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IMMERSIONS AND EMBEDDINGS UP TO COBORDISM

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In 1944 Whitney proved that any differentiable n-manifold ($n \geqq 2$) can be (differentiably) immersed in $\mathbf{R}^{2n-1}$ [**15**] and embedded in $\mathbf{R}^{2n}$ [**14**]. Whitney's results are best possible when $n = 2^r$. (One uses a simple argument involving the dual Stiefel-Whitney classes of real projective space P^n. See [**9**, pp. 14, 15].) However, there is a widely known conjecture that any n-manifold ($n \geqq 2$) immerses in $\mathbf{R}^{2n-\alpha(n)}$ and embeds in $\mathbf{R}^{2n-\alpha(n)+1}$. Here, $\alpha(n)$ denotes the number of ones in the binary expansion of n. We prove (Theorem 5.1) that every compact manifold is cobordant to a manifold that immerses in $(2n - \alpha(n))$-space and embeds in $(2n - \alpha(n) + 1)$-space. (See § 1 for the definition of cobordant manifolds.) It is well known that if the conjecture were true it would be the best possible result. (See Proposition 5.2.) We show that, for $n \neq 3$, our result is also best possible. If conditions involving the vanishing of certain Stiefel-Whitney numbers are placed on the manifold, then it is possible to improve the dimensions of Theorem 5.1. These results are given in § 6. An announcement of our theorems has appeared in [**1**].

The method of proof is to construct enough manifolds, each satisfying the required immersion and embedding condition, to generate all of the cobordism ring. This is carried out in §§ 2, 3, and 4, with proofs of some technical results postponed to § 7.

1. The cobordism ring. By an n-manifold we mean a compact, not necessarily connected, differentiable n-manifold. We denote the boundary of V by ∂V and say that V is closed if $\partial V = \phi$. Two closed n-manifolds M_1^n and M_2^n are said to be cobordant if there is an $(n + 1)$-manifold V such that ∂V is diffeomorphic to the disjoint union $M_1^n \cup M_2^n$. A comprehensive reference on cobordism is Stong's notes [**12**]. The relation of cobordism is an equivalence relation on the class of closed n-manifolds, and $M^n = \partial V$, for some V, if and only if M^n is cobordant to the n-sphere S^n. In this case, $M^n \cup \phi = \partial V$ and we allow the empty manifold as a representative of the cobordism class of S^n. We denote the set of cobordism classes of n-manifolds by $\mathfrak{N}_n$. Disjoint union induces an addition in $\mathfrak{N}_n$, $[\phi] = [S^n]$ serves as the zero element, and, because $M \cup M = \partial(M \times [0, 1])$, every element is its own inverse. Hence, $\mathfrak{N}_n$ is a vector space over the field with two elements, $\mathbf{Z}_2$. Cartesian product induces a multiplication $\mathfrak{N}_n \otimes \mathfrak{N}_m \rightarrow \mathfrak{N}_{n+m}$ which makes $\mathfrak{N}_* = \oplus_{i=1}^{\infty} \mathfrak{N}_i$ into a graded $\mathbf{Z}_2$-algebra. Thom [**13**, Théorème IV.12, p. 79] proved that $\mathfrak{N}_*$ is a polynomial

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algebra over $\mathbf{Z}_2$ with one generator in each dimension, not of the form $2^i - 1$. Thus, $\mathfrak{N}_* = \mathbf{Z}_2[x_2, x_4, x_5, x_6, x_8, \ldots]$ and, for example, a basis for $\mathfrak{N}_6$ is $\{x_6, x_2x_4\}$.

To determine the cobordism class of a given manifold one uses Stiefel-Whitney numbers. We denote the tangent bundle of M^n by τM^n. This bundle is classified by a map $\tau: M^n \to \mathrm{BO}(n)$, where $\mathrm{BO}(n)$ is the classifying space for vector bundles of fibre dimension n. The cohomology ring $H^*(\mathrm{BO}(n))$ is the polynomial algebra $\mathbf{Z}_2[w_1, \ldots, w_n]$, where $w_i \in H^i(BO(n))$. (All our homology will have $\mathbf{Z}_2$ coefficients.) The Stiefel-Whitney classes of M^n are defined by $w_i(M^n) = \tau^*(w_i)$, and we let $w(M^n) = 1 + w_1(M^n) + \ldots + w_n(M^n)$. The dual Stiefel-Whitney classes are defined by $\bar{w}(M) \cdot w(M) = 1$ and $\bar{w}(M) = 1 + \bar{w}_1(M) + \ldots + \bar{w}_n(M)$. Given a polynomial of dimension n in the Stiefel-Whitney classes of M^n, we can evaluate the polynomial on the fundamental homology class to get an element of $\mathbf{Z}_2$ called a Stiefel-Whitney number. The Stiefel-Whitney numbers form a complete set of invariants of the cobordism class of M^n. We say that a manifold is indecomposable if its cobordism class is not in the span of the image of the multiplication map $\mathfrak{N}_* \otimes \mathfrak{N}_* \to \mathfrak{N}_*$. Thom [**13**, p. 79] showed that M^n is indecomposable if and only if a certain Stiefel-Whitney number is nonzero: consider $w_i(M^n)$ as the ith elementary symmetric function on one dimensional variables $t_1, \ldots, t_{n+k}$, express the symmetric function $\sum_{i=1}^{n+k} t_i^n$ as a polynomial $s_{(n)}(M^n)$ in the w_i's, and evaluate $s_{(n)}(M^n)$ on the fundamental homology class. (See also [**3**, pp. 32 and 33; **9**, Corollary 4, p. 93; **12**, pp. 71 and 96].) Thom showed that real projective spaces of even dimension are indecomposable and Dold [**3**] constructed indecomposable manifolds of all dimensions not of the form $2^i - 1$. We will show how to construct such a complete set of generators of $\mathfrak{N}_*$ with the property that each indecomposable manifold embeds and immerses nicely. The main theorems follow easily once this is done.

2. Embedding and immersing products. An immersion of M^n in $\mathbf{R}^{n+k}$ is a differentiable function $f: M^n \to \mathbf{R}^{n+k}$ such that at each $x \in M^n$ the map $df: T_xM^n \to T_{f(x)}\mathbf{R}^{n+k}$ is injective. An embedding is a $1 - 1$ immersion. It is a result of Whitney [**14**; **15**] that any M^n embeds in $\mathbf{R}^{2n}$ and immerses in $\mathbf{R}^{2n-1}$. Also, any two immersions of M^n in $\mathbf{R}^{2n+1}$ are homotopic through immersions. (See [**5**, Theorem 8.4, p. 275].) Given an immersion of M^m in $\mathbf{R}^s$ and an immersion of N^n in $\mathbf{R}^t$ there is the product immersion of $M^m \times N^n$ in $\mathbf{R}^{s+t}$. If $s \leqq 2m - \alpha(m)$ and $t \leqq 2n - \alpha(n)$, then $s + t \leqq 2(m+n) - \alpha(m+n)$ because $\alpha(m+n) \leqq \alpha(m) + \alpha(n)$. We could also take the product embedding of two embeddings, but this will usually not suffice. Hence, we need the following simple result. (See [**11**, p. 319].)

