, the switching operation is that if two marked cells (i, k) and (ℓ, j) make a crossing at (i, j) , then move the marked cells to (i, j) and (ℓ, k) .

We consider a grid configuration G as the vector $\Delta_{M(G)}$. We can then rewrite the relation (3) in terms of grid configurations as follows: For a permutation σ , $E \subseteq \text{Cr}(\sigma)$ and a crossing $c \in \text{Cr}(\sigma) \setminus E$ satisfying the condition (5), we have

$$G(\sigma, E) = G(\sigma, E \cup \{c\}) + G(\sigma', E).$$

For example, let $\sigma = 1324 \in \mathfrak{S}_4$ and $E = \{(1, 3), (1, 4)\}$, and consider the grid configuration $G(\sigma, E)$ which is shown in Figure 4. Resolving a crossing $c = (2, 4) \in \text{Cr}(\sigma) \setminus E$, we have



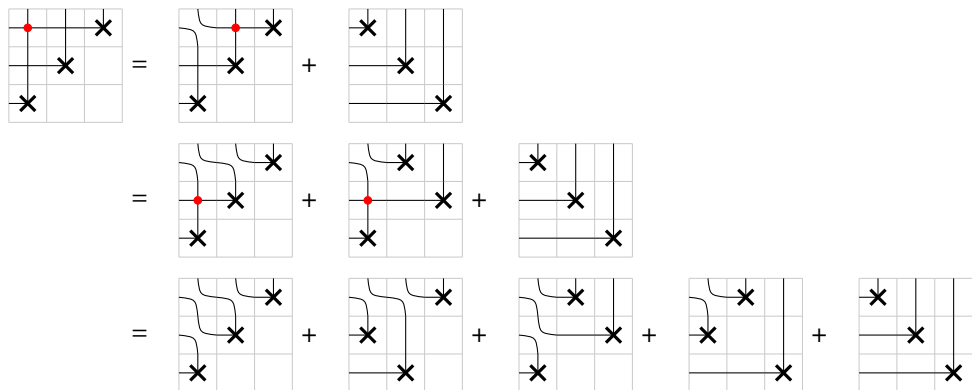
Here, the red dot indicates the crossing c .

In what follows, we always choose a set E as an upper order ideal in $\text{Cr}(\sigma)$, and a crossing c as a maximal crossing in $\text{Cr}(\sigma) \setminus E$. Then $E \cup \{c\}$ is also an upper order ideal in $\text{Cr}(\sigma)$, and furthermore, the condition (5) always will be satisfied.

From the grid configuration $G(id, \emptyset)$, we obtain two grid configurations by resolving the maximal crossing $(1, n)$ by the smoothing and switching operations, respectively. By resolving crossings until there is no crossing left, we get grid configurations of the form $G(\sigma, \text{Cr}(\sigma))$. For each remaining grid configuration $G(\sigma, \text{Cr}(\sigma))$, the permutation σ is called a *web permutation* of $[n]$ and we denote the set of web permutations of $[n]$ by Web_n . We will see that web permutations do not depend on the choice of maximal crossings c in the resolving processes (Proposition 2.1). We would like to highlight the following expression in terms of grid configurations:

$$G(id, \emptyset) = \sum_{\sigma} G(\sigma, \text{Cr}(\sigma)). \tag{6}$$

Here, the summation on the right-hand side is over all permutations σ that arise from resolving crossings in the grid configuration $G(id, \emptyset)$ through the smoothing and switching operations, continuing until no crossings remain. This equation can be regarded as an enhanced representation of (4) with respect to grid configurations. For example, starting from the grid configuration $G(id, \emptyset)$ for $n = 3$, we have



Therefore, we conclude that $\text{Web}_3 = \{123, 213, 132, 231, 321\}$. The following proposition justifies that web permutations are well-defined.

Proposition 2.1. *The expansion in (6) is unique. In other words, the grid configurations appearing in (6) do not depend on the order of resolving procedure (choice of maximal crossings). In addition, the permutations σ in (6) are all distinct.*

Proof. Let $G(\sigma, E)$ be a grid configuration obtained from $G(id, \emptyset)$ through a process of resolving crossings using the smoothing and switching operations. Let c and c' denote two maximal crossings in $G(\sigma, E)$, where the x -coordinate of c is less than the x -coordinate of c' . There are two cases to consider: (a) The y -coordinate of c and the x -coordinate of c' are the same, and (b) they are different. These cases are illustrated in Figure 5, where the crossings c and c' are indicated by red dots.

In the first case, if we resolve both c and c' with the same operation (both by the smoothing operations or both by the switching operations), the order of resolving c and c' becomes irrelevant. Hence, we only need to show that resolving c in a way and c' in the other way leads to the same grid configuration obtained by resolving c' first and then c . This can be directly verified. Furthermore, for the second case, it is evident that the order of resolving crossings c and c' is irrelevant.

In general, any total order that extends the partial order \geq on cells can be derived from another total order by performing a sequence of interchanging the order of two incomparable cells. In our case, it yields that it is sufficient to demonstrate that we can interchange the order of two maximal crossings in $G(\sigma, E)$.

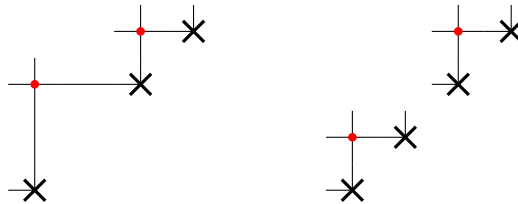


Figure 5. Two cases of maximal crossings.

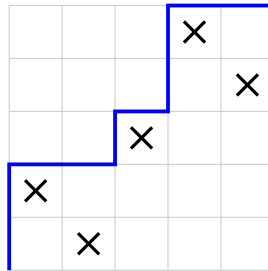


Figure 6. The Dyck path $D(\sigma)$ associated to $\sigma = 21354$ is NNEENENNEE.

