GENERALIZED RADICAL RINGS

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Let R be a ring. We denote by $\circ$ the so-called circle composition on R, defined by $a \circ b = a + b - ab$ for $a, b \in R$. It is well known that this composition is associative and that R is a radical ring in the sense of Jacobson (see **6**) if and only if the semigroup $(R, \circ)$ is a group. We shall say that R is a *generalized radical ring* if $(R, \circ)$ is a union of groups. Such rings might equally appropriately be called generalized strongly regular rings, since every strongly regular ring satisfies this property (see Theorem A below). This definition was in fact partially motivated by the observation of Jiang Luh (**7**) that a ring is strongly regular if and only if its multiplicative semigroup is a union of groups.

After proving that any strongly regular ring is a generalized radical ring, the remainder of the paper is devoted to establishing a characterization of generalized radical rings which possess principal idempotents. An idempotent of a ring will be called *principal* if it is an identity for the ring modulo its (Jacobson) radical. The class of rings containing such idempotents includes all radical rings, all rings with identity, and all rings with the descending chain condition on left ideals. It is shown that a ring R possessing a principal idempotent is a generalized radical ring if and only if R is a splitting extension of its radical N by a strongly regular subring eRe for some idempotent e in R.

1. Preliminaries. A ring R is said to be *strongly regular* if $a \in a^2R$ for all a in R. This implies that $a \in Ra^2$ for all $a \in R$ (see **5**).

Let S be a semigroup, and let e be an idempotent in S. Then there exists a maximal subsemigroup G of S containing e which is a group. Moreover, any two distinct such groups are disjoint (see **4**, p. 22). If every element of S belongs to such a subgroup, then S is said to be a *union of groups*. It is known (**4**, § 4.1, p. 121) that a semigroup S is a union of groups if and only if $a \in a^2S \cap Sa^2$ for all a in S. Thus if a ring R is strongly regular, then its multiplicative semigroup is a union of groups. The converse is trivial.

It is also known that every idempotent of a strongly regular ring lies in the centre of the ring (see **5**). Hence the multiplicative semigroup of a strongly regular ring is an inverse semigroup (see **4**, Theorem 1.17, p. 28) in addition to being a union of groups.

Let N denote the (Jacobson) radical of a ring R. An idempotent e is *principal* if $(1 - e)R + R(1 - e) \subseteq N$. (Here, as below, 1 is used only as a

Received May 30, 1966. This research was supported by the National Science Foundation, Grant 216KO7.

notational device. R may or may not have an identity.) The case $e = 0$ is not excluded; however, this happens only if $R = N$. If R has an identity e, then clearly e is a unique principal idempotent. It is easily seen that a semiprimary SBI ring (see **6**, §§ 8 and 9) always contains a principal idempotent; and, in particular, any ring satisfying the descending chain condition on left ideals contains a principal idempotent. Every principal idempotent is principal in the classical sense (**1**, § 9, p. 25), but in general the converse is not true.

Let $P_e = eR(1 - e) + (1 - e)Re + (1 - e)R(1 - e)$. Then R is an abelian group direct sum $R = eRe \oplus P_e$. This is the so-called two-sided Peirce decomposition of R with respect to the idempotent e. One may easily see that $P_e = (1 - e)R + R(1 - e)$. Thus, an idempotent e is principal if and only if P_e is contained in the radical of R.

A subset T of R will be said to be a *left ideal* of $(R, \circ)$ if $r \circ t \in T$ for all $r \in R$ and all $t \in T$. *Right ideals* and *ideals* are defined analogously.

We now state several results from (**2**) which will be required. If e is a principal idempotent of a ring R with radical N, then it may be shown that $e \circ N \circ e$ is a group. From this it follows rather easily that $N \circ e$ $(e \circ N)$ is a minimal left (right) ideal of $(R, \circ)$. This implies by a result of A. H. Clifford (**3**) that $K = N \circ e \circ N$ is a completely simple ideal of $(R, \circ)$, i.e., K is a minimal ideal and a union of groups. Since $N \circ e$ $(e \circ N)$ is a *minimal* left ideal, we have $N \circ e = R \circ e$ $(e \circ N = e \circ R)$. Whence, $e \circ R \circ e$ is a group and $K = R \circ e \circ R$. Further details concerning K and its role in the structure of R will appear in (**2**).

2. The main results.

THEOREM A. *If a ring R is strongly regular, then it is a generalized radical ring.*

Proof. Since R is strongly regular, its multiplicative semigroup, $(R, \cdot)$, is a union of groups (see preliminaries). We must show that $(R, \circ)$ is a union of groups. If $x \in R$, then x lies in some subgroup G of $(R, \cdot)$. Let e be the identity for G, so that $x = exe$. Now $e - x$ also lies in a subgroup, say H, of $(R, \cdot)$, which contains an identity g. Let y be the inverse of $e - x$ in H. Then $g = (e - x)y = (e^2 - ex)y = eg$. Similarly, $g = ge$. Hence $h = e - g$ is an idempotent. Since $ex = x$, $ge = g$, and $g(e - x) = e - x$, we have $g - gx = g(e - x) = e - x = e - ex$. Hence $e - g = (e - g)x$, implying $h \circ x = x$. A dual argument shows that $x \circ h = x$. Thus h is a two-sided identity for x in $(R, \circ)$. Now let y be the inverse of $e - x$ in H. Since $e(e - x) = e - x$ and $H = (e - x)H$, we have $ey = y$. Similarly, $ye = y$. Now from $(e - x)y = g$, we obtain $ey - xy = g$ and since $ey = y$, $xy = y - g$. Hence $x \circ (e - y) = x + e - y - x + xy = e - g = h$. A similar argument shows that $(e - y) \circ x = h$. This implies that x lies in a subgroup of $(R, \circ)$.

https://doi.org/10.4153/CJM-1968-011-2 Published online by Cambridge University Press

We recall that a ring R is said to be a *splitting extension* of an ideal I if R is a direct sum (*qua* abelian group) $R = S \oplus I$ for some subring S of R. If I is the radical of R, R is sometimes said to be cleft.