LEMMA 2.1. *If M^m immerses in $\mathbf{R}^s$, N^n embeds in $\mathbf{R}^t$, and $s + t \geqq 2m + 1$, then $M^m \times N^n$ embeds in $\mathbf{R}^{s+t}$.*

Proof. We can immerse M^m in $\mathbf{R}^{s+t}$ and the normal bundle will have t linearly independent sections. Because $s + t \geqq 2m + 1$, this immersion is homotopic through immersions to an embedding. The embedding has the same normal bundle which can be realized in $\mathbf{R}^{s+t}$ as a tubular neighbourhood. Because of the t sections, the tubular neighbourhood contains $M^m \times \mathbf{R}^t$, and hence $M^m \times N^n$. This proves the lemma. (For example, the standard embedding of S^1 in $\mathbf{R}^2$ gives a product embedding of the torus $S^1 \times S^1$ in $\mathbf{R}^4$. But actually $S^1 \times \mathbf{R}$ embeds in $\mathbf{R}^2$, so $S^1 \times \mathbf{R}^2 \supset S^1 \times S^1$ embeds in $\mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$.)

An immersion of M^n in $\mathbf{R}^{2n-k}$ is said to have efficiency k.

COROLLARY 2.2. *The cartesian product of k manifolds (not all of dimension one) immerses with efficiency at least k and embeds with efficiency at least* $k - 1$.

Proof. For $k = 1$, the result is just Whitney's theorems. Assume that it is true for $k - 1$ and consider a product of manifolds of dimensions $m_1 \leqq m_2 \leqq \ldots \leqq m_k$, where $m_1 \geqq 1$ and $m_k > 1$. We can immerse the product of the last $k - 1$ factors with efficiency $k - 1$ and embed it with efficiency $k - 2$. If $m_1 > 1$, we can immerse the first manifold with efficiency one, then take the product immersion or apply Lemma 2.1 to complete the proof. If $m_1 = 1$, the first manifold is S^1. Let $m_2 + \ldots + m_k = m$. We can embed $S^1 \times \mathbf{R}$ in $\mathbf{R}^2$ and hence we can embed $S^1 \times \mathbf{R}^{2m-(k-1)}$ in $\mathbf{R}^{2+2m-(k-1)-1} = \mathbf{R}^{2(m+1)-k}$. Because the product of the last $k - 1$ factors can be immersed in $\mathbf{R}^{2m-k+1}$, the complete product can be immersed with efficiency k. The argument for the embedding of efficiency $k - 1$ is similar.

3. Even dimensional generators.

Our even dimensional generators of $\mathfrak{N}_*$ will be submanifolds of products of projective spaces similar to manifolds constructed by Milnor. (See [**12**, pp. 80, 81].) Let n be a positive even integer. If $\alpha(n) = 1$, let $V^n = P^n$, where P^n denotes real projective n-space. If $\alpha(n) > 1$, let $n = r_1 + \ldots + r_k$ be the expansion of n as a sum of distinct powers of 2. Let

$$K^{n+1} = \prod_{j=1}^{k} P^{s_j},$$

where $s_j = r_j$, for $1 \leqq j \leqq k - 1$, and $s_k = r_k + 1$. The cohomology ring of P^n is $\mathbf{Z}_2[\alpha]/(\alpha^{n+1})$, where $\alpha \in H^1(P^n)$. Hence,

$$H^*(K^{n+1}) = \mathbf{Z}_2[\alpha_1, \ldots, \alpha_k]/({\alpha_1}^{s_1+1}, \ldots, {\alpha_k}^{s_k+1}).$$

There is a submanifold $V^n \subset K^{n+1}$ dual to the cohomology class $\sigma = \alpha_1 + \ldots + \alpha_k$. By this we mean that the inclusion $i: V^n \to K^{n+1}$ sends the fundamental homology class $(V^n) \in H_n(V^n)$ to the Poincaré dual of σ. (See [**13**, p. 55; **12**, pp. 78–81].)

PROPOSITION 3.1. *V^n immerses in $\mathbf{R}^{2n-\alpha(n)}$ and embeds in $\mathbf{R}^{2n-\alpha(n)+1}$.*

Proof. If $\alpha(n) = 1$, this is a special case of the Whitney theorems. If $\alpha(n) > 1$, it is sufficient to prove the statement for K^{n+1}. Sanderson [**10**, Theorem 4.1, p. 146, and Theorem 5.3, p. 150] has proved that P^n immerses in $\mathbf{R}^{2n-3}$ if n is odd, $n \geqq 5$. Applying this to P^{s_k}, and immersing the other factors with efficiency one, gives an immersion of K^{n+1} in $\mathbf{R}^{2(n+1)-(k-1+3)} = \mathbf{R}^{2n-\alpha(n)}$. For the embedding, first use Lemma 2.1 to embed the product of the first $k - 1$ factors with efficiency $k - 2$. Then apply Lemma 2.1 again using the efficiency three immersion of P^{s_k} to embed K^{n+1} in $\mathbf{R}^{2n-\alpha(n)+1}$. (The first $k - 1$ factors need at least 4-space, so Lemma 2.1 does apply.)

PROPOSITION 3.2. *$[V^n]$ is an indecomposable element of $\mathfrak{N}_*$.*

Proof. If $\alpha(n) = 1$, $V^n = P^n$. Now, $w(P^n) = (1 + \alpha)^{n+1}$, so $s_{(n)}(P^n) = (n + 1)\alpha^n$. Hence, $\langle s_{(n)}(P^n), (P^n)\rangle = n + 1$. Here, n is even so this Stiefel-Whitney number is nonzero, and $[V^n]$ is indecomposable.