We now verify the distinctness part. Fix a total order on cells which extends the partial order, and we resolve crossings in $G(id, \emptyset)$ to obtain (6) with respect to the total order. Let $G(\sigma, Cr(\sigma))$ and $G(\tau, Cr(\tau))$ be two grid configurations obtained from $G(id, \emptyset)$ through two distinct processes of resolving crossings using the smoothing and switching operations, until no crossings remain. Suppose that the processes of resolving crossings to obtain $G(\sigma, Cr(\sigma))$ and $G(\tau, Cr(\tau))$ are the same until we obtain a grid configuration $G(\rho, E)$, and let $c = (i, j)$ be the maximum crossing in $Cr(\rho) \setminus E$ under the fixed total order. Furthermore, suppose that we resolve c by applying the smoothing operation in the process giving $G(\sigma, Cr(\sigma))$ and by applying the switching operation in the process giving $G(\tau, Cr(\tau))$. As there is an elbow at (i, j) in $G(\sigma, Cr(\sigma))$, we have $\sigma(i) < j$. However, since there is a marking at (i, j) in $G(\tau, Cr(\tau))$ after applying the switching operation to c , we have $\tau(i) = j$. Therefore, we conclude that σ and τ are distinct. \square

For a matching M , record N for openers and E for closers reading M from left to right. This gives the Dyck path $D(M)$ in the $n \times n$ grid. It is well-known that the two restrictions of the map $D : Mat_{2n} \rightarrow Dyck_{2n}$ to NC_{2n} and NN_{2n} are bijections. To a permutation σ , we associate the minimum Dyck path $D(\sigma)$ where every cell $(i, \sigma(i))$ lies below the path; see Figure 6.

Given a nonnesting matching $M \in NN_{2n}$, let $E(M)$ be the set of cells in the $n \times n$ grid which are above the path $D(M)$. Consider the matching $M(id, E(M))$. In the region below $D(M)$, if $\{i, j\}$ is an arc in M , then the i th step and j th step of $D(M)$ are connected by an arc in $G(id, E(M))$. Moreover, in the region above $D(M)$, each i th step is connected to the i th boundary interval. Thus, $\{i, j\}$ is also an arc in $M(id, E(M))$, so we conclude that the matchings M and $M(id, E(M))$ coincide. For example, let M be the first matching in Figure 2. Then the corresponding path $D(M)$ is NENNENEE, and $E(M) = \{(1, 2), (1, 3), (1, 4), (2, 4)\}$. The grid configuration $G(id, E(M))$ is shown in Figure 7, and one can see $M(id, E(M)) = M$. Similar to the definition of Web_n , we consider the equation

$$G(id, E(M)) = \sum_{\sigma} G(\sigma, Cr(\sigma)).$$

Here, the summation on the right-hand side is over all permutations σ that arise from resolving crossings in the grid configuration $G(id, E(M))$ through the smoothing and switching operations, continuing until no crossings remain. We then define Web_M to be the set of permutations σ appearing in the right-hand side of the above equation. In particular, $Web_n = Web_M$, where $M = \{\{1, n+1\}, \{2, n+2\}, \dots, \{n, 2n\}\}$.

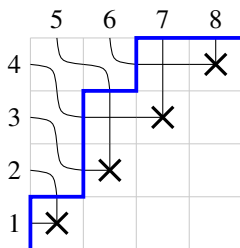


Figure 7. The grid configuration $G(id, E(M))$ and the Dyck path $D(M)$ where $M = \{\{1, 2\}, \{3, 5\}, \{4, 7\}, \{6, 8\}\}$.

Using the above notations, we prove one of our main results that tells us which web permutations contribute to the entry $a_{MM'}$.

Proof of Theorem 1.2. By the definition of web permutations, we have

$$a_{MM'} = |\{\sigma \in \text{Web}_M : M(\sigma) = M'\}|.$$

Hence, it is enough to show that

$$\text{Web}_M = \{\sigma \in \text{Web}_n : D(\sigma) \subseteq D(M)\}. \tag{7}$$

We can obtain the grid configuration $G(id, E(M))$ from $G(id, \emptyset)$ by applying the smoothing operation to all crossings in $E(M)$. Since Proposition 2.1 says that Web_n does not depend on the order of resolving processes, we obtain $\text{Web}_M \subseteq \text{Web}_n$. From this, it is clear that

$$\begin{aligned} \text{Web}_M &= \{\sigma \in \text{Web}_n : E(M) \subseteq \text{Cr}(\sigma)\} \\ &= \{\sigma \in \text{Web}_n : (i, \sigma(i)) \notin E(M) \text{ for all } i\}, \end{aligned}$$

which proves the claim (7). □

3. Characterization of web permutations

In the previous section, we have introduced the new class of permutations which are obtained by tracking the resolving process. In fact, Theorem 1.2 is just a byproduct of the definition of web permutations. In this section, we provide a characterization of these permutations. This characterization depends only on their permutation structures, especially their cycle structures. Using this characterization, we prove the results in [RT19, IZ22] concerning the unitriangularity of the transition matrix and a necessary and sufficient condition for additional vanishing entries.

We begin with recalling two ways to represent permutations. One way is the *one-line notation* which we have already used – that is, regarding a permutation as a word. More precisely, for a permutation $\sigma : [n] \rightarrow [n]$, we write $\sigma = \sigma_1\sigma_2 \dots \sigma_n$, where $\sigma_i = \sigma(i)$. Another way to write permutations is the *cycle notation*. Instead of the precise definition of this notation, we give an example; for the definition, see [Sta12]. Let $\sigma = 564132 \in \mathfrak{S}_6$; then the cycle notation of σ is $(1, 5, 3, 4)(2, 6)$. We always use parentheses and commas when writing cycles.

