ON A CLASS OF ALMOST ALTERNATIVE ALGEBRAS

L. A. KOKORIS

Introduction. In the study of almost alternative algebras (**2**) relative to quasiequivalence an important class called algebras of (γ, δ) type arises. An algebra of (γ, δ) type is a finite dimensional algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ satisfying the identities

$$z(xy) = (zx)y + \gamma(xz)y - \gamma x(zy) + \delta(yz)x - \delta y(zx), \tag{1}$$

and

$$(xy)z = x(yz) + \gamma(xz)y - \gamma x(zy) + (\delta - 1)(yz)x - (\delta - 1)y(zx) \tag{2}$$

where γ and δ are elements of $\mathfrak{F}$ satisfying $\gamma^2 - \delta^2 + \delta = 1$. We shall restrict our study to (γ, δ) type algebras with characteristic $\neq 2$, 3, or 5 and with $\delta \neq 0, 1$. With these restrictions the algebras are power-associative. Also, Albert has shown (**2**, p. 36) that if an algebra $\mathfrak{A}$ of (γ, δ) type has an idempotent e it can be decomposed into a supplementary sum $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{10} + \mathfrak{A}_{01} + \mathfrak{A}_{00}$ where x is in $\mathfrak{A}_{ij}$ if and only if $ex = ix$ and $xe = jx$. The subspaces of our decomposition have the same multiplicative properties as in the case of an associative algebra.

The concepts of a solvable algebra, nilpotent algebra, and nil algebra are equivalent for (γ, δ) type algebras with the restrictions mentioned above (**2**, p. 35). The radical is defined to be the maximal nilideal and it is then proved that a simple algebra is either associative or contains a unity which is an absolutely primitive idempotent. A semisimple algebra is a direct sum of simple algebras.

If $\delta = 0$ or 1 we have the four pairs $(\gamma, \delta) = (1, 1)$, $(-1, 0)$, $(1, 0)$, or $(-1, 1)$. The pair $(-1, 1)$ implies that the algebra is right alternative and $(1, 0)$ implies the left alternative law. In the remaining two cases we are not able to obtain the same multiplicative relations for the subspaces of the decomposition as for the general case and it seems that the results here should be different.

1. Decomposition relative to an idempotent.

Let $\mathfrak{A}$ be an algebra of (γ, δ) type with characteristic $\neq 2$ and with an idempotent e. If $(\gamma, \delta) \neq (-1, 1)$ or $(1, 0)$, it is known that $\mathfrak{A}$ may be decomposed into a vector space direct sum $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{10} + \mathfrak{A}_{01} + \mathfrak{A}_{00}$. This is the decomposition of the theory of associative algebras and we are able to obtain the multiplicative relations of the associative theory when $\delta \neq 0, 1$.

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THEOREM 1. *Let $\mathfrak{A}$ be an algebra of (γ, δ) type with $\delta \neq 0, 1$ and characteristic $\neq 2, 3$. Then $\mathfrak{A}_{ij}\mathfrak{A}_{qt} = 0$ if $j \neq q$ and $\mathfrak{A}_{ij}\mathfrak{A}_{jt} \leqslant \mathfrak{A}_{it}$.*

The proof is made by considering the various cases. Take $z = e$, x in $\mathfrak{A}_{ij}$, and y in $\mathfrak{A}_{qt}$. Then (1) becomes

$$e(xy) = (i + j\gamma - q\gamma)xy + (t\delta - i\delta)yx. \tag{3}$$

Interchanging x and y gives

$$e(yx) = (q + t\gamma - i\gamma)yx + (j\delta - q\delta)xy. \tag{4}$$

With x, y, z as above, (2) becomes

$$(xy)e = (t + j\gamma - q\gamma)xy + (\delta - 1)(t - i)yx. \tag{5}$$

Interchanging the roles of x and y we have

$$(yx)e = (j + t\gamma - i\gamma)yx + (\delta - 1)(j - q)xy. \tag{6}$$

Now consider the case where x and y are in $\mathfrak{A}_{11}$ so that $i = j = q = t = 1$. Relations (3) and (5) yield $e(xy) = xy$, and $(xy)e = xy$. Therefore, $\mathfrak{A}_{11}$ is a subalgebra. The values $i = j = q = t = 0$ in (3) and (5) prove that $\mathfrak{A}_{00}$ is also a subalgebra. When x is in $\mathfrak{A}_{11}$, y is in $\mathfrak{A}_{00}$, (3) and (4) give $e(xy) = (1+\gamma)xy - \delta yx$ and $e(yx) = -\gamma yx + \delta xy$. We now use the fact that $L_e{}^2 = L_e$ (later (cf. **2**, p. 36) we shall also need $R_e{}^2 = R_e$) to see that