Let $\alpha(n) > 1$. (Compare [**12**, pp. 79, 80].) Let ν be the normal line bundle of V^n in K^{n+1}. Then $w(\nu) = i^*(1 + \sigma)$. Now, $i^{-1}\tau K^{n+1} = \tau V^n \oplus \nu$, so $w(V^n)i^*(1 + \sigma) = i^*w(K^{n+1})$, and $w(V^n) = i^*(1 + \sigma)^{-1}w(K^{n+1})$. The total Stiefel-Whitney class of K^{n+1} is the product of the total Stiefel-Whitney classes of its factors. Hence,

$$w(V^n) = i^*(1 + \sigma)^{-1} \prod_{j=1}^{k} (1 + \alpha_j)^{s_j+1}.$$

If $2^r > n + 1$, then

$$(1 + \sigma)^{2^r-1} = (1 + \sigma^{2^r})(1 + \sigma)^{-1} = (1 + \sigma)^{-1},$$

so the above formula expresses $w_j(V^n)$ as the jth elementary symmetric function of 1-dimensional elements. In forming $\sum_{j=1}^{n+k} t_j^n$, recall that $\alpha_j^n = 0$ because $n > s_j$. Hence,

$$\begin{aligned}\langle s_{(n)}(V^n), (V^n)\rangle &= \langle i^*(2^r - 1)\sigma^n, (V^n)\rangle \\ &= \langle i^*\sigma^n, (V^n)\rangle \\ &= \langle \sigma^n, i_*(V^n)\rangle \\ &= \langle \sigma^n, \sigma \cap (K^{n+1})\rangle \\ &= \langle \sigma^{n+1}, (K^{n+1})\rangle.\end{aligned}$$

The relations satisfied by the α_j imply that

$$\sigma^{n+1} = (\alpha_1 + \ldots + \alpha_k)^{n+1} = \{s_1, \ldots, s_k\}\alpha_1^{s_1} \cdot \ldots \cdot \alpha_k^{s_k},$$

where $\{s_1, \ldots, s_k\}$ denotes the multinomial coefficient

$$(s_1 + \ldots + s_k)!/(s_1!) \cdot \ldots \cdot (s_k!).$$

To prove V^n is indecomposable, it is sufficient to prove that $\{s_1, \ldots, s_k\}$ is nonzero modulo 2. This is an immediate consequence of the following result.

LEMMA 3.3. $\{n_1, \ldots, n_k\} \equiv 1 \pmod 2$ *if and only if the binary expansions of $n_1, \ldots, n_k$ intermesh in the sense that no two of them have a* 1 *in the same place. (Equivalently, $\alpha(n_1 + \ldots + n_k) = \alpha(n_1) + \ldots + \alpha(n_k)$.)*

Proof. For any prime p, there is a well known formula for binomial coefficients:

$$\binom{a}{b} \equiv \prod_{i=0}^{s} \binom{a_i}{b_i} \pmod{p},$$

where $a = \sum_{i=0}^{s} a_i p^i$, $b = \sum_{i=0}^{s} b_i p^i$. Thus,

$$\binom{a}{b} \equiv 1 \pmod{2}$$

if and only if $b_i = 1$ implies that $a_i = 1$, for all i. Now,

$$\{a, b\} = \binom{a+b}{b},$$

and the lemma follows easily for $k = 2$. The general case follows by induction from the formula

$$\{n_1, \ldots, n_k\} = \{n_1 + \ldots + n_{k-1}, n_k\}\{n_1, \ldots, n_{k-1}\}.$$

4. Odd dimensional generators. Dold [3] constructed odd dimensional generators of $\mathfrak{N}_*$ as follows. Let $P(m, n)$ be the $(m + 2n)$-manifold formed from the product $S^m \times P^n(C)$ of the m-sphere with complex projective n-space by identifying (u, z) with $(-u, \bar{z})$. If n is odd and not of the form $2^k - 1$, we can write uniquely $n = 2^r(2s + 1) - 1$ $(r > 0, s > 0)$, and Dold proved that $P(2^r - 1, 2^r s)$ is indecomposable. We make a similar construction.

For a space X and a positive integer m let $P(m, X)$ be formed from $S^m \times X \times X$ by identifying (u, x, y) with $(-u, y, x)$. If X is an n-manifold, then $P(m, X)$ is an $(m + 2n)$-manifold. In § 7, we will show that $[P(m, P^n)] = [P(m, n)]$. However, we will need a more general choice of X, and in § 7 we will prove the following result.

PROPOSITION 4.1. *$[P(m, M^k)]$ is indecomposable in $\mathfrak{N}_*$ if and only if $[M^n]$ is indecomposable in $\mathfrak{N}_*$ and the binomial coefficient $\{m - 1, n\}$ is nonzero modulo* 2.

Let n be odd and not of the form $2^k - 1$. Let $n = 2^r(2s + 1) - 1$ $(r > 0, s > 0)$, and let $a = 2^r - 1$, $b = 2^r s$.

COROLLARY 4.2. *$V^n = P(a, V^b)$ determines an indecomposable element in $\mathfrak{N}_*$.*

Proof. Because b is even, V^b has been defined and $[V^b]$ is indecomposable by Proposition 3.2. Also, $\{2^r - 1, 2^r s\}$ is nonzero modulo 2 by Lemma 3.3. Hence, $[V^n]$ is indecomposable by Proposition 4.1.

To find embeddings and immersions of the manifolds $P(a, V^b)$, we can make use of our embeddings and immersions of the manifolds V^b. An immersion (embedding) of V^b in $\mathbf{R}^s$ induces an immersion (embedding) of $P(a, V^b)$ in $P(a, \mathbf{R}^s)$.

PROPOSITION 4.3. *$P(m, \mathbf{R}^k)$ is the total space of the bundle $k\gamma_m \oplus k\epsilon$ where γ_m, ϵ are, respectively, the canonical line bundle and trivial line bundle over P^m.*

Proof. Define $g: S^m \times \mathbf{R}^k \times \mathbf{R}^k \to S^m \times \mathbf{R}^k \times \mathbf{R}^k$ by

$$g(u, x, y) = (u, x - y, x + y).$$

If, in the domain of g, we identify (u, x, y) with $(-u, y, x)$ and, in the range of g, we identify (u, x, y) with $(-u, -x, y)$, then g induces a homeomorphism between these quotient spaces. But the first is $P(m, \mathbf{R}^k)$ and the second is the total space of $k\gamma_m \oplus k\epsilon$.

In order to apply Proposition 4.3, we must be able to immerse and embed sums of line bundles over real projective spaces. We will use the following result of Mahowald and Milgram [**8**, Theorem 4.1, p. 418].

THEOREM 4.4 (Mahowald and Milgram). *Let p and q be odd and let $m = p + q + 1$. Then the total space of $(p + 1)\gamma_q$ immerses in Euclidean space of dimension $2q + p + 1 - \alpha(m) + \alpha(p + 1) - k(p, m)$.*

Here, $k(p, m) = \min(k(p), k(m))$ and $k(t)$ depends on the congruence class of t modulo 8 as follows: $k(t) = 0$ if $t \equiv 1(8)$, $k(t) = 1$ if $t \equiv 3$ or 5 (8), $k(t) = 4$ if $t \equiv 7(8)$.