Let us examine how the two resolving operations on a grid configuration $G(\sigma, E)$ affect the permutation σ , particularly its cycle structure. By definition, the smoothing operation does not change the permutation σ , whereas the switching operation does. Figure 8 provides an illustrative example of the switching operation. The left one in Figure 8 is a grid configuration $G(\sigma, E)$, where $\sigma = (1, 5, 4, 6)(2, 3, 7)$ and $E = \text{Cr}(\sigma) \setminus \{(6, 2)\}$. Applying the switching operation to the crossing $(6, 2)$ in $G(\sigma, E)$ yields the grid configuration $G(\tau, \text{Cr}(\tau))$, where $\tau = (1, 5, 4, 6, 2, 3, 7)$; it is depicted

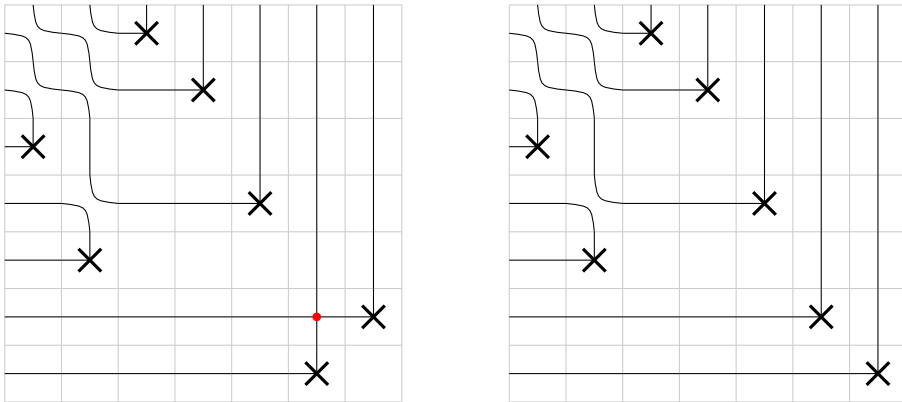


Figure 8. Applying the switching operation to the left grid configuration with the crossing indicated as the red dot gives the right one.

on the right in Figure 8. Note that both permutations σ and τ are web permutations. One can observe that in this example, the switching operation merges two cycles into one cycle. Keeping this example in mind might be helpful for comprehending this section.

To describe our characterization of web permutations, we review the notion of André permutations and define an analogue of them. André permutations were introduced by Foata and Schützenberger [FS73], and have been studied with several applications in the literature; see, for example, [Sta94, FH16]. One of the interesting properties of them is that they are enumerated by Euler numbers; see Section 4.

We now think of permutations as words consisting of distinct positive integers. *André permutations* are defined recursively as follows. First, the empty word and each one-letter word are André permutations. For a permutation $w = w_1 w_2 \cdots w_n$ with $n \geq 2$, let w_k be the smallest letter in w . Then w is an André permutation if both $w_1 \cdots w_{k-1}$ and $w_{k+1} \cdots w_n$ are André permutations and $\max\{w_1, \dots, w_{k-1}\} < \max\{w_{k+1}, \dots, w_n\}$. For example, a word 546237 is an André permutation because the letter 2 is smallest, both two words 546 and 37 are André permutations, and $\max\{5, 4, 6\} < \max\{3, 7\}$. Using this notion, we define a cycle analogue of André permutations.

Definition 3.1. Let $C = (a_1, \dots, a_k)$ be a cycle with $a_1 = \min\{a_1, \dots, a_k\}$. We say that C is an *André cycle* if the permutation $a_2 \cdots a_k$ is an André permutation.

For instance, a cycle $C = (2, 3, 7, 1, 5, 4, 6)$ is an André cycle since $C = (1, 5, 4, 6, 2, 3, 7)$ and the permutation 546237 is an André permutation.

For a cycle $C = (a_1, \dots, a_k)$, we write $\min C = \min\{a_1, \dots, a_k\}$ and $\max C = \max\{a_1, \dots, a_k\}$ for short. The following lemma is useful in the sequel.

Lemma 3.2. Let $C = (a_1, \dots, a_k)$ be an André cycle with $a_1 = \min C$. Then $a_k = \max C$.

Proof. By definition, the last letter of an André permutation is the largest element in the permutation. This fact directly gives the proof. \square

The following lemma gives how to obtain a new André cycle from old André cycles.

Lemma 3.3. Let $C_1 = (a_1, \dots, a_k)$ and $C_2 = (b_1, \dots, b_\ell)$ be André cycles with $a_1 = \min C_1$ and $b_1 = \min C_2$. If $a_1 < b_1$ and $a_k < b_\ell$, then the cycle $(a_1, \dots, a_k, b_1, \dots, b_\ell)$ is also an André cycle.

Proof. We induct on k . First, consider the base case $k = 1$. Since $a_1 = \min\{a_1, b_1, \dots, b_\ell\}$, we only need to show that the permutation $b_1 \cdots b_\ell$ is an André permutation. This follows immediately from the definition of André permutations.

We now suppose $k \geq 2$. Recall that the two permutations $a_2 \cdots a_k$ and $b_2 \cdots b_\ell$ are André permutations. In addition, by Lemma 3.2 and the assumption $a_k < b_\ell$, we have $\max\{a_2, \dots, a_k\} <$

$\max\{b_2, \dots, b_\ell\}$. Thus, if $b_1 < \min\{a_2, \dots, a_k\}$, then the permutation $a_2 \cdots a_k b_1 \cdots b_\ell$ is an André permutation. Otherwise, let $a_p = \min\{a_2, \dots, a_k\}$ for some p , so that $a_p < b_1$ and both $a_2 \cdots a_{p-1}$ and $a_{p+1} \cdots a_k$ are André permutations. By the induction hypothesis, we have that the cycle $(a_p, \dots, a_k, b_1, \dots, b_\ell)$ is an André cycle. It is also clear that $a_{p-1} < a_k < b_\ell$. Again, by the induction hypothesis, we deduce that the cycle $(a_1, \dots, a_k, b_1, \dots, b_\ell)$ is an André cycle, which yields the desired result. \square

We now show another main result of the article, which gives a characterization of web permutations.

