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ABSTRACT

We study a wide class of affine varieties, which we call affine Fano varieties. By analogy with birationally super-rigid Fano varieties, we define super-rigidity for affine Fano varieties, and provide many examples and non-examples of super-rigid affine Fano varieties.

1. Introduction

Throughout the present article, all varieties and morphisms are assumed to be defined over a field \mathbb{k} of characteristic zero.

The automorphism group of an affine surface admitting a smooth projective completion X whose boundary S_X consists of a single smooth curve has been intensively studied during the last few decades [GD75, GD77, BD11, BD15, DL15], inspired by the pioneering work of Gizatullin in [Giz70]. The upshot is that depending on whether the curve S_X is rational or not, the group $\text{Aut}(X \setminus S_X)$ is either an infinite dimensional group which sometimes cannot even be generated by any countable family of its algebraic subgroups, or an affine algebraic group isomorphic to the group $\text{Aut}(X, S_X)$ of automorphisms of the pair (X, S_X) . In higher dimension, given a projective variety X and an ample prime divisor S_X on it, not much is known about geometric conditions on the pair (X, S_X) which ensure that the restriction map $\text{Aut}(X, S_X) \rightarrow \text{Aut}(X \setminus S_X)$ is again an isomorphism. This holds in general when X is smooth and S_X is a non-ruled hypersurface of X [Puk18, Proposition 1], and for certain specific families of singular hypersurfaces S_X in $X = \mathbb{P}^n$ recently studied in [Puk18]. For affine cubic hypersurfaces, there is a folklore as follows.

CONJECTURE 1.1. Let X be a smooth cubic hypersurface in the complex projective space \mathbb{P}^4 , and let S_X be its hyperplane section. The complement $X \setminus S_X$ is an affine cubic hypersurface in \mathbb{A}^4 . Suppose that the cubic surface S_X is smooth. Then

$$\text{Aut}(X \setminus S_X) = \text{Aut}(X, S_X).$$

In particular, the group $\text{Aut}(X \setminus S_X)$ is finite.

In this article, we study a similar problem for a wide class of affine varieties, which we call affine Fano varieties, as follows. Let X be a projective normal variety of Picard rank 1 with \mathbb{Q} -factorial singularities, and let S_X be a prime divisor on X such that the pair (X, S_X) has purely log terminal singularities. In particular, the set $X \setminus S_X$ is an affine variety since the divisor S_X is ample.

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DEFINITION 1.2. If $-(K_X + S_X)$ is an ample divisor, then the affine variety $X \setminus S_X$ is called an affine Fano variety with completion X and boundary S_X .

By analogy with Mori fiber spaces in birational geometry, we introduce the following relative version of Definition 1.2.

DEFINITION 1.3. Let \mathcal{X} and \mathcal{B} be quasi-projective normal varieties such that there exists a dominant non-birational projective morphism $\rho: \mathcal{X} \rightarrow \mathcal{B}$ with connected fibers, and let $\mathcal{S}_{\mathcal{X}}$ be a prime divisor on \mathcal{X} . Denote by \mathcal{X}_{η} the scheme theoretic generic fiber of ρ , so that \mathcal{X}_{η} is defined over the function field of \mathcal{B} . Put $\mathcal{S}_{\mathcal{X}_{\eta}} = \mathcal{S}_{\mathcal{X}}|_{\mathcal{X}_{\eta}}$. We say that $\mathcal{X} \setminus \mathcal{S}_{\mathcal{X}}$ is a (relative) affine Fano variety over \mathcal{B} provided that the following properties are satisfied:

- the generic fiber \mathcal{X}_{η} is a projective variety of Picard rank 1 with \mathbb{Q} -factorial singularities;
- the pair $(\mathcal{X}_{\eta}, \mathcal{S}_{\mathcal{X}_{\eta}})$ has purely log terminal singularities;
- the divisor $-(K_{\mathcal{X}_{\eta}} + \mathcal{S}_{\mathcal{X}_{\eta}})$ is ample.

This definition comprises two important notions: ‘usual’ affine Fano varieties as in Definition 1.2 when the base \mathcal{B} is a point, and \mathbb{A}^1 -cylinders when the dimension of \mathcal{B} is positive. Recall that an \mathbb{A}^1 -cylinder is a variety isomorphic to $\mathcal{B} \times \mathbb{A}^1$ for some smooth quasi-projective variety \mathcal{B} . In this case, the projection $\text{pr}_{\mathcal{B}}: \mathcal{X} = \mathcal{B} \times \mathbb{P}^1 \rightarrow \mathcal{B}$ is a relative affine Fano variety with respect to the divisor $\mathcal{S}_{\mathcal{X}} = \mathcal{B} \times \{\infty\}$, where $\infty = \mathbb{P}^1 \setminus \mathbb{A}^1$. Projective varieties that contain Zariski open \mathbb{A}^1 -cylinders appear naturally in many problems and questions (see [Rus81, KMc99, KPZ13, DK15, CPW16a, CPW16b]). Some of them are still open. For instance, it is not known whether every smooth projective rational variety always contains a Zariski open \mathbb{A}^1 -cylinder or not.

In dimension one, the only affine Fano variety is the affine line \mathbb{A}^1 , and its automorphisms are induced by the automorphisms of its completion \mathbb{P}^1 . This is no longer true in higher dimensions: the complement \mathbb{A}^2 of a line L in \mathbb{P}^2 is an affine Fano variety that contains an open \mathbb{A}^1 -cylinder, for which $\text{Aut}(\mathbb{A}^2) \neq \text{Aut}(\mathbb{P}^2, L)$. Keeping in mind Conjecture 1.1 and the fact that smooth cubic threefolds in \mathbb{P}^4 do not contain open \mathbb{A}^1 -cylinders (see [DK15]) we are primarily interested in affine Fano varieties with the following properties:

- (1) their automorphisms are induced by automorphisms of their completions;
- (2) they do not contain proper relative affine Fano varieties over varieties of positive dimensions. In particular, they do not contain open \mathbb{A}^1 -cylinders.

These resemble basic properties of the so-called birationally super-rigid Fano varieties. We recall from [Cor00, Definition 1.3] that a Fano variety V with terminal \mathbb{Q} -factorial singularities and Picard rank 1 is said to be birationally super-rigid if the following two conditions hold.

- (1) For every Fano variety V' with terminal \mathbb{Q} -factorial singularities and Picard rank 1, if there exists a birational map $\phi: V \dashrightarrow V'$, then ϕ is an isomorphism. In particular, one has

$$\text{Bir}(V) = \text{Aut}(V).$$

- (2) The variety V is not birational to a fibration into Fano varieties over a variety of positive dimension. In particular, the variety V is not birationally ruled.

The first example of birationally super-rigid Fano variety was given by Iskovskikh and Manin in [IM71], where they implicitly showed that every smooth quartic threefold is birationally super-rigid. Since then birational super-rigidity has been proved for many higher-dimensional Fano varieties (see [Puk98, CPR00, Che05, deF13, Puk13, CP17]).

By analogy with birational super-rigidity Fano varieties, we introduce the following.

DEFINITION 1.4. An affine Fano variety $X \setminus S_X$ with completion X and boundary S_X is said to be super-rigid if the following two conditions hold.

- (1) For every affine Fano variety $X' \setminus S'_{X'}$, with completion X' and boundary $S'_{X'}$, if there exists an isomorphism $\phi: X \setminus S_X \rightarrow X' \setminus S'_{X'}$, then ϕ is induced by an isomorphism $X \rightarrow X'$ that maps S_X onto $S'_{X'}$. In particular, one has

$$\text{Aut}(X \setminus S_X) = \text{Aut}(X, S_X).$$

- (2) The affine Fano variety $X \setminus S_X$ does not contain relative affine Fano varieties of the same dimension over varieties of positive dimensions. In particular, $X \setminus S_X$ does not contain open \mathbb{A}^1 -cylinders.

The arguably simplest example of a super-rigid affine Fano surface is given as follows.

Example 1.5 (cf. Corollary 2.6). Let C be an irreducible conic in \mathbb{P}^2 . Then $\mathbb{P}^2 \setminus C$ is an affine Fano variety. If C has a rational point, then $\mathbb{P}^2 \setminus C$ contains an open \mathbb{A}^1 -cylinder, so that $\mathbb{P}^2 \setminus C$ is not super-rigid. On the other hand, if the conic C does not have a rational point, then $\mathbb{P}^2 \setminus C$ is super-rigid.

As in the theory of birational super-rigidity, in general it is arduous to determine whether a given affine Fano variety is super-rigid or not. For a terminal \mathbb{Q} -factorial Fano variety of Picard rank 1, the following theorem, known as a classical Noether–Fano inequality, serves as a criterion for birational super-rigidity.

THEOREM 1.6. Let V be a terminal \mathbb{Q} -factorial Fano variety of Picard rank 1. If for any mobile linear system \mathcal{M}_V on V and a positive rational number λ such that

$$K_V + \lambda \mathcal{M}_V \sim_{\mathbb{Q}} 0,$$

the pair $(V, \lambda \mathcal{M}_V)$ is canonical, then V is birationally super-rigid.

In § 2, we will establish a similar statement for super-rigid affine Fano varieties as below.

THEOREM A. Let $X \setminus S_X$ be an affine Fano variety with completion X and boundary S_X . If for any mobile linear system \mathcal{M}_X on X and a positive rational number λ such that

$$K_X + S_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0,$$

the pair $(X, S_X + \lambda \mathcal{M}_X)$ is log canonical along S_X , then $X \setminus S_X$ is super-rigid.

From the viewpoint of the classical Noether–Fano inequality, a terminal \mathbb{Q} -factorial Fano variety V of Picard rank 1 is birationally super-rigid provided that

$$\text{the pair } (V, D) \text{ is canonical for every effective } \mathbb{Q}\text{-divisor } D \text{ that is } \mathbb{Q}\text{-linearly equivalent to } -K_V. \tag{*}$$

Note that this condition is not a necessary condition for birational super-rigidity.

