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A REFLEXIVE LUR BANACH SPACE THAT LACKS NORMAL STRUCTURE

BY

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ABSTRACT. An example of the type described in the title is given.

A Banach space X is *locally uniformly rotund* (LUR) [4] if the conditions $\|x\| = \|x_n\| = 1$ and $\lim_{n\to\infty}\|x + x_n\| = 2$ imply $\lim_{n\to\infty}\|x - x_n\| = 0$. A Banach space X has *normal structure* [2] if for each closed bounded convex set K in X that contains more than one point, there is a point p in K such that $\sup\{\|p - x\| : x \in K\}$ is less than the diameter of K. The purpose of this note is to answer negatively the following question posed by S. Swaminathan [6]: If X is locally uniformly rotund, does X have normal structure? It should be noted that all uniformly rotund Banach spaces and, more generally, all Banach spaces that are uniformly rotund in every direction have normal structure (see [5] for definitions and pertinent references). Hence, even though certain directionalizations of uniform rotundity are sufficient to imply normal structure, the localization LUR is not sufficient. The example given in this note is a renorming of ℓ^2 that gleans its roundness from Day's LUR norm on c_0; another renorming of ℓ^2 that lacks normal structure is given in [1] but that renorming is not even rotund.

Recall the equivalent norm defined on c_0 by Day [3]. For u in c_0, enumerate the support of u as $\{n_k\}$ such that $|u(n_k)| \geq |u(n_{k+1})|$ for $k = 1, 2, \ldots$, define Du in ℓ^2 by

$$Du(n) = \begin{cases} \dfrac{u(n_k)}{2^k} & \text{if } n = n_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

and define $|||u||| = \|Du\|_2$, where $\|\cdot\|_2$ denotes the usual norm on ℓ^2.

For $x = (x^1, x^2, \ldots)$ in ℓ^2, let

$$u = (\tfrac{1}{2}\|x\|_2, x^1, x^2, x^2, \ldots, \underbrace{x^j, x^j, \ldots, x^j}_{j}, \ldots)$$

be the element of c_0 associated with x and define

$$\|x\|_L = |||u|||.$$

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Then $\|\cdot\|_L$ is a norm on ℓ^2 that is equivalent to $\|\cdot\|_2$.

The norm $\|\cdot\|_L$ was introduced by Smith [5] to provide an example of a reflexive, locally uniformly rotund Banach space that is not uniformly rotund in every direction. The reader is referred to [5] for the proof that $(\ell^2, \|\cdot\|_L)$ is LUR. To achieve the goal of this note, it need only be shown that $(\ell^2, \|\cdot\|_L)$ lacks normal structure. Recall from [2], a space X lacks normal structure if and only if there is a nonconstant, bounded sequence $\{x_j\}$ in X such that the distance from x_{n+1} to the convex hull of $\{x_1, \ldots, x_n\}$ tends to the diameter of the set $\{x_j: j = 1, 2, \ldots\}$ as n tends to ∞ (such a sequence is called a *diametral sequence*). Let $\{e_j\}$ denote the usual unit vector basis in ℓ^2. It will be shown that $\{e_j\}$ is a diametral sequence in $(\ell^2, \|\cdot\|_L)$.

For $m > n$,

$$\|e_m - e_n\|_L = \|\|(\sqrt{2}/2, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{n}, 0, \ldots, 0, \underbrace{1, \ldots, 1}_{m}, 0, \ldots)\|\|$$

$$= \left[\sum_{k=1}^{m+n} 4^{-k} + 4^{-(m+n+1)}/2\right]^{1/2}$$

$$\leq \left[\sum_{k=1}^{\infty} 4^{-k}\right]^{1/2}$$

and thus $\operatorname{diam}\{e_j\} \leq 1/\sqrt{3}$. But, from the computation above, it follows that $\lim_{m\to\infty}\|e_m - e_n\|_L = 1/\sqrt{3}$ and hence $\operatorname{diam}\{e_j\} = 1/\sqrt{3}$.

Suppose $\alpha_1 + \cdots + \alpha_n = 1$ where $0 \leq \alpha_i \leq 1$ for $1 \leq i \leq n$. Then, if $p = 1 + (1 + 2 + \cdots + n)$,

$$\left\|e_{n+1} - \sum_{i=1}^{n} \alpha_i e_i\right\|_L$$

$$= \left\|\left\|\left(\frac{1}{2}\left\|e_{n+1} - \sum_{i=1}^{n} \alpha_i e_i\right\|_2, \quad -\alpha_1, \ldots, \underbrace{-\alpha_n, \ldots, -\alpha_n}_{n}, \underbrace{1, \ldots, 1}_{n+1}, 0, \ldots\right)\right\|\right\|$$

$$\geq \|\|(\underbrace{0, \ldots, 0}_{p}, \underbrace{1, \ldots, 1}_{n+1}, 0, \ldots)\|\|$$

$$= \left[\sum_{k=1}^{n+1} 4^{-k}\right]^{1/2}$$

and hence

$$(*) \qquad \operatorname{dist}(e_{n+1}, co\{e_1, \ldots, e_n\}) \geq \left[\sum_{k=1}^{n+1} 4^{-k}\right]^{1/2}.$$

Also,

$$\left\| e_{n+1} - \sum_{i=1}^{n} \alpha_i e_i \right\|_L \leq |||(\underbrace{1, \ldots, 1}_{p}, \underbrace{1, \ldots, 1}_{n+1}, 0, \ldots)|||$$

$$\leq \left[\sum_{k=1}^{\infty} 4^{-k} \right]^{1/2}$$

and hence

$$\operatorname{dist}(e_{n+1}, co\{e_1, \ldots, e_n\}) \leq 1/\sqrt{3}.$$

From this last inequality and (*), it follows that

$$\lim_{n\to\infty} \operatorname{dist}(e_{n+1}, co\{e_1, \ldots, e_n\}) = 1/\sqrt{3} = \operatorname{diam}\{e_j\}.$$

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