Theorem 3.4. *A permutation $\sigma \in \mathfrak{S}_n$ is a web permutation if and only if each cycle of σ is an André cycle.*

Proof. Recall that the web permutations do not depend on the order of choices of maximal crossings. Hence, we fix the following total order on the cells in the $n \times n$ grid, which completes the partial order, and we assume that our resolving process respects this total order: For two cells (i, j) and (i', j') , we let $(i, j) > (i', j')$ if either $j > j'$, or $j = j'$ and $i < i'$.

We first prove the ‘only if’ part. Let σ be a web permutation, and

$$(G^{(0)} = G(id, \emptyset), G^{(1)}, \dots, G^{(r)} = G(\sigma, Cr(\sigma)))$$

be the sequence of grid configurations where $G^{(k)}$ is obtained from $G^{(k-1)}$ by resolving a single crossing for each k , with respect to the total order. We write $G^{(k)} = G(\sigma^{(k)}, E^{(k)})$. Also let $c^{(k)}$ be the crossing in $Cr(\sigma^{(k-1)}) \setminus E^{(k-1)}$ such that $G^{(k)}$ is obtained from $G^{(k-1)}$ by resolving $c^{(k)}$.

It is obvious that the identity permutation $\sigma^{(0)} = id$ consists of André cycles. We claim that each $\sigma^{(k)}$ also consists of André cycles for $1 \leq k \leq r$; in particular, so does σ . Fix an integer $1 \leq k \leq r$. We use an inductive argument, so suppose that each cycle of $\sigma^{(k-1)}$ is an André cycle. If $G^{(k)}$ is obtained by applying the smoothing operation to the crossing $c^{(k)}$ in $G^{(k-1)}$, then $\sigma^{(k-1)} = \sigma^{(k)}$, and thus there is nothing to prove. Therefore, we assume that $G^{(k)}$ is obtained from $G^{(k-1)}$ by applying the switching operation to the crossing $c^{(k)} = (i, j)$. Then

$$\sigma^{(k-1)}(i) < j \quad \text{and} \quad i < (\sigma^{(k-1)})^{-1}(j). \tag{8}$$

Let C_1, \dots, C_ℓ be cycles of $\sigma^{(k-1)}$. We first observe that for each $1 \leq p \leq \ell$, all entries in C_p except the minimum $\min C_p$ are greater than j . We justify this observation later. From this, we have that i and j are contained in different cycles of $\sigma^{(k-1)}$. Indeed, if i and j lie on the same cycle, then $\sigma^{(k-1)}(i)$ also lies on the cycle, but it is a contradiction to (8). Without loss of generality, let $C_1 = (a_1, \dots, a_s)$ and $C_2 = (b_1, \dots, b_t)$ contain i and j , respectively, with $a_1 = \min C_1$ and $b_1 = \min C_2$. By the first inequality of (8) and the observation, $a_1 = \sigma^{(k-1)}(i)$ and $b_1 = j$, so $a_s = i$ and $b_t = (\sigma^{(k-1)})^{-1}(j)$. By definition, resolving the crossing (i, j) by applying the switching operation merges two cycles C_1 and C_2 into the cycle $C = (a_1, \dots, a_s, b_1, \dots, b_t)$, and leaves other cycles unchanged. It therefore follows from Lemma 3.3 and (8) that the cycle C is an André cycle. Note that $\min C = a_1 = \sigma^{(k-1)}(i) < j$, and there is no crossing on row j in the grid configuration $G^{(k)}$. Hence the crossing $c^{(k+1)}$ lies below row j , which implies the observation inductively.

We now prove the ‘if’ part. It suffices to show that we obtain any André cycle by iterating resolving processes to the identity permutation along the total order. We induct on the length of an André cycle where the base case being trivial. Suppose that $C = (a_1, \dots, a_k)$ is an André cycle with $k \geq 2$ and $a_1 = \min C$. Then, by definition, the permutation $a_2 \cdots a_k$ is an André permutation. Let $a_p = \min\{a_2, \dots, a_k\}$ for some p , so $a_2 \cdots a_{p-1}$ and $a_{p+1} \cdots a_k$ are also André permutations. Thus, the cycles (a_1, \dots, a_{p-1}) and (a_p, \dots, a_k) are André cycles. By the induction hypothesis, we can obtain the web permutation $\sigma = (a_1, \dots, a_{p-1})(a_p, \dots, a_k)$ by resolving processes. More precisely, we can obtain the grid configuration $G(\sigma, E)$ such that for $1 \leq i \leq n$ and $j \leq a_p$, $(i, j) \notin E$. Furthermore, one can easily check that the cell (a_{p-1}, a_p) belongs to $Cr(\sigma)$, so $(a_{p-1}, a_p) \in Cr(\sigma) \setminus E$. We then obtain

the desired cycle C by switching the crossing (a_{p-1}, a_p) in the grid configuration $G(\sigma, E)$, which completes the proof. \square

As an application of the characterization, we show that the transition matrix $(a_{MM'})$ is unitriangular with respect to a certain order on NN_{2n} and NC_{2n} , and we determine which entries $a_{MM'}$ vanish. These are already known due to Russell–Tymoczko [RT19] and Im–Zhu [IZ22].