$$e[e(xy)] = e(xy), \quad e(xy) = (1 + \gamma)[e(xy)] - \delta e(yx).$$

It follows that $(\gamma^2 - \delta^2 + \gamma)xy = 0$. Since $\gamma^2 - \delta^2 + \gamma = 0$ together with the defining relation $\gamma^2 - \delta^2 + \delta = 1$ for an algebra of (γ, δ) type implies $\delta = 0$, we must have $xy = 0$. Also,

$$e[e(yx)] = e(yx), \quad -\gamma e(yx) + \delta e(xy) = e(yx).$$

Consequently $(\gamma^2 - \delta^2 + \gamma)yx = 0$ and so $yx = 0$. Thus $\mathfrak{A}_{11}$ and $\mathfrak{A}_{00}$ are orthogonal subalgebras.

If x is in $\mathfrak{A}_{11}$ and y is in $\mathfrak{A}_{10}$, we have $e(xy) = xy - \delta yx$, $e(yx) = (1 - \gamma)yx = (yx)e$, and $(xy)e = (1 - \delta)yx$. Then $e[e(xy)] = e(xy)$ implies $\delta e(yx) = 0$ and it follows that $yx = 0$. This also proves that xy is in $\mathfrak{A}_{10}$. Next let x be in $\mathfrak{A}_{11}$ and y be in $\mathfrak{A}_{01}$ so that

$$e(xy) = (1 + \gamma)xy = (xy)e, \quad e(yx) = \delta xy, \quad (yx)e = yx + (\delta - 1)xy.$$

The result $xy = 0$ is obtained by noting that $[(yx)e]e = (yx)e$ and $(\delta - 1)[(xy)e] = 0$. Then yx is in $\mathfrak{A}_{01}$.

Consider the case where x and y are both in $\mathfrak{A}_{10}$ and

$$e(xy) = (1 - \gamma)xy - \delta yx, \quad e(yx) = (1 - \gamma)yx - \delta xy, \quad (xy)e = -\gamma xy + (1 - \delta)yx, \quad (yx)e = -\gamma yx + (1 - \delta)xy.$$

From $e[e(xy)] = e(xy)$ and $e[e(yx)] = e(yx)$ we obtain $(\gamma + \delta)[e(xy) + e(yx)] = 0$. Since $\gamma + \delta \neq 0$ by hypothesis,

$$e(xy) + e(yx) = 0 = (1 - \gamma - \delta)(xy + yx).$$

Again $1 - \gamma - \delta \neq 0$ by hypothesis, so $xy + yx = 0$. Thus

$$e(xy) = (1 - \gamma + \delta)xy, \quad (xy)e = (-1 - \gamma + \delta)xy.$$

Moreover,

$$e[e(xy)] = e(xy) = (1 - \gamma + \delta)e(xy), \quad (1 - \gamma + \delta)(-\gamma + \delta)xy = 0.$$

When the characteristic is not 3, $\delta \neq 0, 1$ implies $xy = 0$. The case with x in $\mathfrak{A}_{10}$ and y in $\mathfrak{A}_{01}$ is proved immediately by substituting in (3) to (6). If both x and y are in $\mathfrak{A}_{01}$, $e(xy) = \gamma xy + \delta yx$ and $e(yx) = \gamma yx + \delta xy$. Therefore

$$e[e(xy)] = e(xy) = \gamma e(xy) + \delta e(yx), \quad e[e(yx)] = e(yx) = \gamma e(yx) + \delta e(xy)$$

when added give $(-1 + \gamma + \delta)[e(xy) + e(yx)] = 0$. Hence $(\gamma + \delta)(xy + yx) = 0$ and thus $xy = -yx$. We then have

$$e(xy) = (\gamma - \delta)xy, \quad e[e(xy)] = e(xy) = (\gamma - \delta)[e(xy)].$$

This implies $(\gamma - \delta - 1)(\gamma - \delta)xy = 0$, $xy = 0$.