PROPOSITION 4.5. *$V^n = P(a, V^b)$ immerses in $\mathbf{R}^{2n-\alpha(n)}$ and embeds in $\mathbf{R}^{2n-\alpha(n)+1}$.*

Proof. We will show the existence of the embedding stated above. The immersion is obtained by a similar but somewhat easier argument.

By Proposition 3.1, V^b embeds in $R^{2b-\alpha(b)+1}$. Thus, by Proposition 4.3, $P(a, V^b)$ embeds in the total space of the bundle $(2b - \alpha(b) + 1)(\gamma_a \oplus \epsilon)$. We first apply Theorem 4.4 to $2b\gamma_a$ with $2b = p + 1$, $q = a$, and $m = a + 2b = 2^r - 1 + 2^{r+1}s$. Hence, $\alpha(m) - \alpha(p + 1) = r$, and, certainly, $k(p, m) \geqq 0$. We therefore obtain an immersion of $2b\gamma_a$ in $\mathbf{R}^{2a+2b-r}$. We can consider this as an immersion of P^a in $\mathbf{R}^{2a+2b-r}$ with normal bundle containing $2b\gamma_a$ as a sub-bundle. Because $2a + 2b - r \geqq 2a + 1$, this immersion is homotopic through immersions to an embedding, and a tubular neighbourhood of the embedding contains the total space of $2b\gamma_a$. Now we can take the product with $\mathbf{R}^{2b-\alpha(b)+1}$ to obtain an embedding of $2b\gamma_a \oplus (2b - \alpha(b) + 1)\epsilon$ in $\mathbf{R}^{2a+4b-\alpha(b)-r+1}$. Because $2b \geqq 2b - \alpha(b) + 1$, this bundle contains

$$(2b - \alpha(b) + 1)(\gamma_a \oplus \epsilon).$$

Also, $2a + 4b - (r + \alpha(b)) + 1 = 2n - \alpha(n) + 1$, so we have found the required embedding of $P(a, V^b)$.

5. The main theorem.

We are now ready to prove our main result.

THEOREM 5.1. *Any M^n is cobordant to a manifold that immerses with efficiency $\alpha(n)$ and embeds with efficiency $\alpha(n) - 1$.*

Proof. The manifolds V^n constructed in §§ 3 and 4 generate the cobordism ring $\mathfrak{N}_*$. Thus, any M^n is cobordant to a disjoint union of products of the V^k. As noted at the beginning of § 2, we can take the product of the immersions given by Propositions 3.1 and 4.5 to obtain an immersion of each product in $\mathbf{R}^{2n-\alpha(n)}$. Similarly, we can apply Lemma 2.1 to embed each product in $\mathbf{R}^{2n-\alpha(n)+1}$. (In Lemma 2.1, if $m \leqq n$ then $s + t \geqq 2m + 1$. Thus, we can embed the factor of largest dimension and then immerse the other factors and apply Lemma 2.1 repeatedly.) Finally, we can take the disjoint union of our immersions or embeddings in an obvious way. This completes the proof of Theorem 5.1.

We will now show that for $n \neq 3$, Theorem 1 is the best possible result. Let $n = r_1 + \ldots + r_k$ be the dyadic expansion of n as a sum of distinct powers of 2 and let

$$A^n = \prod_{i=1}^{k} P^{r_i}.$$

PROPOSITION 5.2. *The manifold A^n does not immerse in $\mathbf{R}^{2n-\alpha(n)-1}$ and does not embed in $\mathbf{R}^{2n-\alpha(n)}$. If n is even, no manifold cobordant to A^n immerses in $\mathbf{R}^{2n-\alpha(n)-1}$ or embeds in $\mathbf{R}^{2n-\alpha(n)}$.*

Proof. A necessary condition for M^n to immerse in $\mathbf{R}^{2n-k-1}$ or embed in $\mathbf{R}^{2n-k}$ is that $\bar{w}_i(M^n) = 0$, for $i \geqq n - k$. (See [**9**, Theorem 4, p. 13, Theorem 14, p. 44].) Now,

$$\begin{aligned}\bar{w}(A^n) &= \prod_{i=1}^{k} (1 + \alpha_i)^{-r_i-1} \\ &= \prod_{i=1}^{k} (1 + \alpha_i^{r_i})^{-1}(1 + \alpha_i)^{-1} \\ &= \prod_{i=1}^{k} (1 + \alpha_i)^{r_i-1},\end{aligned}$$

because r_i is a power of 2 and $\alpha_i^s = 0$, for $s > r_i$. Hence,

$$\begin{aligned}\bar{w}_{n-\alpha(n)}(A^n) &= \prod_{i=1}^{k} \alpha_i^{r_i-1} \\ &\neq 0.\end{aligned}$$

This proves the first assertion of the proposition.

Note that if n is odd, the first factor in A^n is $P^1 = S^1$ which is a boundary. Hence, A^n is a boundary if n is odd. Assume that n is even. Then $\bar{w}_{\alpha(n)}(M^n) = \alpha_1 \cdot \ldots \cdot \alpha_k$ plus other terms, where the other terms are each of degree greater than one in some α_j. Hence,

$$\begin{aligned}\bar{w}_{\alpha(n)}\bar{w}_{n-\alpha(n)}(M^n) &= \prod_{i=1}^{k} \alpha_i^{r_i} \\ &\neq 0,\end{aligned}$$

and the corresponding Stiefel-Whitney number is nonzero. Thus, every manifold cobordant to A^n has $\bar{w}_{n-\alpha(n)} \neq 0$, and the second assertion is proved.

Now let n be odd, $n > 3$. Let the dyadic expansion of n be

$$n = 1 + r_1 + \ldots + r_k.$$

(Thus, $\alpha(n) = k + 1$.) For each i $(1 \leqq i \leqq k)$ such that $r_i \neq 2$, define a manifold $B_i^n = P(1, P^s) \times A^t$, where $s = \frac{1}{2}r_i$ and $t = n - (r_i + 1)$. In § 7 we study the Stiefel-Whitney classes of $P(m, M^n)$ and prove the following result.