In order to introduce analogy with this sufficient condition, we use a generalized version of Tian’s α -invariant, which was introduced in [Tia87] using a different language. Let $X \setminus S_X$

be an affine Fano variety with completion X and boundary S_X . Then there exists a uniquely determined effective \mathbb{Q} -divisor $\text{Diff}_{S_X}(0)$ on S_X , usually called the different, which was introduced by Shokurov in [Sho93]. To be precise, let R_1, \dots, R_s be all irreducible components of the singular locus of X that are of codimension 2 and contained in S_X . Then

$$\text{Diff}_{S_X}(0) = \sum_{i=1}^s \frac{m_i - 1}{m_i} R_i,$$

where m_i is the smallest positive integer such that $m_i S_X$ is Cartier at a general point of the divisor R_i . By the adjunction formula, we have

$$K_{S_X} + \text{Diff}_{S_X}(0) \sim_{\mathbb{Q}} (K_X + S_X)|_{S_X}.$$

Moreover, by Adjunction (see [Kol97, Theorem 7.5]), the pair $(S_X, \text{Diff}_{S_X}(0))$ has Kawamata log terminal singularities. Thus, since $X \setminus S_X$ is an affine Fano variety, the pair $(S_X, \text{Diff}_{S_X}(0))$ is a log Fano variety in a usual sense.

The α -invariant of the pair $(S_X, \text{Diff}_{S_X}(0))$ is defined as

$$\alpha(S_X, \text{Diff}_{S_X}(0)) = \sup \left\{ \lambda \in \mathbb{Q} \mid \begin{array}{l} \text{the pair } (S_X, \text{Diff}_{S_X}(0) + \lambda D) \text{ is log canonical for every} \\ \text{effective } \mathbb{Q}\text{-divisor } D \text{ such that } D \sim_{\mathbb{Q}} -(K_{S_X} + \text{Diff}_{S_X}(0)) \end{array} \right\}.$$

Then the condition $\alpha(S_X, \text{Diff}_{S_X}(0)) \geq 1$ will be an analogy of the condition (*).

THEOREM B. *Let $X \setminus S_X$ be an affine Fano variety with completion X and boundary S_X . If $\alpha(S_X, \text{Diff}_{S_X}(0)) \geq 1$, then the affine Fano variety $X \setminus S_X$ is super-rigid.*

This sufficient condition allows us to give a large supply of super-rigid affine Fano varieties in every dimension bigger than or equal to 3. As in birational super-rigidity theory, it can scarcely be expected that the condition in Theorem B is sufficient and necessary to be super-rigid. However, our results (Theorems 3.7, 3.10 and 3.11) show that it plays a role of a criterion for affine Fano varieties of a certain type to be super-rigid.

With Theorem B and some result on non-rationality of Fano varieties we can provide an instructive example that fosters our understanding of Conjecture 1.1. The example below stands in the same stream as Conjecture 1.1, with slightly less difficulty.

Example 1.7. Let X be a smooth hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$ of degree 6 and S_X be a hyperplane section of X by an equation of degree 1 with at most Du Val singularities. The hyperplane section S_X is a del Pezzo surface of anticanonical degree 1. The pair (X, S_X) has purely log terminal singularities, and hence $X \setminus S_X$ is an affine Fano variety. It follows from [Che08a, CK14] that $\alpha(S_X, \text{Diff}_{S_X}(0)) = 1$ if the following two conditions are satisfied:

- (1) the surface S_X has only singular points of types A_1, A_2 and A_3 ;
- (2) the linear system $|-K_{S_X}|$ does not contain cuspidal curves.

In such a case, Theorem B immediately implies that $X \setminus S_X$ is super-rigid. However, in general we do not know whether $X \setminus S_X$ is super-rigid or not. We may only draw a conclusion that $\text{Aut}(X \setminus S_X) = \text{Aut}(X, S_X)$ regardless of how singular the surface S_X is. This follows from the result of Grinenko [Gri03, Gri04] that every birational automorphism of X is biregular, i.e., $\text{Bir}(X) = \text{Aut}(X)$.

It is natural that there should be a lot of conundrums in the border area between super-rigidity and non-super-rigidity. For those affine Fano varieties $X \setminus S_X$ near the border even the problem to determine whether the groups $\text{Aut}(X \setminus S_X)$ and $\text{Aut}(X, S_X)$ coincide or not is subtle. We believe that smooth cubic threefolds lie in this border area from the super-rigidity side. This is one of the reasons why Conjecture 1.1 is far away from its proof.

Meanwhile, as in the theory of birational super-rigidity, the super-rigidity of an affine Fano variety $X \setminus S_X$ may be easily destroyed if we allow some singularities in the boundary S_X . This has been partially observed and investigated in [KPZ11, DK15] from singular cubic surfaces. In §4 we provide a complete and uniform picture for the complements of del Pezzo surfaces in the context of affine Fano varieties. We obtain in particular the following characterization of non-super-rigid complement of del Pezzo surfaces.

THEOREM C. *Let S be a del Pezzo surface of anticanonical degree at most 3 with Du Val singularities. The surface is a hypersurface in a (weighted) projective space \mathbb{P} . If the surface S contains a $(-K_S)$ -polar cylinder, then $\mathbb{P} \setminus S$ is an affine Fano variety that contains an open \mathbb{A}^1 -cylinder. In particular, it is not super-rigid.*

Here we recall from [KPZ13, Definition 0.2] that an open \mathbb{A}^1 -cylinder U in a del Pezzo surface S is called a $(-K_S)$ -polar cylinder if $S \setminus U$ is the support of an effective \mathbb{Q} -divisor \mathbb{Q} -linearly equivalent to the anticanonical divisor $-K_S$ of S .

2. Log Noether–Fano inequalities and α -invariant

In this section, we will establish Theorems A and B. For this goal we generalise the classical Noether–Fano inequality for affine Fano varieties $X \setminus S_X$ in the context of purely log terminal pairs (X, S_X) .

First, we suppose that there is an affine Fano variety $Y \setminus S_Y$ and a birational map $\phi: X \dashrightarrow Y$ such that

- (1) the map ϕ is not an isomorphism in codimension 1;
- (2) the map ϕ induces an isomorphism of $X \setminus S_X$ onto $Y \setminus S_Y$.

Let \mathcal{M}_Y be a very ample complete system on Y , and let \mathcal{M}_X be the proper transform of \mathcal{M}_Y by ϕ . We consider the following resolution of indeterminacy of ϕ :

$$\begin{array}{ccc} & W & \\ f \swarrow & & \searrow g \\ X & \dashrightarrow \phi & Y \end{array}$$

where f and g are birational morphisms and W is a smooth projective variety. Let \tilde{S}_X and \tilde{S}_Y be the proper transforms of S_X and S_Y by f and g , respectively. Due to the conditions above, the divisor \tilde{S}_X is g -exceptional, the divisor \tilde{S}_Y is f -exceptional, and $f(\tilde{S}_Y)$ is contained in S_X .

By analogy with the notion of ϵ -Kawamata log terminal singularities, we say that a pair (Y, S_Y) with irreducible and reduced boundary S_Y has ϵ -purely log terminal singularities, if for every resolution of singularities $h: Z \rightarrow Y$, where Z is a smooth projective variety, the log discrepancy of every exceptional divisor of h is bigger than ϵ .

LEMMA 2.1 (Log Noether–Fano inequality I). *Suppose that the pair (Y, S_Y) is ϵ -purely log terminal for some positive rational number ϵ . Let μ and λ be non-negative rational numbers*

such that $1 - \epsilon \leq \mu \leq 1$ and

$$K_X + \mu S_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0.$$

Then the log discrepancy of \tilde{S}_Y with respect to $(X, \mu S_X + \lambda \mathcal{M}_X)$ is less than $1 - \mu$.

Proof. We have

$$K_W + \mu \tilde{S}_X + \lambda \mathcal{M}_W \sim_{\mathbb{Q}} f^*(K_X + \mu S_X + \lambda \mathcal{M}_X) + a \tilde{S}_Y + \sum a_i E_i,$$

and

$$K_W + \mu \tilde{S}_Y + \lambda \mathcal{M}_W \sim_{\mathbb{Q}} g^*(K_Y + \mu S_Y + \lambda \mathcal{M}_Y) + b \tilde{S}_X + \sum b_i E_i,$$

where each E_i is simultaneously f -exceptional and g -exceptional. The hypothesis that (Y, S_Y) is ϵ -purely log terminal immediately implies that $\mu + b$ is positive.

Suppose $a + \mu \geq 0$. Since $\mu + b$ is positive, applying the negativity lemma [Sho93, 1.1] to

$$(\mu + a) \tilde{S}_Y - g^*(K_Y + \mu S_Y + \lambda \mathcal{M}_Y) \sim_{\mathbb{Q}} (\mu + b) \tilde{S}_X + \sum (b_i - a_i) E_i$$

we see that $K_Y + \mu S_Y + \lambda \mathcal{M}_Y$ is positive.

Let λ' be the number such that $K_Y + \mu S_Y + \lambda' \mathcal{M}_Y \sim_{\mathbb{Q}} 0$. Note that $\lambda' < \lambda$. We have

$$K_W + \mu \tilde{S}_X + \lambda' \mathcal{M}_W \sim_{\mathbb{Q}} f^*(K_X + \mu S_X + \lambda' \mathcal{M}_X) + a' \tilde{S}_Y + \sum a'_i E_i,$$

and

$$K_W + \mu \tilde{S}_Y + \lambda' \mathcal{M}_W \sim_{\mathbb{Q}} g^*(K_Y + \mu S_Y + \lambda' \mathcal{M}_Y) + b \tilde{S}_X + \sum b_i E_i,$$

where $a' \geq a$ and $a'_i \geq a_i$. We then get

$$(\mu + b) \tilde{S}_X - f^*(K_X + \mu S_X + \lambda' \mathcal{M}_X) \sim_{\mathbb{Q}} (\mu + a') \tilde{S}_Y + \sum (a'_i - b_i) E_i.$$

Since $\mu + b > 0$ and $K_X + \mu S_X + \lambda' \mathcal{M}_X$ is negative, the negativity lemma implies $\mu + a' \leq 0$. This is absurd. Therefore, $a + 1 < 1 - \mu$. □

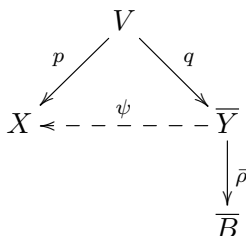
COROLLARY 2.2. *Let λ be a non-negative rational number such that*

$$K_X + S_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0.$$

Then the log discrepancy of \tilde{S}_Y with respect to the pair $(X, S_X + \lambda \mathcal{M}_X)$ is negative.