Before we give the vanishing condition, we first show that the set Web_n includes a well-studied class of permutations. For a permutation $\sigma = \sigma_1 \cdots \sigma_n$, we say that σ contains a 312-pattern if there exist three indices $1 \leq i < j < k \leq n$ such that $\sigma_j < \sigma_k < \sigma_i$. A permutation is 312-avoiding if it does not contain a 312-pattern. Note that 312-avoiding permutations are a Catalan object. Furthermore, the restriction of $D : \mathfrak{S}_n \rightarrow \text{Dyck}_{2n}$ to the set of 312-avoiding permutations of $[n]$ is a bijection.

Corollary 3.5. *A 312-avoiding permutation is a web permutation.*

Proof. By Theorem 3.4, it suffices to show the following: For a permutation σ , if there is a cycle $C = (a_1, \dots, a_\ell)$ which is not an André cycle in σ , then σ contains a 312-pattern which consists of a_i 's.

We use induction on the length of C . Since any cycle of length less than 3 is an André cycle, the base case is when the length of C is 3. The only case is of the form (a_1, a_2, a_3) , where $a_1 < a_3 < a_2$. Thus, σ contains a 312-pattern $a_1 < a_3 < a_2$.

Now assume that the length of C is larger than 3. Write $C = (a_1, a_2, \dots, a_\ell, b_1, b_2, \dots, b_r)$, where a_1 and b_1 is the smallest and the second smallest elements of C , respectively. Then one of the following holds:

- i) $C_1 = (a_1, a_2, \dots, a_\ell)$ is not an André cycle.
- ii) $C_2 = (b_1, b_2, \dots, b_r)$ is not an André cycle.
- iii) both C_1 , and C_2 are André cycles and $\max C_1 = a_\ell > b_r = \max C_2$.

For the first case, by the induction hypothesis, there exist integers $0 \leq i, j, k \leq \ell$ such that $a_i < a_j < a_k$ and $a_{j+1} < a_{k+1} < a_{i+1}$, where the subscripts are interpreted modulo ℓ . Note that $\sigma_{a_i} = a_{i+1}$ except for $\sigma_{a_\ell} = b_1$. Since b_1 is the second smallest element, replacing a_1 with b_1 does not change the pattern of $a_{i+1}a_{j+1}a_{k+1}$. Therefore, σ contains a 312-pattern as we claimed. The second case can be proved similarly to the first case. For the last case, $a_1 < b_r < a_\ell$ forms a 312-pattern. Indeed, $\sigma_{a_1} = a_2, \sigma_{b_r} = a_1, \sigma_{a_\ell} = b_1$ and $a_1 < b_1 < a_2$. \square

Recall that the set Dyck_{2n} has a partial order \subseteq , and there are bijections D from NN_{2n} and from NC_{2n} to Dyck_{2n} . Then the maps D induce a partial order on NN_{2n} and NC_{2n} . Furthermore, when we choose a total order on Dyck_{2n} that completes the partial order \subseteq , the maps D give a total order on NN_{2n} and NC_{2n} .

Remark 3.6. In [RT19], Russell and Tymoczko defined a directed graph Γ on NC_{2n} and defined a partial order on NC_{2n} using the digraph. The graph Γ is an edge-labeled directed graph whose vertex set is the set of noncrossing matchings, and its labeled edges are given as follows. For $M, M' \in \text{NC}_{2n}$, assign a labeled, directed edge $M \xrightarrow{i} M'$ if both of the following conditions hold:

- i) M has arcs $\{j, k\}$ and $\{i, i + 1\}$, while M' has arcs $\{j, i\}$ and $\{i + 1, k\}$, where $j < i < k$.
- ii) Other arcs in M and M' are the same.

The graph Γ defines a partial order on NC_{2n} by letting $M' \leq M$ if there is a directed path from M to M' in Γ . Russell and Tymoczko also defined a partial order on NN_{2n} via a well-known bijection between NN_{2n} and NC_{2n} .

In terms of Dyck paths, the labeled, directed edges defined by Russell and Tymoczko can be described as follows: For $D, D' \in \text{Dyck}_{2n}$, assign a labeled, directed edge $D \xrightarrow{i} D'$ if both of the following conditions hold:

- i) D has N at i th step and E at $(i + 1)$ st step, while D' has E at i th step and N at $(i + 1)$ st step.
- ii) Other steps in D and D' are the same.

In other words, we obtain D' from D by removing the cell containing i th and $(i + 1)$ st steps as edges in D . Thus, the two partial orders on Dyck paths defined by the cover relation and inclusion are the same. Therefore, the partial order defined by Russell and Tymoczko in terms of NC_{2n} , as well as NN_{2n} , coincides with ours.

We now take a total order on Dyck_{2n} which completes the partial order \subseteq , and thus we have the induced total order on NN_{2n} and NC_{2n} . We assume that orderings of rows and columns of the transition matrix $(a_{MM'})$ are the decreasing orders with respect to the total order on NN_{2n} and NC_{2n} . Then the entry $a_{MM'}$ is on the diagonal if and only if $D(M) = D(M')$.

We are now ready to prove the unitriangularity of the transition matrix $(a_{MM'})$ and the conjecture of Russell and Tymoczko [RT19, Conjecture 5.8] concerning the condition of the vanishing entries, which is later proved by Im and Zhu [IZ22, Theorem 1.1].