PROPOSITION 5.3. *No manifold cobordant to B_i^n immerses in $\mathbf{R}^{2n-\alpha(n)-1}$ or embeds in $\mathbf{R}^{2n-\alpha(n)}$.*

6. Vanishing Stiefel-Whitney numbers. A necessary condition for M^n to be cobordant to a manifold that immerses in $\mathbf{R}^{2n-k-1}$ or embeds in $\mathbf{R}^{2n-k}$ is that all Stiefel-Whitney numbers involving $\bar{w}_{n-i}$, for $0 \leqq i \leqq k$, should vanish. We will now show that this condition is sometimes also sufficient.

First, we will define some new generators W^n of $\mathfrak{N}_*$. If n is odd (and $n \neq 2^k - 1$), let $W^n = V^n$, as defined in § 4. If n is even and $\alpha(n) \geqq 3$, let $n = r_1 + \ldots + r_k$ be the dyadic expansion of n, and define

$$K^{n+1} = \prod_{i=1}^{k-1} P^{s_i},$$

where $s_i = r_i (1 \leqq i \leqq k - 3)$, $s_{k-2} = r_{k-2} + 1$, and $s_{k-1} = r_{k-1} + r_k$, and let $W^n \subset K^{n+1}$ be a submanifold dual to $\alpha_1 + \ldots + \alpha_{k-1}$. (See § 3.) If $\alpha(n) \leqq 2$, let $W^n = P^n$.

PROPOSITION 6.1. *$[W^n]$ is indecomposable in $\mathfrak{N}_*$.*

Proof. If n is odd, then Corollary 4.2 applies. If n is even and $\alpha(n) \leqq 2$, $W^n = P^n$ and $[P^n]$ is indecomposable whenever n is even. (Because $w(P^n) = (1 + \alpha)^{n+1}$, $s_{(n)}(P^n) = (n + 1)\alpha^n \neq 0$.) Finally, if n is even and $\alpha(n) \geqq 3$, we use the proof of Proposition 3.2 including Lemma 3.3.

It is clear that when n is odd, W^n immerses in $\mathbf{R}^{2n-\alpha(n)}$ and embeds in $\mathbf{R}^{2n-\alpha(n)+1}$.

PROPOSITION 6.2. *If n is even and $\alpha(n) \geqq 2$, then W^n immerses in $\mathbf{R}^{2n-\alpha(n)-1}$ and (for $n \neq 6$) embeds in $\mathbf{R}^{2n-\alpha(n)}$.*

Proof. If $\alpha(n) = 2$, $W^n = P^n$. Suppose that $n = 2^r + 2$ $(r \geqq 2)$. Then Sanderson [**10**, Theorem 4.1, p. 146 and Theorem 5.3, p. 150] gives the required immersion, and Handel [**4**, Theorem 4.1, p. 129] the required embedding. If $n = 2^r + 2^s$ $(r > s > 1)$, then $n = 4t$, where t is not a power of 2 and Mahowald [**7**, Theorem 7.2.2, p. 346] gives an embedding of P^n in $\mathbf{R}^{2n-3}$.

If $\alpha(n) \geqq 3$, $s_{k-1} = 4t$, where t is not a power of 2. Thus, we can embed the last factor in K^{n+1} with efficiency 3. Using Sanderson's result [**10**, Theorems

4.1, 5.3], we can immerse the second last factor with efficiency 3 unless $s_{k-2} = 3$; however, in the latter case we can immerse $P^3 \times \mathbf{R}$ in $\mathbf{R}^4$. If now we apply Lemma 2.1 repeatedly, we can embed K^{n+1} in $\mathbf{R}^{2n-\alpha(n)-1}$.

THEOREM 6.3. *Let n be even. If the Stiefel-Whitney number of M^n corresponding to $\bar{w}_{\alpha(n)}\bar{w}_{n-\alpha(n)}$ vanishes, then M^n is cobordant to a manifold that immerses in $\mathbf{R}^{2n-\alpha(n)-1}$ and (for $n \neq 6$) embeds in $\mathbf{R}^{2n-\alpha(n)}$.*

Proof. The classes $[W^k]$ generate $\mathfrak{N}_*$ so M^n is cobordant to a disjoint union of products of the W^k. One product which may or may not appear is A^n. (See § 5.) We claim that every other product immerses in $\mathbf{R}^{2n-\alpha(n)-1}$ and (for $n \neq 6$) embeds in $\mathbf{R}^{2n-\alpha(n)}$. Clearly, this is true for W^n; for, either $\alpha(n) = 1$, in which case $W^n = A^n$, or $\alpha(n) \geqq 2$, in which case Proposition 6.2 applies. Any other product can be written in the form $U^s \times U^t$, where $\alpha(s+t) < \alpha(s) + \alpha(t)$, and this strict inequality implies the product immersion or embedding using Lemma 2.1 gives the required result.

The Stiefel-Whitney number corresponding to $\bar{w}_{\alpha(n)}\bar{w}_{n-\alpha(n)}$ is zero on all products except A^n. Hence, the hypothesis of Theorem 6.3 implies that A^n does not appear. This completes the proof of Theorem 6.3.

Note that if n is odd, the argument fails. If n is odd, $n \neq 2^r + 1$, we can immerse W^n in $\mathbf{R}^{2n-\alpha(n)-1}$ and embed W^n in $\mathbf{R}^{2n-\alpha(n)}$. (Indeed we need only consider $k(p, m)$ more carefully in the proof of Proposition 4.5.) However, the difficulty is that there may be more than one product B_i^n. (See § 5.) The hypothesis of Theorem 6.3 implies only that the number of products B_i^n occurring in the expansion of $[M^n]$ is even.

THEOREM 6.4. *Let $n = 2^r$ or $2^r + 1$, and let $0 \leqq s \leqq 3$. Then M^n is cobordant to a manifold that immerses in $\mathbf{R}^{2n-s-1}$ and embeds in $\mathbf{R}^{2n-s}$ if and only if the Stiefel-Whitney numbers of M^n corresponding to $\bar{w}_i\bar{w}_{n-i}$ vanish, for $0 \leqq i \leqq s$.*

Proof. We can represent the cobordism class of M^n by a disjoint union of products of the form

$$\prod_{i=1}^{k} W^{n_i}.$$

Terms with $\sum_{i=1}^{k} \alpha(n_i) \geqq 4$ immerse and embed with efficiencies 4 and 3, respectively. If $\sum_{i=1}^{k} \alpha(n_i) \leqq 3$, then $k \leqq 3$. Also, $\alpha(n_i) \leqq 2$ because $n = 2^r$ or $2^r + 1$. The only terms satisfying these conditions are given by:

(i) $n = 2^r + 1, \quad n_1 = n,$
(ii) $n = 2^r + 1, \quad n_1 = 2^{r-1} + 1, \quad n_2 = 2^{r-1},$
(iii) $n = 2^r, \quad n_1 = n,$
(iv) $n = 2^r, \quad n_1 = 2^{r-1}, \quad n_2 = 2^{r-1},$
(v) $n = 2^r, \quad n_1 = 2^{r-1}, \quad n_2 = 2^{r-2}, \quad n_3 = 2^{r-2} \ (r > 2),$
(vi) $n = 2^r, \quad n_1 = 2^{r-1} + 2^{r-2}, \quad n_2 = 2^{r-2} \ (r > 2).$

It is easy to check that the first five terms are all detected by the Stiefel-Whitney numbers of Theorem 6.4. We claim that the sixth embeds with efficiency 3 and immerses with efficiency 4. If $r > 3$, we use Mahowald's result [**7**, Theorem 7.2.2, p. 346] to embed the first factor with efficiency 3, and if $r = 3$, the first factor is P^6, which immerses with efficiency 4. (See [**10**, Theorem 4.1].) Theorem 6.4 is now proved.