Now let Y be a quasi-projective variety, let S_Y be a prime divisor on the variety Y , and let $\rho: Y \rightarrow B$ be a dominant non-birational projective morphism with connected fibers such that the complement $Y_\eta \setminus S_{Y_\eta}$ is an affine Fano variety over the function field of the variety B . Here Y_η is a generic fiber of ρ , and $S_{Y_\eta} = S_Y|_{Y_\eta}$. We projectivize Y and B into projective varieties \bar{Y} and \bar{B} respectively. We may assume that $\rho: Y \rightarrow B$ extends to a projective morphism $\bar{\rho}: \bar{Y} \rightarrow \bar{B}$.

Suppose that $\dim(B) > 0$, and that there exists a birational map $\psi: \bar{Y} \dashrightarrow X$ that induces an embedding $Y \setminus S_Y$ into $X \setminus S_X$. As above, we consider the following resolution of indeterminacy of ψ :



where p and q are birational morphisms and V is a smooth projective variety. Let \bar{S}_Y be the closure of S_Y in \bar{Y} . Let \tilde{S}_X and \tilde{S}_Y be the proper transforms of S_X and \bar{S}_Y by p and q , respectively. Then \tilde{S}_Y is p -exceptional, because X has \mathbb{Q} -factorial singularities, and its Picard group is of rank 1. Moreover, since ψ induces an embedding $Y \setminus S_Y$ into $X \setminus S_X$, $q(\tilde{S}_X)$ is contained either in \bar{S}_Y or in $\bar{Y} \setminus Y$. Thus, the divisor \tilde{S}_X is either q -exceptional or mapped by $\bar{\rho} \circ q$ to a proper subvariety of \bar{B} . Since S_X is ample, the latter also implies that $p(\tilde{S}_Y)$ is contained in S_X .

Let H be a very ample Cartier divisor on \bar{B} . Let $\mathcal{H}_{\bar{Y}}$ be the complete linear system $|\bar{\rho}^*(H)|$, and let \mathcal{H}_X be the proper transform of $\mathcal{H}_{\bar{Y}}$ on the variety X by ψ .

LEMMA 2.3 (Log Noether–Fano inequality II). *Let μ and λ be non-negative rational numbers such that $\mu \leq 1$ and*

$$K_X + \mu S_X + \lambda \mathcal{H}_X \sim_{\mathbb{Q}} 0.$$

Then the log discrepancy of \tilde{S}_Y with respect to $(X, \mu S_X + \lambda \mathcal{H}_X)$ is less than $1 - \mu$.

Proof. As in the proof of Lemma 2.1, we have

$$K_V + \mu \tilde{S}_X + \lambda \mathcal{H}_V \sim_{\mathbb{Q}} p^*(K_X + \mu S_X + \lambda \mathcal{H}_X) + a \tilde{S}_Y + \sum a_i E_i,$$

and

$$K_V + \mu \tilde{S}_Y + \lambda \mathcal{H}_V \sim_{\mathbb{Q}} q^*(K_{\bar{Y}} + \mu \bar{S}_Y + \lambda \mathcal{H}_{\bar{Y}}) + b \tilde{S}_X + \sum b_j F_j,$$

where each E_i is p -exceptional, and each F_j is q -exceptional. Then

$$(a + \mu) \tilde{S}_Y \sim_{\mathbb{Q}} q^*(K_{\bar{Y}} + \mu \bar{S}_Y + \lambda \mathcal{H}_{\bar{Y}}) + (b + \mu) \tilde{S}_X + \sum b_j F_j - \sum a_i E_i.$$

Thus, we have

$$(a + \mu) \bar{S}_Y \sim_{\mathbb{Q}} K_{\bar{Y}} + \mu \bar{S}_Y + \lambda \mathcal{H}_{\bar{Y}} + (b + \mu) q(\tilde{S}_X) - \sum a_i q(E_i).$$

On the other hand, each p -exceptional divisor E_i is either q -exceptional or mapped by $\bar{\rho} \circ q$ to a proper subvariety of \bar{B} . This shows that

$$-(a + \mu) S_{Y_\eta} \sim_{\mathbb{Q}} -(K_{Y_\eta} + \mu S_{Y_\eta}) = -(K_{Y_\eta} + S_{Y_\eta}) + (1 - \mu) S_{Y_\eta}.$$

Since $-(K_{Y_\eta} + S_{Y_\eta})$ and S_{Y_η} are ample, we obtain $a + \mu < 0$. Therefore, the log discrepancy of \tilde{S}_Y with respect to $(X, \mu S_X + \lambda \mathcal{H}_X)$ is less than $1 - \mu$. □

COROLLARY 2.4. *Let λ be a non-negative rational number such that*

$$K_X + S_X + \lambda \mathcal{H}_X \sim_{\mathbb{Q}} 0.$$

Then the log discrepancy of \tilde{S}_Y with respect to the pair $(X, S_X + \lambda \mathcal{H}_X)$ is negative.

THEOREM 2.5 (Theorem A). *Let $X \setminus S_X$ be an affine Fano variety with completion X and boundary S_X . If for any mobile linear system \mathcal{M}_X on X and a positive rational number λ such that*

$$K_X + S_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0,$$

the pair $(X, S_X + \lambda \mathcal{M}_X)$ is log canonical along S_X , then $X \setminus S_X$ is super-rigid.

Proof. This immediately follows from Corollaries 2.2 and 2.4. □

As an application of Theorem A, we obtain the following positive solution to Conjecture 1.1 for smooth cubic surfaces without rational points.

COROLLARY 2.6. *Let S be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{k}}$ without \mathbb{k} -rational points. Then $\mathbb{P}^3_{\mathbb{k}} \setminus S$ is a super-rigid affine Fano variety.*

Proof. Indeed, otherwise Theorem A implies that there exists a mobile linear system \mathcal{M} on $\mathbb{P}^3_{\mathbb{k}}$ of degree m such that the singularities of the pair

$$\left(\mathbb{P}^3, S + \frac{1}{m} \mathcal{M} \right)$$

are not log canonical along S . Write $D = (1/m)\mathcal{M}|_S$. Then the pair (S, D) is not log canonical by Inversion of adjunction (see [Kol97, Theorem 7.5]), and D is an effective \mathbb{Q} -divisor on S such that $D \sim_{\mathbb{Q}} -K_S$.

Let $S_{\bar{\mathbb{k}}}$ be the surface obtained from S by base change to an algebraic closure $\bar{\mathbb{k}}$ of \mathbb{k} . Similarly, let $D_{\bar{\mathbb{k}}}$ be the \mathbb{Q} -divisor obtained from D by the same base change. Then $D_{\bar{\mathbb{k}}}$ is an effective \mathbb{Q} -divisor on $S_{\bar{\mathbb{k}}}$ such that

$$D_{\bar{\mathbb{k}}} \sim_{\mathbb{Q}} -K_{S_{\bar{\mathbb{k}}}}.$$

Moreover, the pair $(S_{\bar{\mathbb{k}}}, D_{\bar{\mathbb{k}}})$ is not log canonical at some point $P \in S_{\bar{\mathbb{k}}}$. Furthermore, it follows from [Che08a, Lemma 3.7] that this pair is log canonical outside of the point P . Thus, the point P must be invariant under the action of the Galois group $\text{Gal}(\bar{\mathbb{k}}/\mathbb{k})$ on $S_{\bar{\mathbb{k}}}$, and hence corresponds to a \mathbb{k} -rational point of S . This is a contradiction. \square

Example 2.7. Let X be the cubic hypersurface in $\mathbb{P}^4_{\mathbb{Q}}$ that is given by

$$5x^3 + 9y^3 + 10z^3 + 12w^3 + uf_2(x, y, z, w, u) = 0,$$

where f_2 is a general homogeneous polynomial of degree 2. Let S_X be its hyperplane section cut by $u = 0$. Then X has at most canonical singularities, the surface S_X is smooth, and $X \setminus S_X$ is an affine cubic hypersurface in $\mathbb{A}^4_{\mathbb{Q}}$. Moreover, Cassels and Guy proved in [CG66] that the surface S_X violates the Hasse principle. In particular, it does not contain any \mathbb{Q} -rational points. Then the proof of Corollary 2.6 shows verbatim that $X \setminus S_X$ is super-rigid.

THEOREM 2.8 (Theorem B). *Let $X \setminus S_X$ be an affine Fano variety with completion X and boundary S_X . If $\alpha(S_X, \text{Diff}_{S_X}(0)) \geq 1$, then the affine Fano variety $X \setminus S_X$ is super-rigid.*

Proof. Suppose that $X \setminus S_X$ is not super-rigid. Then Theorem A above implies that the variety X must carry a mobile linear system \mathcal{M}_X such that the singularities of the pair $(X, S_X + \lambda \mathcal{M}_X)$ are not log canonical along S_X for a positive rational number λ satisfying

$$K_X + S_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0.$$

However, the condition $\alpha(S_X, \text{Diff}_{S_X}(0)) \geq 1$ and Inversion of adjunction (see [Kol97, Theorem 7.5]) immediately show that this is impossible. \square

Applying Theorem B to quasi-smooth well-formed (see [Ian00]) Fano hypersurfaces in well-formed weighted projective spaces, we are able to provide many examples of super-rigid affine Fano varieties. To be precise, let X be a well-formed weighted projective space $\mathbb{P}(a_0, a_1, \dots, a_n)$, where a_0, a_1, \dots, a_n are positive integers such that $a_0 \leq a_1 \leq \dots \leq a_n$. Let S_X be a quasi-smooth well-formed hypersurface in X of degree $d < \sum_{i=0}^n a_i$. Then $X \setminus S_X$ is an affine Fano variety of dimension n , and $\text{Diff}_X(0) = 0$.