Corollary 3.7 [RT19, IZ22]. *Let $M \in \text{NN}_{2n}$ and $M' \in \text{NC}_{2n}$. Then $a_{MM'} > 0$ if and only if $D(M') \subseteq D(M)$. In particular, the transition matrix $(a_{MM'})$ is upper-triangular with ones along the diagonal.*

Proof. Recall that by the argument in the proof of Theorem 1.2, we have

$$a_{MM'} = \{\sigma \in \text{Web}_M : M(\sigma) = M'\}.$$

We first show that there are ones along the diagonal (i.e., $a_{MM'} = 1$ if $D(M) = D(M')$). Let σ be a permutation in Web_M satisfying $M(\sigma) = M'$. Denote the set of cells above the Dyck path $D(\sigma)$ by $E(\sigma)$. We claim that $E(\sigma) = \text{Cr}(\sigma)$. Since $E(\sigma) \subseteq \text{Cr}(\sigma)$ is obvious, suppose that we have $E(\sigma) \subsetneq \text{Cr}(\sigma)$, and let c be a maximal crossing in $\text{Cr}(\sigma) \setminus E(\sigma)$. Note that

$$D(M(\sigma, E(\sigma) \cup \{c\})) \subsetneq D(M(\sigma, E(\sigma))) = D(M).$$

Thus, if we resolve all crossings by applying the smoothing operation to obtain $G(\sigma, \text{Cr}(\sigma))$, the associated Dyck path $D(M') = D(M(\sigma))$ lies strictly below $D(M)$, which is a contradiction.

It is well-known that 312-avoiding permutations are only permutations satisfying the condition $\text{Cr}(\sigma) = E(\sigma)$, and the map $D : \text{Web}_n \rightarrow \text{Dyck}_{2n}$ is a bijection when restricted to 312-avoiding permutations (see [Sta12, §1.2]). Here, the restriction makes sense by Corollary 3.5. Combining these facts, it follows that each 312-avoiding permutation represents each one on the diagonal in the transition matrix.

To show the ‘only if’ part of the first assertion, assume that $D(M') \not\subseteq D(M)$. Then there exists a cell below the Dyck path $D(M')$ and above the Dyck path $D(M)$ (i.e., $E(M) \setminus E(M') \neq \emptyset$). Choose a maximal cell $c = (i, j)$ in $E(M) \setminus E(M') \neq \emptyset$. Let σ be a web permutation. If σ is in Web_M , then we have $c \in \text{Cr}(\sigma)$; thus, $\sigma_i < j$. However, if $M(\sigma) = M'$, then we have $\sigma_i = j$, which is a contradiction. Therefore, we have $a_{MM'} = 0$.

For the ‘if’ part, let M'' be the nonnesting matching such that $D(M'') = D(M')$. Then we have

$$\begin{aligned} a_{MM'} &= |\{\sigma \in \text{Web}_n : D(\sigma) \subseteq D(M), M(\sigma) = M'\}| \\ &\geq |\{\sigma \in \text{Web}_n : D(\sigma) \subseteq D(M') = D(M''), M(\sigma) = M'\}| \\ &= a_{M''M'} = 1. \end{aligned}$$

This completes the proof. □

4. Enumeration of web permutations

In this section, we focus on the number of web permutations. More precisely, we give a relation between web permutations and André cycles (Theorem 4.1), and we show that the numbers of web permutations equal Euler numbers. We also conjecture that the Seidel triangle can be recovered completely from the certain classes of web permutations.

We have characterized web permutations using André cycles (Theorem 3.4). We now present another relationship between web permutations and André cycles. Let us first review the Foata transformation $\widehat{\cdot} : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$. For a permutation $\sigma \in \mathfrak{S}_n$, the *canonical cycle notation* of σ is a cycle notation of σ such that its cycles are sorted based on the smallest elements of the cycles, and the smallest element of each cycle is written in the last place of the cycle. We define $\widehat{\sigma}$ to be the permutation obtained by dropping the parentheses in the canonical cycle notation of σ . A *right-to-left minimum* is an element σ_i such that $\sigma_i < \sigma_j$ for all $j > i$. Using right-to-left minima of σ , one can easily construct the inverse of the Foata transformation. Note that the number of cycles of σ equals the number of right-to-left minima of $\widehat{\sigma}$.

We now introduce a map $\phi : \mathfrak{S}_n \rightarrow \mathfrak{S}_{n+2}$ as a slight modification of the Foata transformation. For a permutation $\sigma \in \mathfrak{S}_n$, define the one-cycle permutation $\phi(\sigma) \in \mathfrak{S}_{n+2}$ by

$$\phi(\sigma) := (1, \widehat{\sigma}_1 + 1, \dots, \widehat{\sigma}_n + 1, n + 2).$$

It follows immediately from the bijectivity of the Foata transformation that the map ϕ is injective, and its image $\phi(\mathfrak{S}_n)$ is the set of one-cycle permutations $\sigma \in \mathfrak{S}_{n+2}$ with $\sigma(n + 2) = 1$. For instance, let $\sigma = 568479312 \in \mathfrak{S}_9$. In the canonical cycle notation, $\sigma = (5, 7, 3, 8, 1)(6, 9, 2)(4)$, so $\widehat{\sigma} = 573816924$. Then we have

$$\phi(\sigma) = (1, 6, 8, 4, 9, 2, 7, 10, 3, 5, 11) \in \mathfrak{S}_{11}.$$

The right-to-left minima of $\widehat{\sigma}$ are 1, 2 and 4, which are the minima of cycles of σ . Note that the permutation σ is a web permutation, and the cycle $\phi(\sigma)$ is an André cycle. Surprisingly, this is not an accident.