7. $P(m, X)$**.** We will begin by computing the modulo 2 cohomology ring of $P(m, X)$.

The projection of $S^m \times X^2$ on S^m induces a bundle map of $P(m, X)$ on P^m with fibre X^2. If we choose $x \in X$, we can define a section of this bundle by setting $s(u) = (u, x, x)$. It follows that the cohomology ring $H^*(P^m)$ is a direct summand of $H^*(P(m, X))$. We denote the generator of this summand by $c \in H^1(P(m, X))$. Then $c^{m+1} = 0$ and $\pi^*c = 0$, where $\pi: S^m \times X^2 \rightarrow P(m, X)$ is the identification projection.

Let $T: S^m \times X^2 \rightarrow S^m \times X^2$ be defined by $T(u, x, y) = (-u, y, x)$. Then $\pi \circ T = \pi$, so $(1 + T^*) \circ \pi^* = 0$, and the image of π^* is contained in the kernel of $1 + T^*$. Let $t: X^2 \rightarrow X^2$ be the interchange $t(x, y) = (y, x)$. Let $N \subset H^*(X^2) = H^*(X) \otimes H^*(X)$ be the image of $1 + t^*$ and let $D \subset H^*(X^2)$ be the set of diagonal elements of the form $x \otimes x$. Then the kernel of $1 + t^*$ is $D + N$. (Note that D is not closed under addition but that $D + N$ is closed under addition because, for example, $x \otimes x + y \otimes y = (x + y) \otimes (x + y) + (1 + t^*)(x \otimes y)$.) Let g_m generate $H^m(S^m)$. Then the kernel of $1 + T^*$ is $D + N + g_m \otimes (D + N)$. (Here, we are writing $D + N$ instead of $1 \otimes (D + N)$.)

THEOREM 7.1. *The cohomology ring* $H^*(P(m, X))$ *is isomorphic to*

$$(\mathbf{Z}_2[c]/(c^{m+1}) \otimes D) + N + g_m \otimes N,$$

where $c^0 \otimes D = D$, *and multiplication is determined by the multiplication in* $D + N + g_m \otimes N$ *and the relation* $c \otimes N = 0$. *Also,* π^* *is the identity on* $D + N + g_m \otimes N$.

Proof. We will use the exact sequence of the pair $(P(m, X), P(m - 1, X))$ to prove the theorem by induction on m. We will use π to denote the identification maps for both spaces and for the pair of spaces. Recall that $m \geqq 1$.

Let $S^{m-1} \times I$ be a band around the equator of S^m with upper and lower boundaries S_+^{m-1} and S_-^{m-1}. Let D_+^m and D_-^m be the top and bottom caps of S^m with boundaries S_+^{m-1} and S_-^{m-1}, respectively. The inclusion

$$((D_+^m, S_+^{m-1}) \cup (D_-^m, S_-^{m-1})) \times X^2 \rightarrow (S^m, S^{m-1} \times I) \times X^2$$

is an excision. There is an induced excision after identification, and, because π identifies the two pieces on the left, we obtain an isomorphism

$$H^*(P(m, X), P(m - 1, X)) \cong H^*((D^m, S^{m-1}) \times X^2).$$

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We will denote the generator of $H^m((D^m, S^{m-1}))$ by g_m again. The long exact sequence of the pair now becomes

$$\ldots \xrightarrow{j^*} H^k(P(m,X)) \xrightarrow{i^*} H^k(P(m-1,X)) \xrightarrow{\delta} g_m \otimes H^{k+1-m}(X^2) \xrightarrow{j^*} \ldots .$$

LEMMA 7.2. *Let $\hat{\imath}: S^{m-1} \times X^2 \to S^m \times X^2$ be the inclusion and let δ' be the coboundary in the sequence of the pair $(D^m, S^{m-1}) \times X^2$. Then $i^*c = c, j^*g_m = c^m$, $\pi^*i^* = \hat{\imath}^*\pi^*$, $\pi^*j^*(g_m \otimes x \otimes y) = g_m \otimes (x \otimes y + y \otimes x)$, and $\delta = \delta'\pi^*$.*

Proof. The first two statements follow by comparison with the exact sequence of the pair (P^m, P^{m-1}). The other statements follow from the commutative diagram below:

$$\begin{tikzcd}
S^{-1} \times X^2 \arrow[r] \arrow[d, "id"] & D^m \times X^2 \arrow[rr] \arrow[d, "h"] & & (D^m, S^{m-1}) \times X^2 \arrow[d, "f"] \\
S^{m-1} \times X^2 \arrow[r, "\hat{\imath}"] \arrow[d, "\pi"] & S^m \times X^2 \arrow[r] \arrow[d, "\pi"] & (S^m, S^{m-1}) \times X^2 \arrow[d, "\pi"] & \begin{array}{c} (D_+{}^m, S_+{}^{m-1}) \times X^2 \\ \cup \\ (D_-{}^m, S_-{}^{m-1}) \times X^2 \end{array} \arrow[l, "e_1"'] \arrow[d, "\pi"] \\
P(m-1, X) \arrow[r, "i"] & P(m, X) \arrow[r, "\bar{j}"] & (P(m, X), P(m-1, X)) & (D^m, S^{m-1}) \times X^2 \arrow[l, "e_2"']
\end{tikzcd}$$

In this diagram, e_1 and e_2 are excisions, πf is the identity, and $j^* = \bar{j}^* e_2{}^{*-1}$.