The examples below are motivated by Conjecture 1.1.

Example 2.9. Suppose that $X = \mathbb{P}^n$ and $d = n \geq 4$. Then

$$\text{Aut}(X \setminus S_X) = \text{Aut}(X, S_X),$$

because S_X is not ruled (see [IM71, Puk98, deF13]). We do not know whether the affine Fano variety $X \setminus S_X$ is super-rigid or not. However, if $n \geq 6$ and S_X is a general hypersurface of degree n , then $\alpha(S_X) = 1$ by [Puk05], so that $X \setminus S_X$ should be super-rigid by Theorem B.

Smooth cubic surfaces in \mathbb{P}^3 , smooth quartic surfaces in $\mathbb{P}(1, 1, 1, 2)$, and smooth sextic surfaces in $\mathbb{P}(1, 1, 2, 3)$ form three special families of a much larger class of quasismooth well-formed two-dimensional del Pezzo hypersurfaces in three-dimensional weighted projective spaces. They provide a lot of examples of super-rigid affine Fano threefolds.

Example 2.10. Suppose that $n = 3$, so that $X \setminus S_X$ should be an affine Fano threefold. Then it follows from [CPS10, CS13] that $\alpha(S_X) > 1$ if and only if (a_0, a_1, a_2, a_3, d) is one of the following the quintuples:

- | | | | |
|------------------------|-------------------------|-------------------------|-------------------------|
| (2, 3, 5, 9, 18), | (3, 3, 5, 5, 15), | (3, 5, 7, 11, 25), | (3, 5, 7, 14, 28), |
| (3, 5, 11, 18, 36), | (5, 14, 17, 21, 56), | (5, 19, 27, 31, 81), | (5, 19, 27, 50, 100), |
| (7, 11, 27, 37, 81), | (7, 11, 27, 44, 88), | (9, 15, 17, 20, 60), | (9, 15, 23, 23, 69), |
| (11, 29, 39, 49, 127), | (11, 49, 69, 128, 256), | (13, 23, 35, 57, 127), | (13, 35, 81, 128, 256), |
| (3, 4, 5, 10, 20), | (3, 4, 10, 15, 30), | (5, 13, 19, 22, 57), | (5, 13, 19, 35, 70), |
| (6, 9, 10, 13, 36), | (7, 8, 19, 25, 57), | (7, 8, 19, 32, 64), | (9, 12, 13, 16, 48), |
| (9, 12, 19, 19, 57), | (9, 19, 24, 31, 81), | (10, 19, 35, 43, 105), | (11, 21, 28, 47, 105), |
| (11, 25, 32, 41, 107), | (11, 25, 34, 43, 111), | (11, 43, 61, 113, 226), | (13, 18, 45, 61, 135), |
| (13, 20, 29, 47, 107), | (13, 20, 31, 49, 111), | (13, 31, 71, 113, 226), | (14, 17, 29, 41, 99), |
| (5, 7, 11, 13, 33), | (5, 7, 11, 20, 40), | (11, 21, 29, 37, 95), | (11, 37, 53, 98, 196), |
| (13, 17, 27, 41, 95), | (13, 27, 61, 98, 196), | (15, 19, 43, 74, 148), | (9, 11, 12, 17, 45), |
| (10, 13, 25, 31, 75), | (11, 17, 20, 27, 71), | (11, 17, 24, 31, 79), | (11, 31, 45, 83, 166), |
| (13, 14, 19, 29, 71), | (13, 14, 23, 33, 79), | (13, 23, 51, 83, 166), | (11, 13, 19, 25, 63), |
| (11, 25, 37, 68, 136), | (13, 19, 41, 68, 136), | (11, 19, 29, 53, 106), | (13, 15, 31, 53, 106), |
| (11, 13, 21, 38, 76). | | | |

In all these cases, the affine Fano variety $X \setminus S_X$ is super-rigid by Theorem B.

Example 2.11. With the notation and assumptions of Example 2.10, one has $\alpha(S_X) = 1$ if (a_0, a_1, a_2, a_3, d) is one of the following the quintuples:

- | | | | |
|---|-------------------|--|--------------------|
| (1, 3, 5, 8, 16), | (2, 3, 4, 7, 14), | (5, 6, 8, 9, 24), | (5, 6, 8, 15, 30), |
| (2, 2n + 1, 2n + 1, 4n + 1, 8n + 4), | | (3, 3n, 3n + 1, 3n + 1, 9n + 3), | |
| (3, 3n + 1, 3n + 2, 3n + 2, 9n + 6), | | (3, 3n + 1, 3n + 2, 6n + 1, 12n + 5), | |
| (3, 3n + 1, 6n + 1, 9n, 18n + 3), | | (3, 3n + 1, 6n + 1, 9n + 3, 18n + 6), | |
| (4, 2n + 1, 4n + 2, 6n + 1, 12n + 6), | | (4, 2n + 3, 2n + 3, 4n + 4, 8n + 12), | |
| (6, 6n + 3, 6n + 5, 6n + 5, 18n + 15), | | (6, 6n + 5, 12n + 8, 18n + 9, 36n + 24), | |
| (6, 6n + 5, 12n + 8, 18n + 15, 36n + 30), | | (8, 4n + 5, 4n + 7, 4n + 9, 12n + 23), | |
| (9, 3n + 8, 3n + 11, 6n + 13, 12n + 35), | | | |

where n is any positive integer. Moreover, we also have $\alpha(S_X) = 1$ if S_X is a general hypersurface of degree d in $\mathbb{P}(a_0, a_1, a_2, a_3)$ and (a_0, a_1, a_2, a_3, d) is one of the following quintuples: $(1, 1, 2, 3, 6)$, $(1, 2, 3, 5, 10)$, $(1, 3, 5, 7, 15)$, $(2, 3, 4, 5, 12)$. In all these cases, the affine Fano variety $X \setminus S_X$ is super-rigid by Theorem B.

Smooth quartic threefolds in \mathbb{P}^4 form one family among the famous 95 families of Reid and Iano-Fletcher in [Ian00]. They also provide many examples of super-rigid affine Fano fourfolds through Theorem B.

Example 2.12. For the case where $n = 4$, $d = \sum_{i=0}^4 a_i - 1$, Iano-Fletcher verified that there are exactly 95 quintuples $(a_0, a_1, a_2, a_3, a_4)$ that define the hypersurface S_X with only terminal singularities and listed such quintuples in [Ian00]. Moreover, it follows from [CPR00, CP17] that

$$\text{Aut}(X \setminus S_X) = \text{Aut}(X, S_X).$$

Furthermore, if $-K_{S_X}^3 \leq 1$ and the hypersurface S_X is general, then $\alpha(S_X) = 1$ by [Che08b, Che09], so that the corresponding affine Fano fourfold $X \setminus S_X$ should be super-rigid by Theorem B.

Meanwhile, Johnson and Kollár completely described the quintuples $(a_0, a_1, a_2, a_3, a_4)$ that define quasi-smooth hypersurface S_X of degree $\sum_{i=0}^4 a_i - 1$ in $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ in [JK01]. They also show that $\alpha(S_X) \geq 1$ in many of these cases. In such cases, the corresponding affine Fano variety $X \setminus S_X$ is super-rigid by Theorem B.

3. Global finite quotients of affine spaces

As seen in the previous section, it is hardly expected that the α -invariant plays a role of a criterion (a sufficient and necessary condition) for an affine Fano variety to be super-rigid. However, it is able to serve as a criterion for affine Fano varieties of a certain type. In this section, we study affine Fano varieties that are quotients of \mathbb{A}^n by actions of finite subgroups of $\text{GL}_n(\mathbb{C})$. We verify that the α -invariant completely determines super-rigidity of such affine Fano varieties of dimensions up to 4.

Let G be a finite subgroup in $\text{GL}_n(\mathbb{C})$, where $n \geq 2$. Put $\mathbb{V} = \mathbb{A}^n$ and consider \mathbb{V} as a linear representation of the group G . In addition, let us identify $\text{GL}_n(\mathbb{C})$ with a subgroup of $\text{PGL}_{n+1}(\mathbb{C}) = \text{Aut}(\mathbb{P}^n)$ by means of the natural embedding $\text{GL}_n(\mathbb{C}) \hookrightarrow \text{GL}_{n+1}(\mathbb{C})$ and the quotient homomorphism $\text{GL}_{n+1}(\mathbb{C}) \rightarrow \text{PGL}_{n+1}(\mathbb{C})$. In this way, we thus consider G as a subgroup of $\text{Aut}(\mathbb{P}^n)$.

Let H be the $\text{GL}_n(\mathbb{C})$ -invariant hyperplane $\mathbb{P}^n \setminus \mathbb{V}$. The action of G on H is not necessarily faithful. Denote by \overline{G} the image of G in $\text{Aut}(H) = \text{PGL}_n(\mathbb{C})$, and denote by Z the kernel of the group homomorphism $G \rightarrow \overline{G}$. Then Z is a cyclic group of order $m \geq 1$, and G is a central extension of the group \overline{G} by Z . Letting

$$X = \mathbb{P}^n / G,$$

we can identify the quotient $S_X = H / \overline{G}$ with a prime divisor in X , so that by construction

$$X \setminus S_X \cong \mathbb{A}^n / G. \tag{3.1}$$

Recall that an element $g \in G \subset \text{GL}_n(\mathbb{C})$ is called a quasi-reflection if there is a hyperplane in $H \cong \mathbb{P}^{n-1}$ that is pointwise fixed by the image of g in \overline{G} . If G is generated by quasi-reflections, then by virtue of the Chevalley–Shephard–Todd theorem \mathbb{A}^n / G is isomorphic to \mathbb{A}^n , hence is in particular an affine Fano variety. More generally, we have the following.