Theorem 4.1. *For $n \geq 1$, let $AC_{n+2} \subset \mathfrak{S}_{n+2}$ be the set of André cycles consisting of $[n + 2]$. Then we have $\phi(\text{Web}_n) = AC_{n+2}$. In particular, the number of web permutations of $[n]$ is equal to the number of André cycles consisting of $[n + 2]$.*

Proof. Let σ be a web permutation of $[n]$. In the canonical cycle notation, we write

$$\sigma = C_1 C_2 \cdots C_k,$$

where $C_i = (c_1^{(i)}, \dots, c_{n_i}^{(i)})$ is a cycle with $\min C_i = c_{n_i}^{(i)}$ for each $i = 1, \dots, k$, and $1 = c_{n_1}^{(1)} < \dots < c_{n_k}^{(k)}$. Note that by Theorem 3.4, each cycle C_i is an André cycle; that is, each word $c_1^{(i)} \cdots c_{n_i}^{(i)}$ is an André permutation. We claim that the word

$$c_1^{(1)} \cdots c_{n_1}^{(1)} c_1^{(2)} \cdots c_{n_2}^{(2)} \cdots c_{n_k}^{(k)} (n + 1)$$

obtained by appending $n + 1$ to the end of $\widehat{\sigma}$ is an André permutation. Since $c_{n_1}^{(1)}$ is the minimum in the word, and $c_1^{(1)} \cdots c_{n_1}^{(1)}$ is an André permutation, it suffices to show that the suffix $c_1^{(2)} \cdots c_{n_2}^{(2)} \cdots c_{n_k}^{(k)} (n + 1)$ is an André permutation. Then an appropriate inductive argument shows the claim. We now consider the cycle $\phi(\sigma)$. Using the canonical cycle notation of σ , we have

$$\phi(\sigma) = (1, c_1^{(1)} + 1, \dots, c_{n_1}^{(1)} + 1, c_1^{(2)} + 1, \dots, c_{n_2}^{(2)} + 1, \dots, c_1^{(k)} + 1, \dots, c_{n_k}^{(k)} + 1, n + 2).$$

Thus, by the claim, $\phi(\sigma)$ is an André permutation of $[n + 2]$, as desired.

Conversely, let τ be an André cycle of $[n + 2]$. By Lemma 3.2, τ forms $(1, a_1, \dots, a_n, n + 2)$. Let a_{n_1}, \dots, a_{n_k} be the right-to-left minima of the permutation $a_1 \cdots a_n$ with $n_1 < \dots < n_k = n$ so that $a_{n_1} < \dots < a_{n_k}$. Then we only need to show that the permutation $(a_1, \dots, a_{n_1})(a_{n_1+1}, \dots, a_{n_2}) \cdots (a_{n_{k-1}+1}, \dots, a_{n_k})$ is a web permutation, or equivalently, due to Theorem 3.4, each cycle $(a_{n_{i-1}+1}, \dots, a_{n_i})$ is an André cycle. It is easily verified by a similar argument as in the previous claim and using the right-to-left minima. Hence, we leave the details to the reader. \square

4.1. Euler and Entringer numbers

In this subsection, we give various enumerative properties of web permutations using Theorem 4.1.

We start with recalling Euler numbers. The Euler numbers E_n are defined via the exponential generating function

$$E(z) := \sum_{n \geq 0} E_n \frac{z^n}{n!} = \sec z + \tan z.$$

The first few Euler numbers are 1, 1, 1, 2, 5, 16, 61; see [Slo20] with ID number A000111. There are numerous combinatorial objects enumerated by Euler numbers E_n (e.g., alternating permutations, complete increasing binary trees, etc.). For example, the Euler number E_n counts André permutations of $[n]$. For details, we refer to [Sta10], which is a wonderful survey of Euler numbers and related topics. We provide another occurrence of Euler numbers.

Corollary 4.2. *The Euler number E_{n+1} enumerates the number of web permutations of $[n]$.*

Proof. By definition, the number of André permutations of $[n]$ is equal to the number of André cycles of $[n + 1]$. Then Theorem 4.1 implies the desired result. □

Remark 4.3. One can prove the corollary without using the fact that the number of André permutations is equal to the Euler number. Indeed, let w_n be the number of web permutations of $[n]$, ac_n the number of André cycles of $[n]$, and

$$W(z) = \sum_{n \geq 0} w_n \frac{z^n}{n!}, \quad \text{and} \quad AC(z) = \sum_{n \geq 1} ac_n \frac{z^n}{n!},$$

where we set $w_0 = 1$. Then by a standard fact of generating functionology [Sta99, Corollary 5.1.6] and Theorem 3.4, we have

$$W(z) = \exp AC(z).$$

Meanwhile, Theorem 4.1 gives the ODE

$$W(z) = \frac{d^2}{dz^2} AC(z) = \frac{d^2}{dz^2} \log W(z),$$

whose unique solution is $W(z) = \sec z \tan z + \sec^2 z = E'(z)$, which implies $w_n = E_{n+1}$.

For a permutation σ , let $c(\sigma)$ be the number of cycles of σ , and $\text{rlmin}(\sigma)$ the number of right-to-left minima of σ . By convention, we set $c(\emptyset) = 0$, where \emptyset is the empty permutation, and $\text{Web}_0 = \{\emptyset\}$. Since the Foata transformation gives the equidistribution of the two statistics $c(\sigma)$ and $\text{rlmin}(\sigma)$, we have the following corollary concerning the distribution of $c(\sigma)$ on Web_n .

Corollary 4.4. *We have*

$$\left(\frac{1}{1 - \sin z} \right)^t = \sum_{n \geq 0} \sum_{\sigma \in \text{Web}_n} t^{c(\sigma)} \frac{z^n}{n!}.$$

Proof. In [Dis13, Proposition 1], the author showed that

$$\left(\frac{1}{1 - \sin z} \right)^t = \sum_{n \geq 1} \sum_{\sigma} t^{\text{rlmin}(\sigma)-1} \frac{z^{n-1}}{(n-1)!},$$

where the inner sum is over all André permutations of $[n]$. Therefore, the proof follows immediately from Theorem 4.1. □

We also recall Entringer numbers. The *Entringer* numbers are given by the generating function

$$\frac{\cos x + \sin x}{\cos(x + y)} = \sum_{m,n \geq 0} E_{m+n, [m,n]} \frac{x^m}{m!} \frac{y^n}{n!},$$

where $[m, n]$ is m if $m + n$ is odd, and n otherwise. These numbers refine Euler numbers in the following sense: For $n \geq 1$,

$$\sum_{k=1}^n E_{n,k} = E_{n+1}.$$

We have a counterpart of this refinement.