Now we are ready to prove the theorem for $m = 1$. Because $P(0, X) = X^2$, we have an exact sequence.

$$\ldots \xrightarrow{j^*} H^k(P(1,X)) \xrightarrow{i^*} H^k(X^2) \xrightarrow{\delta} g_1 \otimes H^k(X^2) \xrightarrow{j^*} H^{k+1}(P(1,X)) \xrightarrow{i^*} \ldots$$

in which $\delta(x \otimes y) = \delta'\pi^*(x \otimes y) = g_1 \otimes (x \otimes y + y \otimes x)$. Thus,

$$H^*(P(1, X))/\text{image}(j^*) = \text{kernel}(\delta) = D + N.$$

Because $\text{kernel}(j^*) = \text{image}(\delta) = g_1 \otimes N$, it follows that

$$j^*(g_1 \otimes x \otimes y) = j^*(g_1 \otimes y \otimes x)$$

and that $\pi^*j^*(g_1 \otimes x \otimes y) = g_1 \otimes (x \otimes y + y \otimes x)$ in $H^*(S^1 \times X^2)$. Thus, the image of π^* is exactly $D + N + g_1 \otimes N$. Also, j^* is injective on $g_1 \otimes D$, and $\pi^*j^*(g_1 \otimes D) = 0$. We can denote $j^*(g_1 \otimes D)$ by $c \otimes D$. For, if $u \in H^*(P(1, X))$ with $\pi^*u = x \otimes x \in D$, then $h^*\pi^*u = x \otimes x$ and, because πf is the identity and $H^*(Y, A)$ is a module over $H^*(Y)$, $g_1 \otimes (x \otimes x) = g_1 \otimes u$ in $H^*((D^1, S^0) \times X^2)$. Thus, $j^*(g_1 \otimes x \otimes x) = j^*(g_1) \otimes u = c \otimes u$.

Similarly, if $\pi^*u = x \otimes y + y \otimes x$, then

$$c \otimes u = j^*(g_1) \otimes u = j^*(g_1 \otimes (x \otimes y + y \otimes x)) = j^*\delta(g_1 \otimes x \otimes y) = 0.$$

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Thus, $c \otimes N = 0$.

This completes the proof for $m = 1$.

Assume now that the theorem is proved for $m - 1$. By Lemma 7.2, $\delta(c) = 0$, $\delta(D + N) = 0$, and $\delta(g_{m-1} \otimes N) = g_m \otimes N$. Thus, j^* maps $g_m \otimes D$ injectively to $c^m \otimes D$. Also, $j^*(g_m \otimes x \otimes y) = j^*(g_m \otimes y \otimes x)$ and

$$\pi^* j^*(g_m \otimes x \otimes y) = g_m \otimes (x \otimes y + y \otimes x).$$

Hence, the image of j^* is $c^m \otimes D + g_m \otimes N$. The rest of $H^*(P(m, X))$ is isomorphic to image(i^*) = kernel(δ) = $\{c^i \otimes D + N, 0 \leqq i \leqq m - 1\}$. This completes the inductive proof of the theorem.

Recall that a map $f:X \to Y$ induces a map $P(m, f)$ from $P(m, X)$ to $P(m, Y)$ defined by $P(m, f)(u, x, y) = (u, f(x), f(y))$. The next result is immediate from Theorem 7.1.

COROLLARY 7.3. *If $f:X \to Y$ is such that the induced map in cohomology is injective, then the cohomology map of $P(m, f):P(m, X) \to P(m, Y)$ is also injective.*

Given x, $y \in H^*(X)$, we denote $x \otimes x \in H^*(P(m, X))$ by $d(x)$ and $x \otimes y + y \otimes x \in H^*(P(m, X))$ by $e(x \otimes y)$. Let $e(x) = e(x \otimes 1) = e(1 \otimes x)$.

If γ is a vector bundle over X with fibre $\mathbf{R}^n$, total space $E(\gamma)$, and projection $p:E(\gamma) \to X$, then $P(m, p):P(m, E(\gamma)) \to P(m, X)$ is the projection of a vector bundle (denoted by $P(m, \gamma)$) over $P(m, X)$ with fibre $\mathbf{R}^{2n}$. It is straightforward to verify that $P(m, \gamma \oplus \gamma') = P(m, \gamma) \oplus P(m, \gamma')$.

PROPOSITION 7.4. *If γ is a line bundle over X with total Stiefel-Whitney class $w(\gamma) = 1 + \alpha$, then $w(P(m, \gamma)) = 1 + c + e(\alpha) + d(\alpha)$.*

Proof. The pullback of $P(m, \gamma)$ to $S^m \times X^2$ is just $\gamma \times \gamma$. Hence,

$$\pi^* w(P(m, \gamma)) = 1 + \alpha \otimes 1 + 1 \otimes \alpha + \alpha \otimes \alpha.$$

If $s:P^m \to P(m, X)$ is a section, then the pullback of $P(m, \gamma)$ to P^m is just the sum of the canonical line bundle and the trivial line bundle. Hence, $s^* w(P(m, \gamma)) = 1 + c$. It follows that $w(P(m, \gamma))$ must be as stated.

COROLLARY 7.5. *$P(m, P^n)$ is cobordant to the Dold manifold $P(m, n)$.*

Proof. If γ_n is the canonical line bundle and ϵ the trivial line bundle over P^n, then $\tau(P^n) \oplus \epsilon = (n + 1)\gamma_n$. (See [**9**, p. 11].) Hence, $P(m, \tau(P^n)) \oplus P(m, \epsilon) = (n + 1)P(m, \gamma_n)$. Now in general, $\tau(P(m, M^n)) = \tau(P^m) \oplus P(m, \tau(M^n))$. Observe also that $P(m, \epsilon) = \gamma_m \oplus \epsilon$. Hence,

$$w(P(m, P^n))(1 + c) = (1 + c)^{m+1}(1 + c + e(\alpha) + d(\alpha))^{n+1}.$$

According to [**3**, Satz 1, p. 29 and Satz 2, p. 30] the formula for the total Stiefel-Whitney class of $P(m, n)$ is

$$w(P(m, n)) = (1 + c)^m (1 + c + d)^{n+1},$$

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where $c^{m+1} = 0$, $d^{n+1} = 0$, $c \in H^1(P(m, n))$, $d \in H^2(P(m, n))$. Because $ce(\alpha) = 0$, $P(m, P^n)$ and $P(m, n)$ will have the same Stiefel-Whitney numbers and hence be cobordant.