LEMMA 3.2. *The quotient (3.1) is an affine Fano variety with completion $X = \mathbb{P}^n/G$ and boundary $S_X = H/\overline{G}$.*

Proof. Since the subgroup of G generated by quasi-reflections is normal, we may assume that G does not contain non-trivial quasi-reflection. Note that G when considered as a subgroup in $\text{GL}_{n+1}(\mathbb{C})$ may contain quasi-reflections. These are just elements of Z that are different from identity. To take these quasi-reflexions into account, we consider the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{q} & \mathbb{P}(1^n, m) \\ & \searrow \pi & \swarrow \bar{\pi} \\ & X & \end{array}$$

where π is the quotient map by the group G , the morphism q is the quotient map by the group Z , and $\bar{\pi}$ is the quotient map by the group \overline{G} . By construction, the finite morphism q is branched over $q(H)$, which is a smooth hypersurface in $\mathbb{P}(1^n, m)$ of degree n , that does not contain the singular point of the weighted projective space $\mathbb{P}(1^n, m)$. Moreover, the finite morphism $\bar{\pi}$ is unramified in codimension one. This implies that the divisor

$$-(K_X + S_X)$$

is ample, and (X, S_X) has purely log terminal singularities. Thus, the quotient (3.1) is an affine Fano variety as desired. □

In view of the above discussion, to address the question whether the quotient (3.1) is a super-rigid affine Fano variety, we may and will assume in the sequel that G is small, i.e., does not contain any non-trivial quasi-reflection. Let $\alpha_{\overline{G}}(H)$ be the number defined as

$$\sup \left\{ \lambda \in \mathbb{Q} \mid \begin{array}{l} \text{the pair } (H, \lambda D) \text{ is log canonical for every effective} \\ \overline{G}\text{-invariant } \mathbb{Q}\text{-divisor } D \text{ on } H \text{ such that } D \sim_{\mathbb{Q}} -K_H \end{array} \right\}.$$

Then $\alpha_{\overline{G}}(H)$ is the \overline{G} -equivariant α -invariant of the projective space H . Moreover, it follows from the proof of [CS11, Theorem 3.16] that

$$\alpha(S_X, \text{Diff}_{S_X}(0)) = \alpha_{\overline{G}}(H). \tag{3.3}$$

Furthermore, one has $\alpha_{\overline{G}}(H) \geq 1$ if and only if the quotient singularity \mathbb{A}^n/G is weakly-exceptional in the notation of [CS11, CS14, Sak12, Sak14]. In particular, if $\alpha_{\overline{G}}(H) \geq 1$, then \mathbb{V} must be an irreducible representation of the group G by [CS11, Theorem 3.18]. In this case, the subgroup $\overline{G} \subset \text{Aut}(H)$ is usually called transitive.

LEMMA 3.4. *One has $\alpha_{\overline{G}}(H) \geq 1 \Rightarrow (3.1)$ is super-rigid $\Rightarrow \overline{G}$ is transitive.*

Proof. The first implication follows from Theorem B and (3.3). For the second one, suppose on the contrary that \mathbb{V} splits as the direct sum of two non-trivial representations

$$\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2.$$

Then the projection $\text{pr}_1: \mathbb{V} \rightarrow \mathbb{V}_1$ descends to a fibration $\rho: X \setminus S_X \rightarrow \mathbb{V}_1/G$, whose general fibers are isomorphic to \mathbb{V}_2/G' , where $G' \subset G$ denotes the stabilizer of a general fiber of pr_1 . Since \mathbb{V}_2/G' is an affine Fano variety by Lemma 3.2, we see that the generic fiber of ρ is an affine Fano variety. In other words, $\rho: X \setminus S_X \rightarrow \mathbb{V}_1/G$ is a relative affine Fano variety in contradiction to the super-rigidity hypothesis. □

COROLLARY 3.5. *Suppose that for every irreducible \overline{G} -invariant subvariety $Z \subset H$, there exists no hypersurface in H of degree $\dim(Z) + 1$ that contains Z . Then $X \setminus S_X$ is super-rigid.*

Proof. Indeed, the conditions imply that $\alpha_{\overline{G}}(H) \geq 1$ by virtue of [CS14, Theorem 1.12]. □

One can show that super-rigid affine Fano quotients (3.1) exist in all dimensions.

Example 3.6. Suppose that n is an odd prime. Let G be a subgroup in $SL_n(\mathbb{C})$ that is isomorphic to the Heisenberg group of order n^3 . Then $\alpha_{\overline{G}}(H) \geq 1$ by [CS14, Theorem 1.15], so that $X \setminus S_X$ should be super-rigid by Lemma 3.4.

We now give a complete classification of super-rigid affine Fano varieties (3.1) of dimensions at most 4.

THEOREM 3.7. *Suppose that $n = 2$. Then the following conditions are equivalent:*

- (1) *the affine Fano variety (3.1) is super-rigid;*
- (2) *the inequality $\alpha_{\overline{G}}(H) \geq 1$ holds;*
- (3) *the line H does not have G -fixed points;*
- (4) *the representation \mathbb{V} is irreducible;*
- (5) *the group G is not abelian;*
- (6) *the group G is not cyclic.*

Proof. We have $S_X \cong H \cong \mathbb{P}^1$. For every point $P \in S_X$, denote by n_P the order of the stabilizer in \overline{G} of any point on H that is mapped to P by the quotient map $\pi: \mathbb{P}^2 \rightarrow X$. Then

$$\text{Diff}_{S_X}(0) = \sum_{P \in S_X} \frac{n_P - 1}{n_P} P.$$

Note that $\text{Diff}_X(0) \neq 0$ provided that \overline{G} is not trivial. Let Q be the point in S_X with the largest n_Q . Then it follows from [CS11, Example 3.3] that

$$\alpha(S_X, \text{Diff}_{S_X}(0)) = \frac{1 - (n_Q - 1)/n_Q}{2 - \sum_{P \in S_X} (n_P - 1)/n_P} = \begin{cases} 6 & \text{if } \overline{G} \cong \mathfrak{A}_5, \\ 3 & \text{if } \overline{G} \cong \mathfrak{S}_4, \\ 2 & \text{if } \overline{G} \cong \mathfrak{A}_4, \\ 1 & \text{if } \overline{G} \text{ is a dihedral group,} \\ 1 & \text{if } \overline{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \\ \frac{1}{2} & \text{if } \overline{G} \text{ is cyclic.} \end{cases}$$

Furthermore, since we assumed that G is small, it is abelian if and only if it is cyclic. Finally, observe that $\alpha_{\overline{G}}(H)$ is the length of the smallest \overline{G} -orbit in H divided by 2, so that

$$\alpha(S_X, \text{Diff}_{S_X}(0)) = \alpha_{\overline{G}}(H) = \frac{|\overline{G}|}{2n_Q}$$

by virtue of (3.3). Now the conclusion follows from Lemma 3.4 together with the fact that the representation \mathbb{V} splits when G is cyclic. □

Using Theorem 3.7 together with the results of Miyanishi and Sugie on \mathbb{Q} -homology planes with quotient singularities [MS91], we obtain the following classification of all two-dimensional super-rigid affine Fano varieties.

THEOREM 3.8. *Let $S \setminus C$ be an affine Fano variety of dimension 2 with completion S and boundary C . Then $S \setminus C$ is super-rigid if and only if the pair (S, C) is isomorphic to a pair*

$$(\mathbb{P}^2/G, H/\overline{G})$$

for some non-cyclic small finite subgroup $G \subset \mathrm{GL}_2(\mathbb{C})$.

Proof. Since (S, C) has purely log terminal singularities, $-(K_S + C)$ is ample and the Picard rank of S is equal to 1, it follows that $S \setminus C$ is a logarithmic \mathbb{Q} -homology plane with smooth locus of negative Kodaira dimension. By [MS91, Theorems 2.7 and 2.8], the surface $S \setminus C$ either contains an open \mathbb{A}^1 -cylinder, which is impossible as it is super-rigid by the hypothesis, or is isomorphic to a quotient \mathbb{A}^2/G for a small finite subgroup G of $\mathrm{GL}_2(\mathbb{C})$.

Since $S \setminus C$ is super-rigid, the group G is not cyclic by Theorem 3.7.

The quotient space \mathbb{P}^2/G is the natural projective completion of \mathbb{A}^2/G with boundary H/\overline{G} . The isomorphism $S \setminus C \cong \mathbb{A}^2/G$ extends to a birational map $S \dashrightarrow \mathbb{P}^2/G$, which must be an isomorphism of pairs $(S, C) \cong (\mathbb{P}^2/G, H/\overline{G})$ by the definition of super-rigidity. \square

A consequence of Theorem 3.7 is that for $n = 2$ all three conditions of Lemma 3.4 are actually equivalent. This is no longer true for $n \geq 3$ as illustrated by the following example.

Example 3.9. Suppose that $n = 3$. Let $G = \mathfrak{A}_5$ and let \mathbb{V} be an irreducible three-dimensional representation of G . Then $\overline{G} \cong G$, and the center Z is trivial. Moreover, there exists a \overline{G} -invariant smooth conic C in H . Let $\pi: W \rightarrow \mathbb{P}^3$ be the blow up of the conic C with exceptional divisor E , and let \tilde{H} be the proper transform of the plane H on the threefold W . Then there exists a G -equivariant commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\eta} & Q \\ & \searrow \pi & \swarrow \psi \\ & & \mathbb{P}^3 \end{array}$$

where Q is a smooth quadric threefold in \mathbb{P}^4 , the morphism η is the contraction of the surface \tilde{H} to a smooth point of Q , and ψ is a linear projection from this point. Then $(Q, \eta(E))$ is purely log terminal and $-(K_Q + \eta(E))$ is ample, so that $(Q, \eta(E))$ is an affine Fano variety. By construction, we have $Q \setminus \eta(E) \cong \mathbb{P}^3 \setminus H$. Let $Y = Q/G$ and $S_Y = \eta(E)/G$. Then (Y, S_Y) is an affine Fano variety and $Y \setminus S_Y \cong X \setminus S_X$, so that (X, S_X) could not be super-rigid.