Corollary 4.5. *The Entringer number $E_{n,k}$ is equal to the number of web permutations σ of $[n]$ with $\sigma_1 = n + 1 - k$.*

Proof. In [FH16, Theorem 1.1], the authors showed that $E_{n,k}$ equals the number of André permutations σ of $[n + 1]$ with $\sigma_1 = n + 1 - k$. Combining this fact and Theorem 4.1 gives the proof. □

4.2. Genocchi numbers and the Seidel triangle

The Genocchi numbers are well-studied numbers with various combinatorial properties; see, for example, [Dum74, LW20]. The Genocchi numbers can be defined by the *Seidel triangle* as follows [Sei77]. The Seidel triangle is an array of integers $(s_{i,j})_{i,j \geq 1}$ such that $s_{1,1} = s_{2,1} = 1$ and

$$\begin{cases} s_{2i+1,j} = s_{2i+1,j-1} + s_{2i,j} & \text{for } j = 1, 2, \dots, i + 1 \\ s_{2i,j} = s_{2i,j+1} + s_{2i-1,j} & \text{for } j = i, i - 1, \dots, 1, \end{cases}$$

where $s_{i,j} = 0$ for $j < 0$ or $j > [i/2]$. The *Genocchi numbers* g_n are defined by

$$g_{2n-1} = s_{2n-1,n} \quad \text{and} \quad g_{2n} = s_{2n,1}.$$

In fact, the sequence (g_n) is the interleaving of the Genocchi numbers of the first kind and the median Genocchi numbers. The first values of the Seidel triangle and Genocchi numbers (in red) are given in the following sequence:

$n \backslash k$	1	2	3	4	5
1	1				
2	1				
3	1	1			
4	2	1			
5	2	3	3		
6	8	6	3		
7	8	14	17	17	
8	56	48	34	17	
9	56	104	138	155	155

Recall that we denote by M_0 for the unique matching which is simultaneously noncrossing and nonnesting (i.e., $M_0 = \{\{1, 2\}, \dots, \{2n - 1, 2n\}\}$). To emphasize the size of the matching, we denote this unique matching of $[2n]$ by $M_0^{(n)}$. Let $f(n)$ be the number of web permutations σ of $[n]$ with $M(\sigma) = M_0^{(n)}$. In [Nak20], Nakamigawa studied expansions of certain chord diagrams. The author

computed the multiplicity of the n -necklace in the expansion of the n -crossing chord diagram through an iterative process of resolving crossings. Here, the resolving rule employed in this process coincides with the resolving rule (3). In the context of matchings, this is equivalent to the problem of determining the number of occurrences of the noncrossing matching M_0 obtained by iteratively resolving crossings in $M = \{\{1, n+1\}, \{2, n+2\}, \dots, \{n, 2n\}\}$ via the rule (3). In our terminology, a result of Nakamigawa can be stated as follows.

Theorem 4.6 [Nak20, Theorem 3.1]. *For $n \geq 1$, we have $f(n) = g_n$.*

Let $f(n, k)$ be the number of web permutations σ of $[n]$ such that $M(\sigma) = M_0^{(n)}$ and $\sigma_1 = k$. Obviously, $f(n) = \sum_{1 \leq k \leq n} f(n, k)$. Some of these numbers vanish in the following cases.

Proposition 4.7. *For $n \geq 1$ and $1 \leq k \leq \lfloor n/2 \rfloor$, we have $f(n, 2k) = 0$.*

Proof. Let σ be a web permutation of $[n]$ with $\sigma_1 = 2k$. Then considering the grid configuration $G(\sigma, \text{Cr}(\sigma))$, the associated matching $M(\sigma)$ has an arc connecting $2k$ and some j with $2k < j$. Since there is the arc connecting $2k - 1$ and $2k$ in $M_0^{(n)}$, we deduce $M(\sigma) \neq M_0^{(n)}$. \square

Proposition 4.8. *For $n > 1$, we have $f(n, n) = 0$.*

Proof. Let σ be a web permutation of $[n]$ with $\sigma_1 = n$. Since the elements 1 and n are contained in the same cycle, we have $\sigma_n = 1$ by Lemma 3.2 and Theorem 3.4. Then there is a marking at $(n, 1)$ in the grid configuration $G(\sigma, \text{Cr}(\sigma))$. Observe that the vertical line and horizontal line starting from the cell $(n, 1)$ do not make a crossing. Hence, we deduce $\{1, 2n\} \in M(\sigma)$, which implies that $M(\sigma) \neq M_0^{(n)}$. \square

By Propositions 4.7 and 4.8, we have

$$f(n) = \sum_{1 \leq k \leq \lfloor n/2 \rfloor} f(n, 2k - 1).$$

We now propose a conjecture that the values appearing in the Seidel triangle are $f(n, k)$.

Conjecture 4.9 (Verified up to $n = 6$). *For $n \geq 1$, we have*

$$\begin{cases} f(2n - 1, 2k - 1) = s_{2n-2, k}, \\ f(2n, 2k - 1) = s_{2n-1, n-k+1}. \end{cases}$$

This conjecture includes Nakamigawa's result. To elaborate, let σ be a web permutation of $[n]$ such that $M(\sigma) = M_0^{(n)}$ and $\sigma_1 = 1$. Deleting the cycle (1) from σ and decreasing each letter by 1, the resulting permutation is a web permutation of $[n - 1]$ with $M(\sigma) = M_0^{(n-1)}$. In addition, this correspondence is bijective, so we deduce $f(n, 1) = f(n - 1)$. Thus, the conjecture implies $f(n - 1) = g_{n-1}$, which is Nakamigawa's result.

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Competing interest. The authors have no competing interest to declare.

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