Take x in $\mathfrak{A}_{10}$ and y in $\mathfrak{A}_{00}$. Then $e(xy) = xy - \delta yx$, $(xy)e = (1 - \delta)yx$, and $(yx)e = -\gamma yx$. We have $[(xy)e]e] = (1 - \delta)[(yx)e] = (xy)e$ and $(1 - \delta)(1 + \gamma)yx = 0$. Our hypothesis on δ implies $yx = 0$ and it follows that xy is in $\mathfrak{A}_{10}$. The last case is with x in $\mathfrak{A}_{01}$ and y in $\mathfrak{A}_{00}$. Relations (3) to (6) become

$$e(xy) = \gamma xy, \quad e(yx) = \delta xy, \quad (xy)e = \gamma xy, \quad (yx)e = yx + (\delta - 1)xy.$$

Also $e[e(yx)] = e(yx) = \delta e(xy)$ and so $\delta(1 - \gamma)xy = 0$. Since $\delta(1 - \gamma) \neq 0$, $xy = 0$ and yx is in $\mathfrak{A}_{01}$. This completes the proof of Theorem 1.

2. Power-associativity. When $x = y = z$, relation (1) becomes $(1+\gamma+\delta)(xx^2 - x^2x) = 0$ and (2) yields $(2 - \gamma - \delta)(xx^2 - x^2x) = 0$. Addition of the two expressions gives $xx^2 = x^2x$ if the characteristic $\neq 3$. Assume that $\mathfrak{A}$ is an algebra of (γ, δ) type with characteristic $\neq 2, 3$ and let $z = x^2$, $y = x$ in (1) and (2) to obtain

$$x^2x^2 = (1 + \gamma + \delta)x^3x - (\gamma + \delta)xx^3 = (-1 + \gamma + \delta)x^3x - (-2+\gamma+\delta)xx^3.$$

It follows that $2x^3x = 2xx^3$ and $x^2x^2 = x^3x = xx^3$. If also $\mathfrak{A}$ has characteristic $\neq 5$, it satisfies the hypotheses of the known **(1**, Lemma 4**)**:

LEMMA 1. *Let $\mathfrak{A}$ be an algebra with characteristic $\neq 2, 3, 5$ and $x^\lambda x^\mu = x^{\lambda+\mu}$ for $\lambda + \mu < n$, $n \geqslant 5$. Then*

$$x^{n-\alpha}x^\alpha = x^{n-1}x + \frac{\alpha - 1}{2}[x^{n-1}, x] \qquad (\alpha = 1, \ldots, n - 1) \tag{7}$$

where $[x^{n-1}, x] = x^{n-1}x - xx^{n-1}$. Also, $n[x^{n-1}, x] = 0$.

The Lemma will be used to show that an algebra of (γ, δ) type is power-associative if its characteristic $\neq 2, 3, 5$. Write x^α for x, x^β for y and $x^{n-\alpha-\beta}$ for z in (1) where α, β are positive integers such that $\alpha + \beta < n$ and assume that $x^\lambda x^\mu = x^{\lambda+\mu}$ for $\lambda + \mu < n$ to obtain

$$x^{n-\alpha-\beta}x^{\alpha+\beta} = x^{n-\beta}x^\beta + \gamma x^{n-\beta}x^\beta - \gamma x^\alpha x^{n-\alpha} + \delta x^{n-\alpha}x^\alpha - \delta x^\beta x^{n-\beta}.$$

By (7) we have after multiplying by 2,

$$(\alpha+\beta-1)[x^{n-1},x] = (1+\gamma)(\beta-1)[x^{n-1},x] - \gamma(n-\alpha-1)[x^{n-1},x] \\ +\delta(\alpha-1)[x^{n-1},x] - \delta(n-\beta-1)[x^{n-1},x].$$

Thus either $[x^{n-1}, x] = 0$ or

$$a+\beta-1 = \beta-1+\gamma\beta-\gamma+\gamma\alpha+\gamma+\delta\alpha-\delta+\delta\beta+\delta.$$

If $[x^{n-1}, x] = 0$, (7) implies $\mathfrak{A}$ is power-associative. Otherwise $\alpha = (\gamma+\delta)(\alpha+\beta)$. Since α and β are any positive integers, restricted only by $\alpha+\beta < n$, interchange α and β to obtain $\beta = (\gamma+\delta)(\alpha+\beta)$. Adding,

$$\alpha+\beta = 2(\gamma+\delta)(\alpha+\beta)$$

and $\alpha = \beta = 1$ implies $2(\gamma+\delta) = 1$. But it is impossible for γ and δ to satisfy both this equation and $\gamma^2 - \delta^2 + \delta = 1$.

THEOREM 2. *An algebra $\mathfrak{A}$ of (γ, δ) type whose characteristic $\neq 2, 3, 5$ is power-associative.*

3. Simple algebras. From this point on we shall consider algebras of (γ, δ) type with $\delta \neq 0, 1$ and with characteristic $\neq 2, 3, 5$ so that we may use the results of Theorems 1 and 2. We shall make use of the associator (x, y, z) which is defined by $(x, y, z) = (xy)z - x(yz)$. If $\mathfrak{A}$ is an algebra with an idempotent e we may prove the following result.