In fact, for $n = 3$ and 4, we can obtain criteria for $X \setminus S_X$ to be super-rigid which are similar to Theorem 3.7. For example, we have the following.

THEOREM 3.10. *Suppose that $n = 3$. Then the following conditions are equivalent:*

- (1) *the affine Fano variety $X \setminus S_X$ is super-rigid;*
- (2) *one has $\alpha_{\overline{G}}(H) \geq 1$;*
- (3) *the plane H contains neither \overline{G} -invariant lines nor \overline{G} -invariant conics.*

Proof. By [CS11, Theorem 3.23], the conditions (2) and (3) are equivalent. By Theorem B and (3.3), the condition (2) implies (1). Thus, it is enough to show that (1) implies (3). If H contains a \overline{G} -invariant line, then $X \setminus S_X$ is not super-rigid by Lemma 3.4. Similarly, if H contains a \overline{G} -invariant conic, then the construction presented in Example 3.9 shows that $X \setminus S_X$ is not super-rigid. \square

Similarly, for $n = 4$ we have the following.

THEOREM 3.11. *Suppose that $n = 4$. Then the following conditions are equivalent:*

- (1) *the affine Fano variety $X \setminus S_X$ is super-rigid;*
- (2) *one has $\alpha_{\overline{G}}(H) \geq 1$;*
- (3) *the group \overline{G} is transitive, and the hyperplane H does not contain \overline{G} -invariant quadrics, \overline{G} -invariant cubic surfaces or \overline{G} -invariant twisted cubic curves.*

Proof. By [CS11, Theorem 4.3], Theorem B and (3.3), it is enough to show that (1) implies (3). If the group \overline{G} is not transitive, then $X \setminus S_X$ is not super-rigid by Lemma 3.4. Thus, we may assume that \overline{G} is transitive. Let us show that $X \setminus S_X$ is not super-rigid in the case where the hyperplane H contains one of the following \overline{G} -invariant subvarieties: a quadric surface, a cubic surface, or a twisted cubic curve.

Suppose first that the hyperplane H contains a \overline{G} -invariant surface S_d of degree $d = 2$ or 3 . Because \overline{G} is transitive, this surface must be smooth. This is obvious in the case when $d = 2$ while in the case when $d = 3$, the conclusion follows from the classification of singular cubic surfaces.

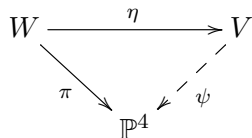
We may assume that H is given by $u = 0$, and S_d is defined by

$$\begin{cases} f_d(x, y, z, w) = 0, \\ u = 0, \end{cases}$$

where $f_d(x, y, z, w)$ is a homogeneous polynomial of degree d , and x, y, z, w, u are homogeneous coordinates on \mathbb{P}^4 . Let V be a hypersurface in $\mathbb{P}(1^5, d - 1)$ given by

$$ut = f_d(x, y, z, w),$$

where x, y, z, w, u, t are quasi-homogeneous coordinates on $\mathbb{P}(1^5, d - 1)$ such that t is a coordinate of weight $d - 1$. Then there exists a G -equivariant commutative diagram



where π is the blow up of the surface S_d with exceptional divisor E , the morphism η is the contraction of the proper transform of the hyperplane H to the point $[0 : 0 : 0 : 0 : 0 : 1]$, and ψ is the linear projection from this point. Then $\eta(E)$ is cut out on V by $u = 0$. Let $Y = V/G$ and $S_Y = \eta(E)/G$. Then (Y, S_Y) is an affine Fano variety and $Y \setminus S_Y \cong X \setminus S_X$. In particular, we see that $X \setminus S_X$ is not super-rigid.

Next suppose that H contains a \overline{G} -invariant twisted cubic curve C . Then the group \overline{G} acts faithfully on the curve C . Using the classification of finite subgroups in $\text{PGL}_2(\mathbb{C})$, we see that \overline{G} is isomorphic to one of the three groups $\mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5$, because we assumed that \overline{G} is transitive.

Let $\sigma: U \rightarrow \mathbb{P}^4$ be the blow up of the curve C with exceptional divisor F , and denote by \tilde{H} the proper transform of the hyperplane H on the fourfold U . Then there exists a G -equivariant morphism $\nu: \tilde{H} \rightarrow \mathbb{P}^2$ that is a \mathbb{P}^1 -bundle (see [SW90, Application 1]). Its fibers are the proper transforms of the secants of the curve C . It follows from [PZ16, Proposition 3.9] that there exists a G -equivariant commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\nu} & W_5 \\ & \searrow \sigma & \swarrow \phi \\ & \mathbb{P}^4 & \end{array}$$

where W_5 is a smooth del Pezzo fourfold in \mathbb{P}^7 of degree 5, the morphism ν is a contraction of the proper transform of \tilde{H} to a plane in W_5 such that the restriction $\nu|_{\tilde{H}}: \tilde{H} \rightarrow \nu(\tilde{H})$ is the \mathbb{P}^1 -bundle ν , and ϕ is the linear projection from the plane $\nu(\tilde{H})$. Then $\nu(F)$ is a singular hyperplane section of the fourfold W_5 such that the pair $(W_5, \nu(F))$ has purely log terminal singularities. As above, we let $M = W_5/G$ and $S_M = \nu(F)/G$. Then (M, S_M) is an affine Fano variety and $M \setminus S_M \cong X \setminus S_X$, so that $X \setminus S_X$ could not be super-rigid. \square

4. Complements to del Pezzo surfaces

Let S be a del Pezzo surface of anticanonical degree at most 3, i.e., $K_S^2 \leq 3$, with at worst Du Val singularities. If the anticanonical degree is 3, then S is a cubic surface in \mathbb{P}^3 . Similarly, if the anticanonical degree is 2, then S is a quartic surface in $\mathbb{P}(1, 1, 1, 2)$. Finally, if the anticanonical degree is 1, then S is a sextic surface in $\mathbb{P}(1, 1, 2, 3)$.

In this section, we study automorphism groups of affine Fano varieties whose boundary is S and whose completion is the corresponding ambient weighted projective space. Using the result obtained in this section we are able to yield many examples of non-trivial non-super-rigid affine Fano threefolds.

Let \mathbb{P} be the ambient spaces \mathbb{P}^3 , $\mathbb{P}(1, 1, 1, 2)$ and $\mathbb{P}(1, 1, 2, 3)$ in the cases of anticanonical degrees 3, 2 and 1, respectively. We have

$$\mathbb{P} = \text{Proj}(\mathbb{C}[x, y, z, w]),$$

where the variables x, y, z and w are of weights $(1, 1, 1, 1)$, $(1, 1, 1, 2)$ and $(1, 1, 2, 3)$ according to anticanonical degrees 3, 2 and 1. Denote by d the degree of the surface S as a hypersurface in \mathbb{P} . Then d is equal to 3, 4 and 6 according to anticanonical degrees 3, 2 and 1, respectively.

The following theorem summarizes the current knowledge towards the structure of the automorphism groups of the complements of smooth del Pezzo surfaces in \mathbb{P} .

THEOREM 4.1. *Suppose that S is smooth. If its anticanonical degree is 1, then*

$$\text{Aut}(\mathbb{P} \setminus S) = \text{Aut}(\mathbb{P}, S).$$

In particular, $\text{Aut}(\mathbb{P} \setminus S)$ is a finite group. If its anticanonical degree is either 2 or 3, then $\text{Aut}(\mathbb{P} \setminus S)$ does not contain non-trivial connected algebraic groups.

Proof. We follow the same strategy as [DK15, § 2.2], which consists in shifting the question to a suitable finite étale cover of $\mathbb{P} \setminus S$. Namely, let $\pi: V \rightarrow \mathbb{P}$ be a cyclic Galois cover of degree d branched along S and étale elsewhere. Then V is a smooth Fano threefold of index 2, isomorphic

to a hypersurfaces of degree d in \mathbb{P}^4 , $\mathbb{P}(1, 1, 1, 1, 2)$ and $\mathbb{P}(1, 1, 1, 2, 3)$ in the cases $K_S^2 = 3, 2$ and 1 , respectively. Furthermore, the ramification divisor of π coincides with a hyperplane section H of V . We then have a split exact sequence of groups

$$0 \rightarrow \text{Aut}(V \setminus H, \pi) \rightarrow \text{Aut}(V \setminus H) \rightarrow \text{Aut}(\mathbb{P} \setminus S) \rightarrow 0,$$

where $\text{Aut}(V \setminus H, \pi) \cong \mathbb{Z}/d\mathbb{Z}$ denotes the group of the deck transformations of the induced étale Galois cover $\pi|_{V \setminus H}: V \setminus H \rightarrow \mathbb{P} \setminus S$. The surjectivity of the right hand side homomorphism follows from the fact that

$$\pi_* \mathcal{O}_{V \setminus H} \cong \bigoplus_{i=0}^{d-1} (\omega_{\mathbb{P} \setminus S}^{-1})^{\otimes i},$$

where $\omega_{\mathbb{P} \setminus S}^{-1} = \omega_{\mathbb{P}}^{-1}|_{\mathbb{P} \setminus S} \cong \mathcal{O}_{\mathbb{P}}(d+1)|_{\mathbb{P} \setminus S} \cong \mathcal{O}_{\mathbb{P}}(1)|_{\mathbb{P} \setminus S}$ denotes the anticanonical sheaf of the variety $\mathbb{P} \setminus S$.

If $K_S^2 = 1$, then $\text{Bir}(V) = \text{Aut}(V)$ by [Gri03, Gri04], so that $\text{Aut}(V \setminus H) = \text{Aut}(V, H)$. This implies in turn that $\text{Aut}(\mathbb{P} \setminus S) = \text{Aut}(\mathbb{P}, S)$, which is a finite group.