Remark 7.6. There is another way of proving Corollary 7.5 which depends on fixed point sets of involutions. Define an involution s on S^m by

$$s(x_0, \ldots, x_m) = (-x_0, x_1, \ldots, x_m).$$

Extend to $S^m \times M^n \times M^n$ by setting $s(u, x, y) = (s(u), x, y)$. There is an induced involution (again denoted by s) on $P(m, M^n)$ and the fixed point set F of s is $F = P(m - 1, M^n) \cup \Delta$, where Δ is the diagonal of $M^n \times M^n$. If ν is the normal bundle of F in $P(m, M^n)$, then $P(m, M^n)$ is cobordant to the real projective bundle of $\nu \oplus \epsilon$, $[P(m, M^n)] = [RP(\nu \oplus \epsilon)]$. (See [**2**, Theorem 24.2].) Over $P(m - 1, M^n)$, this bundle has fibre $P^1 = S^1$ and hence is a boundary. (It bounds a bundle with fibre D^2.) The normal bundle to Δ is $m\epsilon \oplus \tau(M^n)$. Hence, $[P(m, M^n)] = [RP(\tau(M^n) \oplus (m + 1)\epsilon)]$. This result offers an alternative method for proving Proposition 4.1. Now, if we define s on $P(m, n)$ by $s(u, z) = (s(u), z)$, then a similar argument shows that $[P(m, n)] = [RP(\tau(P^n) \oplus (m + 1)\epsilon)]$. Hence, $[P(m, n)] = [P(m, P^n)]$.

Proof of Proposition 5.3. It is slightly more convenient to work with $P(1, s)$ rather than with $P(1, P^s)$. Recall that s is a power of 2 and that $s \geqq 2$. Hence,

$$\begin{aligned} \bar{w}(P(1, s)) &= (1 + c)^{-1}(1 + c + d)^{-s-1} \\ &= (1 + c)(1 + c + d)^{s-1}, \end{aligned}$$

because $c^2 = 0$. Thus, $\bar{w}_i(P(1, s)) = 0$, for $i \geqq 2s$, and $\bar{w}_{2s-1}(P(1, s)) = cd^{s-1} \neq 0$. Also, $\bar{w}_2(P(1, s)) = d$, so $\bar{w}_2\bar{w}_{2s-1}(P(1, s)) = cd^s \neq 0$. This covers the case $t = 0$ of Proposition 5.3. The general case is a straightforward consequence of these calculations and the calculations given in Proposition 5.2.

Proof of Proposition 4.1. Let $f{:}X \to M^n$ be a splitting map for $\tau(M^n)$. (See [**6**, Proposition 5.1, p. 235].) That is, the pullback $f^{-1}\tau(M^n)$ is a direct sum of line bundles, and the cohomology map f^* is injective. Suppose that

$$f^*w(M^n) = w(f^{-1}\tau M^n) = \prod_{i=1}^{n} (1 + \alpha_i).$$

Then the pullback of $P(m, \tau M^n)$ to $P(m, X)$ under the map $P(m, f)$ is a direct sum of $\mathbf{R}^2$-bundles and

$$P(m, f)^*w(P(m, \tau M^n)) = \prod_{i=1}^{n} (1 + c + e(\alpha_i) + d(a_i)).$$

Because $\tau P(m, M^n) = \tau P^m \oplus P(m, \tau M^n)$, we obtain the relation

$$P(m, f)^*w(P(m, M^n)) = (1 + c)^{m+1} \prod_{i=1}^{n} (1 + c + e(\alpha_i) + d(\alpha_i)).$$

Let $1 + c + e(\alpha_i) + d(\alpha_i) = (1 + u_i)(1 + v_i)$ so that $u_i + v_i = c + e(\alpha_i)$ and $u_i v_i = d(\alpha_i)$. Because $P(m, f)^*$ is injective, $[P(m, M^n)]$ is indecomposable if and only if the polynomial in $c + e(\alpha_i)$ and $d(\alpha_i)$ corresponding to the symmetric function

$$s_{(m+2n)} = (m+1)c^{m+2n} + \sum_{i=1}^{n} (u_i^{\,m+2n} + v_i^{\,m+2n})$$

is nonzero. But $c^{m+2n} = 0$ and

$$u_i^{\,m+2n} + v_i^{\,m+2n} = \sum_{k+2j=m+2n} \{k-1, j\} (u_i + v_i)^k (u_i v_i)^j.$$

If we now substitute and use the relations $c^{m+1} = 0$, $ce(\alpha_i) = 0$, and $e(\alpha_i)^k d(\alpha_i)^j = 0$, if $k + 2j > 2n$, we obtain the result

$$\begin{aligned} s_{(m+2n)} &= \{m-1, n\} c^m \sum_{i=1}^{n} d(\alpha_i)^n \\ &= \{m-1, n\} c^m d\left(\sum_{i=1}^{n} \alpha_i^{\,n}\right). \end{aligned}$$

This is nonzero if and only if $\{m-1, n\} \equiv 1 \pmod 2$ and $\sum_{i=1}^{n} \alpha_i^{\,n} \neq 0$; the latter condition holds if and only if $[M^n]$ is indecomposable.

References

1. R. L. W. Brown, *Imbeddings, immersions, and cobordism of differentiable manifolds*, Bull. Amer. Math. Soc. *76* (1970), 763–766.
2. P. E. Conner and E. E. Floyd, *Differentiable periodic maps* (Springer-Verlag, Berlin, 1964).
3. A. Dold, *Erzengende der Thomschen Algebra* $\mathfrak{N}$, Math. Z. *65* (1956), 25–35.
4. D. Handel, *An embedding theorem for real projective spaces*, Topology *7* (1968), 125–130.
5. M. W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. *93* (1959), 242–276.
6. D. Husemoller, *Fibre bundles* (McGraw-Hill, New York, 1966).
7. M. Mahowald, *On obstruction theory in orientable fibre bundles*, Trans. Amer. Math. Soc. *110* (1964), 315–349.
8. M. Mahowald and R. J. Milgram, *Embedding real projective spaces*, Ann. of Math. *87* (1968), 411–422.
9. J. Milnor, *Lectures on characteristic classes*, mimeographed notes (Princeton University, Princeton, N.J., 1957).
10. B. J. Sanderson, *Immersions and embeddings of projective spaces*, Proc. London Math. Soc. *14* (1964), 137–153.
11. B. J. Sanderson and R. L. E. Schwarzenberger, *Non-immersion theorems for differentiable manifolds*, Proc. Cambridge Philos. Soc. *59* (1963), 319–322.
12. R. E. Stong, *Notes on cobordism theory* (Princeton University Press, Princeton, N.J., 1968).
13. R. Thom, *Quelques propriétés globales des variétés différentiables*. Comment. Math. Helv. *28* (1954), 17–86.
14. H. Whitney, *The self-intersections of a smooth n-manifold in 2n-space*, Ann. of Math. *45* (1944), 220–246.
15. ——— *The singularities of a smooth n-manifold in (2n − 1)-space*, Ann. of Math. *45* (1944), 247–293.

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