LEMMA 2. *The associator (x, y, z) is 0 if one of the elements x, y, z is in $\mathfrak{A}_{10}$ or $\mathfrak{A}_{01}$.*

First consider the possible ordered triples with x_{10} in $\mathfrak{A}_{10}$ on the left and y, z in the decomposition subspaces. It is clear that by linearity we need only consider elements in the subspaces of the decomposition. By Theorem 1 it is clear that the only triples with x_{10} on the left giving nonzero products are

$$x_{10},\ y_{01},\ z_{11};\ x_{10},\ y_{01},\ z_{10};\ x_{10},\ y_{00},\ z_{01};\ x_{10},\ y_{00},\ z_{00},$$

where the subscripts indicate the subspaces in which the elements lie. Let $x = x_{10}$, $y = y_{01}$, $z = z_{11}$ in (2) and use the fact that our decomposition is supplementary to obtain $x_{10}(y_{01}z_{11}) = (x_{10}y_{01})y_{11}$. Similarly we prove the result for the second and third triples. For the last triple we use (1) with $z = x_{10}$, $x = y_{00}$, $y = z_{00}$ to get $x_{10}(y_{00}z_{00}) = (x_{10}y_{00})z_{00}$.

Triples with y_{10} in the middle giving nonzero products are

$$x_{11},\ y_{10},\ z_{01};\quad x_{11},\ y_{10},\ z_{00};\quad x_{01},\ y_{10},\ z_{01};\quad x_{01},\ y_{10},\ z_{00}.$$

The result of the Theorem is proved by making the obvious substitutions in (1) for the first two of these triples and in (2) for the last two.

There are also four triples with z_{10} on the right giving non-zero products. These are

$$x_{11},\ y_{11},\ z_{10};\ x_{01},\ y_{11},\ z_{10};\ x_{10},\ y_{01},\ z_{10};\ x_{00},\ y_{01},\ z_{10}.$$

For the first three substitute in (2) and use (1) for the last triple. By symmetry we have the result for elements in $\mathfrak{A}_{01}$.

COROLLARY. *The algebra $\mathfrak{A}$ is associative if and only if $\mathfrak{A}_{11}$ and $\mathfrak{A}_{00}$ are associative.*

Now let $\mathfrak{A}$ be a simple algebra. There must be a nonnilpotent element x in $\mathfrak{A}$ and the subalgebra generated by x must be associative since $\mathfrak{A}$ is power-associative. Since an associative algebra not a nilalgebra has an idempotent, $\mathfrak{A}$ has an idempotent e. Decompose $\mathfrak{A}$ relative to e. Then the sets

$$\mathfrak{B} = \mathfrak{A}_{11} + \mathfrak{A}_{10} + \mathfrak{A}_{01} + \mathfrak{A}_{01}\mathfrak{A}_{10}, \quad \mathfrak{C} = \mathfrak{A}_{00} + \mathfrak{A}_{10} + \mathfrak{A}_{01} + \mathfrak{A}_{10}\mathfrak{A}_{01}$$

can easily be seen to be ideals of $\mathfrak{A}$. Since e is in $\mathfrak{B}$, $\mathfrak{B} = \mathfrak{A}$ and thus $\mathfrak{A}_{00} = \mathfrak{A}_{01}\mathfrak{A}_{10}$. It follows from this and Lemma 2 that $\mathfrak{A}_{00}$ is zero or an associative algebra. In the latter case $\mathfrak{A}_{00}$ is simple for if $\mathfrak{B}_{00}$ were a proper ideal of $\mathfrak{A}_{00}$, then $\mathfrak{B}_{00}$ would generate the proper ideal

$$\mathfrak{B}_{00} + \mathfrak{A}_{10}\mathfrak{B}_{00} + \mathfrak{B}_{00}\mathfrak{A}_{01} + \mathfrak{A}_{10}\mathfrak{B}_{00}\mathfrak{A}_{01}$$

of $\mathfrak{A}$. The ideal $\mathfrak{C} = \mathfrak{A}$ or 0. If $\mathfrak{C} = \mathfrak{A}$, $\mathfrak{A}_{11} = \mathfrak{A}_{10}\mathfrak{A}_{01}$ and $\mathfrak{A}_{11}$ is a simple associative algebra. If $\mathfrak{C} = 0$, $\mathfrak{A} = \mathfrak{A}_{11}$ and e is the unity element of $\mathfrak{A}$. In case $e = u + v$ is not primitive, we can get a proper decomposition with respect to u and with the new $\mathfrak{A}_{00} \neq 0$. Then $\mathfrak{A}$ is associative. When e is not absolutely primitive we can find a scalar extension $\mathfrak{K}$ of the base field $\mathfrak{F}$ such that $e = u + v$ for pairwise orthogonal idempotents u, v in $\mathfrak{A}_{\mathfrak{K}}$. Consequently $\mathfrak{A}_{\mathfrak{K}}$ is associative and $\mathfrak{A}$ is associative.