If $K_S^2 = 2$ or $K_S^2 = 3$, then V is not rational by [Voi88, Corollaire 4.8] and [CG72], respectively. Now suppose that $\text{Aut}(\mathbb{P} \setminus S)$ contains a non-trivial connected algebraic subgroup G . First note that G is necessarily an affine algebraic group. Indeed, otherwise by Chevalley’s theorem, there would exist a maximal proper normal linear algebraic subgroup G' of G such that G/G' is abelian variety, and so the affine variety $\mathbb{P} \setminus S$ would inherit a non-trivial action of a proper connected algebraic group, which is absurd. Since it is affine and connected, G thus contains either \mathbb{G}_m or \mathbb{G}_a , which implies in turn that $V \setminus H$ admits a non-trivial action of either \mathbb{G}_m or \mathbb{G}_a . By virtue of Rosenlicht’s theorem [Ros56], it would follow that V is birational to $Z \times \mathbb{P}^1$ for some rational surface Z . But then, the threefold V itself would be rational, which is a contradiction. \square

A consequence of Theorem 4.1 is the following generalization of [DK15, Proposition 10].

COROLLARY 4.2. *If S is smooth, then $\mathbb{P} \setminus S$ does not contain open \mathbb{A}^1 -cylinders.*

Proof. Observe that the divisor class group of $\mathbb{P} \setminus S$ is isomorphic to the finite group $\mathbb{Z}/d\mathbb{Z}$. Thus, it follows from [DK15, Proposition 2] that for every open \mathbb{A}^1 -cylinder $B \times \mathbb{A}^1$ in $\mathbb{P} \setminus S$, there exists an action of \mathbb{G}_a on $\mathbb{P} \setminus S$ whose general orbits coincide with the general fibers of the projection $\text{pr}_B: B \times \mathbb{A}^1 \rightarrow B$. But on the other hand, the group $\text{Aut}(\mathbb{P} \setminus S)$ does not contain any non-trivial connected algebraic group by Theorem 4.1. \square

It is known that a smooth del Pezzo surface of anticanonical degree at most 3 never contains any $(-K_S)$ -polar cylinder. For the singular case, we obtain a complete description for $(-K_S)$ -polar cylinders from [CPW16a, Theorem 1.5].

THEOREM 4.3. *The surface S does not contain any $(-K_S)$ -polar cylinder if and only if one of the following conditions is satisfied:*

- (1) *its anticanonical degree is 1 and it has only singular points of types A_1, A_2, A_3, D_4 if any;*
- (2) *its anticanonical degree is 2 and it allows only singular points of type A_1 if any;*
- (3) *its anticanonical degree is 3 and it allows no singular point.*

Existence of open \mathbb{A}^1 -cylinders in the complements of singular normal cubic surfaces with singularities strictly worse than A_2 was first established in [KPZ11, Proposition 3.7]. The first example of a single nodal cubic surface whose complement contains an open \mathbb{A}^1 -cylinder was

constructed later on by Lamy (unpublished). Combined with the above results on $(-K_S)$ -polar cylinders of singular del Pezzo surfaces and Example 1.7, this leads to anticipate that the complement of a del Pezzo surface with a $(-K_S)$ -polar cylinder contains an open \mathbb{A}^1 -cylinder. Ideas evolving from the proof of [CPW16a, Theorem 4.1] and the study of open ‘vertical’ \mathbb{A}^1 -cylinders in del Pezzo fibrations [DK18] turn out to confirm this expectation, as follows.

THEOREM 4.4 (Theorem C). *If the surface S contains a $(-K_S)$ -polar cylinder, then the affine Fano variety $\mathbb{P} \setminus S$ contains an open \mathbb{A}^1 -cylinder.*

Before we proceed the proof, let us first prepare setups for the proof. Due to Theorem 4.3 above the surface S contains a singular point P . Furthermore, we may assume that the singular point P is not of type A_1 if the anticanonical degree of S is 2; it is not of types A_1, A_2, A_3, D_4 if the anticanonical degree is 1. By suitable coordinate changes, we may assume that the point P is located at $[1 : 0 : 0 : 0]$. Under such conditions, we immediately observe the following.

LEMMA 4.5. *Under the condition above, the surface S is defined in \mathbb{P} by one quasi-homogenous equation of the following types.*

- Case $K_S^2 = 3$.

$$x f_2(y, z, w) + f_3(y, z, w) = 0, \tag{4.6}$$

where $f_2(y, z, w)$ and $f_3(y, z, w)$ are homogenous polynomials of degrees 2 and 3.

- Case $K_S^2 = 2$.

$$w^2 + x(ayw + f_3(y, z)) + f_4(y, z) = 0, \tag{4.7}$$

where $f_3(y, z)$ and $f_4(y, z)$ are homogenous polynomials of degrees 3 and 4, respectively, and a is a constant.

- Case $K_S^2 = 1$.

$$w^2 + x(ay^2w + f_5(y, z)) + f_6(y, z) = 0 \tag{4.8}$$

or

$$w^2 + x(zw + f_5(y, z)) + f_6(y, z) = 0, \tag{4.9}$$

where $f_5(y, z)$ and $f_6(y, z)$ are quasi-homogenous polynomials of degrees 5 and 6, respectively, and a is a constant.

Proof. We omit the proof since it is easy. □

Let $\pi : S \dashrightarrow \Pi$ be the projection from the point P to the hyperplane Π defined by $x = 0$ in \mathbb{P} , i.e., $\pi([x : y : z : w]) = [y : z : w]$. The hyperplane Π is isomorphic to $\mathbb{P}^2, \mathbb{P}(1, 1, 2), \mathbb{P}(1, 2, 3)$ according to the anticanonical degrees, 3, 2, 1, respectively. We denote by $g(y, z, w)$ the coefficient quasi-homogenous polynomial of x in each of the quasi-homogenous equations of (4.6), (4.7), (4.8) and (4.9), i.e., for the case $K_S^2 = 1$

$$g(y, z, w) = ay^2w \text{ (or } zw) + f_5(y, z),$$

for the case $K_S^2 = 2$

$$g(y, z, w) = ayw + f_3(y, z),$$

and for the case $K_S^2 = 3$

$$g(y, z, w) = f_2(y, z, w).$$

Let D be the divisor on S cut by the equation $g(y, z, w) = 0$. In the case of $K_S^2 = 3$, the divisor D consists of the lines in S that pass through P . There are at most six such lines and they are defined by the system of homogenous equations

$$\begin{cases} g(y, z, w) = 0, \\ f_3(y, z, w) = 0, \end{cases}$$

in \mathbb{P}^3 . In the case of $K_S^2 = 2$, D consists of at most six curves passing through the point P . They are defined by the system of quasi-homogeneous equations

$$\begin{cases} g(y, z, w) = 0, \\ w^2 + f_4(y, z) = 0, \end{cases}$$

in $\mathbb{P}(1, 1, 1, 2)$. Finally, in the case of $K_S^2 = 1$, it consists of at most five curves passing through the point P , which are defined by the system of quasi-homogeneous equations

$$\begin{cases} g(y, z, w) = 0, \\ w^2 + f_6(y, z) = 0, \end{cases}$$

in $\mathbb{P}(1, 1, 2, 3)$. In each case, the number of curves in D is completely determined by the number of points determined by the system of quasi-homogeneous equations on Π . Denote these curves by L_1, \dots, L_r in each case. The map π contracts each curve L_i to a point on Π .

Now we are ready to prove Theorem C.

Proof of Theorem C. Lemma 4.5 immediately implies that the projection π is a birational map. Moreover, it induces an isomorphism

$$\tilde{\pi} : S \setminus (L_1 \cup \dots \cup L_r) \rightarrow \text{Im}(\tilde{\pi}) \subset \Pi.$$

Let \mathcal{C} be the curve on Π defined by

$$g(y, z, w) = 0.$$

Note that this can be reducible or non-reduced.

CLAIM 1. If $K_S^2 = 3$, then $\text{Im}(\tilde{\pi}) = \Pi \setminus \mathcal{C}$ and there is a hyperplane section H of S such that $S \setminus (H \cup L_1 \cup \dots \cup L_r)$ is an \mathbb{A}^1 -cylinder.

Let $\varphi : \bar{S} \rightarrow S$ be the blow up at the point P . Then there exists a commutative diagram

$$\begin{array}{ccc} & \bar{S} & \\ \varphi \swarrow & & \searrow \phi \\ S & \overset{\pi}{\dashrightarrow} & \Pi \end{array}$$

where ϕ is the birational morphism that contracts exactly the proper transforms of the lines L_1, \dots, L_r . Let E be the exceptional divisor of the blow up φ . Then the image of the curve E by ϕ in Π is the conic curve \mathcal{C} . Then \mathcal{C} contains all the points $\pi(L_i)$. If P is an ordinary double point of the surface S , then \mathcal{C} is a smooth conic. Moreover, if P is a singular point of the surface S of type A_n for $n \geq 2$, then \mathcal{C} consists of two distinct lines. Finally, if P is either of type D_n or of type E_6 , then \mathcal{C} is a double line.

If \mathcal{C} is smooth, let ℓ be a general line in Π that is tangent to \mathcal{C} . If \mathcal{C} is singular, let ℓ be a general line in Π that passes through a singular point of the conic \mathcal{C} . By a suitable coordinate

change, we may assume that ℓ is defined by the equation $y = 0$ on Π . Let H be the divisor on S cut by the equation $y = 0$ in \mathbb{P}^3 .

Then

$$S \setminus (H \cup L_1 \cup \dots \cup L_r) \cong \Pi \setminus (\mathcal{C} \cup \ell) \cong (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1,$$

so that $S \setminus (H \cup L_1 \cup \dots \cup L_r)$ is an \mathbb{A}^1 -cylinder.

For the rest of the cases we also denote by ℓ the curve defined by $y = 0$ on Π . In addition let H be the divisor on S cut by $y = 0$ in \mathbb{P} .

CLAIM 2. If $K_S^2 = 2$ or if $K_S^2 = 1$ and the surface S is defined by a quasi-homogenous equation of type (4.8), then $S \setminus (H \cup L_1 \cup \dots \cup L_r)$ is an \mathbb{A}^1 -cylinder.

As in the case of $K_S^2 = 3$, the isomorphism $\tilde{\pi}$ maps $S \setminus (H \cup L_1 \cup \dots \cup L_r)$ onto $\Pi \setminus (\mathcal{C} \cup \ell)$. Meanwhile, the projection π maps $S \setminus H$ onto $\Pi \setminus \ell \cong \mathbb{A}^2$. The affine variety $S \setminus (H \cup L_1 \cup \dots \cup L_r)$ is therefore isomorphic to the complement of the curve defined by

$$aw + f_3(1, z) = 0$$

for type (4.7) (respectively $aw + f_5(1, z) = 0$ for type (4.8)) in $\Pi \setminus \ell \cong \mathbb{A}^2$. This immediately implies the claim.

CLAIM 3. If $K_S^2 = 1$ and the surface S is defined by a quasi-homogenous equation of type (4.9), let H_z be the hyperplane section of S cut by the equation $z = 0$. Then $S \setminus (H_z \cup L_1 \cup \dots \cup L_r)$ is an \mathbb{A}^1 -cylinder. In particular, $S \setminus H_z$ contains an open \mathbb{A}^1 -cylinder.

In this case, the isomorphism $\tilde{\pi}$ maps $S \setminus (H_z \cup L_1 \cup \dots \cup L_r)$ onto $\Pi \setminus (\mathcal{C} \cup \ell_z)$, where ℓ_z is the curve on Π defined by $z = 0$. The projection π maps $S \setminus H_z$ onto $\Pi \setminus \ell_z$. The affine variety $\Pi \setminus \ell_z$ is isomorphic to the quotient space \mathbb{A}^2 / μ_2 , where the μ_2 -action is given by $(y, w) \mapsto (-y, -w)$. Notice that we abuse our notations y and w . The affine open subset $\Pi \setminus (\mathcal{C} \cup \ell_z)$ is an \mathbb{A}^1 -cylinder. To see this, notice that $\Pi \setminus (\mathcal{C} \cup \ell_z)$ is the complement of the image of the curve defined by

$$w + f_5(y, 1) = 0$$

in $\Pi \setminus \ell_z \cong \mathbb{A}^2 / \mu_2$. Since $f_5(y, 1)$ is an odd polynomial in y , the isomorphism $\psi : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ defined by $(y_1, w_1) = \psi(y, w) = (y, w + f_5(y, 1))$ is μ_2 -equivariant for the action $(y_1, w_1) \mapsto (-y_1, -w_1)$ so that we have a μ_2 -equivariant diagram

$$\begin{array}{ccc} \mathbb{A}^2 & \xrightarrow{\psi} & \mathbb{A}^2 \\ & \searrow & \swarrow \\ & \mathbb{A}^2 / \mu_2 & \end{array}$$

where the morphisms from \mathbb{A}^2 to \mathbb{A}^2 / μ_2 are the respective quotient maps. This shows that $\Pi \setminus (\mathcal{C} \cup \ell_z)$ is isomorphic to the complement in \mathbb{A}^2 / μ_2 of the image of the curve defined by $w_1 = 0$. Letting $s = w_1^2$, $t = y_1^2$ and $u = w_1 y_1$, \mathbb{A}^2 / μ_2 is isomorphic the affine variety defined by the equation $st = u^2$ in \mathbb{A}^3 . The image in \mathbb{A}^2 / μ_2 of the curve defined by $w_1 = 0$ coincides with the curve defined by $s = u = 0$. Its complement is thus isomorphic to $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1$, which is an \mathbb{A}^1 -cylinder as desired.

From now on, let \mathcal{H} (respectively \mathcal{G}) be the hyperplane defined by $y = 0$ (respectively $z = 0$) in \mathbb{P} and let \mathcal{Q} be the (weighted) hypersurface (possibly reducible or non-reduced) defined by $g(y, z, w) = 0$ in \mathbb{P} .

We first consider the surface S dealt with in Claims 1 and 2.

From Claims 1 and 2 we obtain the property that

$$S \setminus (\mathcal{H} \cup \mathcal{Q}) \cong \mathbb{P}^1 \setminus (\mathcal{C} \cup \ell)$$

is an \mathbb{A}^1 -cylinder. In addition we suppose that the weighted surface S is defined by $f(x, y, z, w) = 0$ in \mathbb{P} . The equation must be one of the types in Lemma 4.5.

The pencil on \mathbb{P} generated by the surfaces S and $\mathcal{H} + \mathcal{Q}$ consists of (weighted) surfaces with equations of the form

$$\alpha f(x, y, z, w) + \beta yg(y, z, w) = 0,$$

where $[\alpha : \beta] \in \mathbb{P}^1$. This pencil gives a rational map $\rho: \mathbb{P} \dashrightarrow \mathbb{P}^1$. Its generic fiber S_η is the weighted surface defined by equation

$$f(x, y, z, w) + \lambda yg(y, z, w) = 0$$

in the corresponding weighted projective space $\mathbb{P}_{\mathbb{k}(\lambda)}$ over the field $\mathbb{k}(\lambda)$, where λ is a parameter. It is singular at the point $[1 : 0 : 0 : 0]$, and the intersection of S_η with the surface in $\mathbb{P}_{\mathbb{k}(\lambda)}$ given by $f(x, y, z, w) = 0$ consists of the $\mathbb{k}(\lambda)$ -rational hyperplane section

$$\mathcal{H}_\eta = S_\eta \cap \{y = 0\}$$

and the $\mathbb{k}(\lambda)$ -rational (weighted) hypersurface section

$$\mathcal{Q}_\eta = S_\eta \cap \{g(y, z, w) = 0\}.$$

As in the case of $S \setminus (\mathcal{H} \cup \mathcal{Q})$, the complement $S_\eta \setminus (\mathcal{H}_\eta \cup \mathcal{Q}_\eta)$ is an \mathbb{A}^1 -cylinder in S_η , defined over the field $\mathbb{k}(\lambda)$, and hence $\mathbb{P} \setminus S$ contains an open \mathbb{A}^1 -cylinder (see [DK18, Lemma 3]).

Now we consider the surface S defined by a quasi-homogenous equation of type (4.9). Letting S_η be the generic member of the pencil of hypersurfaces of degree 6 generated by S and $3\mathcal{G}$, we deduce from Claim 3 in the same way as in the previous cases that the complement of $S_\eta \setminus \mathcal{G}$ contains an open \mathbb{A}^1 -cylinder defined over the field $\mathbb{k}(\lambda)$, and hence that $\mathbb{P} \setminus S$ contains an open \mathbb{A}^1 -cylinder. □

COROLLARY 4.10. *If S contains a $(-K_S)$ -polar cylinder, then $\text{Aut}(\mathbb{P} \setminus S) \neq \text{Aut}(\mathbb{P}, S)$.*

Proof. Suppose that S contains a $(-K_S)$ -polar cylinder. Using Theorems 4.3 and C, we see that the affine Fano variety $\mathbb{P} \setminus S$ contains an open \mathbb{A}^1 -cylinder. Then we conclude in the same way as in the proof of Corollary 4.2 that the existence of an open \mathbb{A}^1 -cylinder in $\mathbb{P} \setminus S$ implies the existence of a non-trivial action of \mathbb{G}_a on $\mathbb{P} \setminus S$. This implies that $\text{Aut}(\mathbb{P} \setminus S) \neq \text{Aut}(\mathbb{P}, S)$, because the right hand side is a finite group. □

Remark 4.11. The proof of Theorem C reproves [CPW16a, Theorem 1.5] in a uniform manner. The original proof for anticanonical degrees 1 and 2 in [CPW16a] has been given in a case-by-case way.

Combined with Corollary 4.2, Theorem C fully settles the question of existence of open \mathbb{A}^1 -cylinders in the complements to cubic surfaces with at worst Du Val singularities. Since the cubic surface S has a $(-K_S)$ -polar cylinder if and only if S is singular, it follows from Corollary 4.2 and Theorem C that the following two conditions are equivalent:

- the affine Fano variety $\mathbb{P} \setminus S$ contains an open \mathbb{A}^1 -cylinder;
- the cubic surface S contains a $(-K_S)$ -polar cylinder.

It is natural to expect that the same holds in the cases of lower anticanonical degrees.

CONJECTURE 4.12. Let S be a del Pezzo surface of anticanonical degree at most 2 with at worst Du Val singularities. The affine Fano variety $\mathbb{P}\backslash S$ contains an open \mathbb{A}^1 -cylinder if and only if S contains a $(-K_S)$ -polar cylinder.

Remark 4.13. Using the idea of the proof of Theorem 4.1, one can try to prove that the affine Fano variety $\mathbb{P}\backslash S$ does not contain open \mathbb{A}^1 -cylinders provided that $K_S^2 \leq 2$ and S does not contain $(-K_S)$ -polar cylinders. The crucial difference, in this case, is that the threefold V in the proof of Theorem 4.1 would no longer be smooth, because S may be singular. Hence, we cannot use [Gri03, Gri04] and [Voi88, Corollaire 4.8] anymore. For instance, if $K_S^2 = 2$ and S is a singular quartic surface in $\mathbb{P}(1, 1, 1, 2)$ that has at most ordinary double points, then the threefold V is a singular double cover of \mathbb{P}^3 ramified in a quartic surface, which has singular points of type A_3 . The rationality problem for such singular quartic double solids has never been addressed as far as we are aware. *A priori*, one can adapt [Voi88, CPS15] to prove their irrationality in some cases.

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¹ The published version in Osaka J. Math. **51**(4) (2014) does not carry Proposition 3.7 as in [arXiv:1106.1312v1](https://arxiv.org/abs/1106.1312v1).

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