THEOREM 3. *Let $\mathfrak{A}$ be a simple algebra of (γ, δ) type with $\delta \neq 0, 1$ and with characteristic $\neq 2, 3, 5$. Then $\mathfrak{A}$ is an associative algebra or $\mathfrak{A}$ has a unity quantity which is an absolutely primitive idempotent.*

4. Semisimple algebras.

The study of semisimple algebras begins with

THEOREM 4. *Let e be a principal idempotent of an algebra $\mathfrak{A}$ of (γ, δ) type with $\delta \neq 0, 1$ and characteristic $\neq 2, 3, 5$. Then $\mathfrak{A}_{10} + \mathfrak{A}_{01} + \mathfrak{A}_{00}$ is contained in the radical $\mathfrak{N}$ of $\mathfrak{A}$.*

The proof is made by an induction on the order of $\mathfrak{A}$. The result is clear when $\mathfrak{A}$ has order one. Assume the Theorem for all algebras of order less than n and let $\mathfrak{A}$ have order n. If $\mathfrak{A}$ is not semisimple we consider $\mathfrak{B} = \mathfrak{A} - \mathfrak{N}$ which has order $m < n$. The principal idempotent e of $\mathfrak{A}$ corresponds to a principal idempotent u of $\mathfrak{B}$. Decompose $\mathfrak{B}$ relative to u. Since $\mathfrak{B}$ is semisimple our induction hypothesis simplies $\mathfrak{B}_{10} + \mathfrak{B}_{01} + \mathfrak{B}_{00} = 0$. This implies that in the decomposition of $\mathfrak{A}$ relative to e, $\mathfrak{A}_{10} + \mathfrak{A}_{01} + \mathfrak{A}_{00} \subseteq \mathfrak{N}$.

If $\mathfrak{A}$ is simple, Theorem 3 implies $\mathfrak{A}$ has a unity e and an algebra with a unity has no other principal idempotent. Thus we may pass to the consideration of a semisimple algebra $\mathfrak{A}$ with a proper ideal $\mathfrak{B}$. The ideal $\mathfrak{B}$ can not be a nilideal so it must contain an idempotent and hence a principal idempotent e. Then $\mathfrak{B} = \mathfrak{B}_{11} + \mathfrak{B}_{10} + \mathfrak{B}_{01} + \mathfrak{B}_{00}$ and we may also decompose $\mathfrak{A}$ relative to e so that $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{10} + \mathfrak{A}_{01} + \mathfrak{A}_{00}$. The idempotent e is in $\mathfrak{B}$ and so if

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$ex = x$ or $xe = x$, it follows that x is also in $\mathfrak{B}$. Consequently, $\mathfrak{A} = \mathfrak{B}_{11} + \mathfrak{B}_{10} + \mathfrak{B}_{01} + \mathfrak{A}_{00}$. By the induction $\mathfrak{B}$ has radical $\mathfrak{M} = \mathfrak{M}_{11} + \mathfrak{B}_{10} + \mathfrak{B}_{01} + \mathfrak{B}_{00}$ where $\mathfrak{M}_{11}$ is the part of $\mathfrak{M}$ in $\mathfrak{B}_{11}$. Since $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ it follows that $\mathfrak{M}$ is a nilideal of $\mathfrak{A}$ and that $\mathfrak{M} = 0$. Therefore $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{A}_{00}$ and e is the unity quantity of $\mathfrak{B}$. The subalgebra $\mathfrak{A}_{00}$ is an ideal of $\mathfrak{A}$ and by a repetition of the above argument $\mathfrak{A}_{00}$ has a unity f. Then $u = e + f$ is a unity for $\mathfrak{A}$ and is therefore the only principal idempotent of $\mathfrak{A}$. This completes the proof[1] of Theorem 4. We have also proved

THEOREM 5. *A semisimple algebra of* (γ, δ) *type with* $\delta \neq 0, 1$ *and with characteristic* $\neq 2, 3, 5$ *has a unity quantity and is a direct sum of simple algebras.*

[1]The reader should notice that our proofs follow those of Theorems 7 and 8 of Albert **(3)**.

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