The remainder of this paper will be devoted to a proof of the following theorem.

THEOREM B. *A ring R is a splitting extension of its radical N by a strongly regular subring eRe for some idempotent e in R if and only if R is a generalized radical ring possessing a principal idempotent.*

The proof will be divided into a series of lemmas.

LEMMA 1. *Let R be a ring with identity e and let y be an element of the radical N of R. Then $e - y$ lies in a subgroup of $(R, \circ)$ only if $y = 0$.*

Proof. We first note that $x \to e - x$ is an isomorphism from $(R, \circ)$ onto the multiplicative semigroup $(R, \cdot)$ of R. Therefore, if $e - y$ lies in a subgroup of $(R, \circ)$, then $y = e - (e - y)$ lies in a subgroup, say G, of $(R, \cdot)$. Let f be the identity of G, and let z be the inverse of y in G. Then $f = zy \in N$. But, as is well known, the only idempotent in the radical is zero; whence $f = y = 0$.

LEMMA 2. *Let R be a ring containing a principal idempotent e and let $K = R \circ e \circ R$. Then the additive subgroup of R generated by $K - e$ is an ideal contained in the radical of R.*

Proof. Let I denote the additive subgroup of R generated by $K - e$. Let $t = \sum n_i(k_i - e) \in I$ where $k_i \in K$ and the n_i are integers. Now, for any $r \in R$, we have

$$\begin{aligned} rt = r\sum n_i(k_i - e) &= \sum n_i(k_i - r \circ k_i + r \circ e - e) \\ &= \sum n_i(k_i - e) - \sum n_i(r \circ k_i - e) + \sum n_i(r \circ e - e). \end{aligned}$$

Since K is an ideal of $(R, \circ)$, this implies that $rt \in I$. Similarly, $tr \in I$. Hence I is an ideal of R.

To show that I is contained in the radical N of R it suffices to show that $K - e \subseteq N$. By the remarks in the preliminaries above, $K = N \circ e \circ N$ which is clearly contained in $N + e$. This implies that $K - e \subseteq N$.

LEMMA 3. *Let R be a generalized radical ring containing a principal idempotent e, then P_e is an ideal.*

Proof. Let I denote the ideal generated by $K - e$ where $K = R \circ e \circ R$. Since $R(1 - e) = R \circ e - e \subseteq K - e$ and similarly $(1 - e)R \subseteq K - e$, it is clear that $P_e \subseteq I$. To show the opposite inclusion it suffices to show that $eIe = 0$ (since $R = eRe \oplus P_e$ *qua* abelian group). On the other hand, since by Lemma 2 I is the *additive group* generated by $K - e$, it suffices to show that $e(K - e)e = 0$ or equivalently that $eKe = e$. To show this we consider any element exe for $x \in K$. Since $(R, \circ)$ is a union of groups, there exists an

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idempotent $g \in R$ such that $g \circ exe = exe \circ g = exe$. From this one easily obtains that $eg = ge = g$. Therefore $g \in eRe$. Let z be the inverse of exe relative to g. Then $g = z \circ exe = z + exe - zexe$; so $z = g - exe + zexe$. This yields $ze = z$. Similarly, $ez = z$. Consequently $z \in eRe$. This proves that exe lies in a subgroup of $(eRe, \circ)$.

Now let $y = exe - e$. Clearly $y = eye \in eRe$. Since

$$exe = x - x \circ e - e \circ x + e \circ x \circ e + e,$$

we have

$$\begin{aligned} y &= x - x \circ e - e \circ x + e \circ x \circ e \\ &= (x - e) - (x \circ e - e) - (e \circ x - e) + (e \circ x \circ e - e). \end{aligned}$$

This implies that $y \in I \subseteq N$ and since $y \in eRe$, y lies in eNe, which is the radical of eRe (see **6**, § 3.7, p. 48). But by Lemma 1, $exe = e + y$ cannot lie in a subgroup of eRe unless $y = 0$. Since we have shown in the above paragraph that exe does lie in a subgroup of $(eRe, \circ)$, we must have $y = 0$; consequently, $exe = e$.

LEMMA 4. *If R is a generalized radical ring containing a principal idempotent e, then R is a splitting extension of its radical $N = P_e$ by the strongly regular subring eRe.*

Proof. Since the property of being a generalized radical ring is clearly preserved under homomorphisms, and since by Lemma 3 P_e is an ideal, we have that $eRe \cong R/P_e$ is a generalized radical ring with identity e. As in the proof of Lemma 1, $(eRe, \circ)$ is isomorphic to the multiplicative semigroup $(eRe, \cdot)$ of eRe. This implies that $(eRe, \cdot)$ is a union of groups and therefore eRe is strongly regular and semisimple. Since eRe is semisimple, $N \subseteq P_e$. On the other hand, $P_e \subseteq N$ since e is principal (see preliminaries). Therefore $P_e = N$. Since $R = eRe + P_e$ is always a direct sum decomposition of the additive group of R, we are done.

Lemma 4 completes the proof of Theorem B in one direction. We now work toward the converse.

LEMMA 5. *Let R be a ring which is a splitting extension of its radical N by a subring eRe for $e = e^2 \in R$. Then e is a principal idempotent and $N = P_e$.*

Proof. By hypothesis $R = eRe \oplus N$ is an abelian group direct sum. This clearly implies that e is a principal idempotent; whence $P_e \subseteq N$. Since also $R = eRe \oplus P_e$ is an abelian group direct sum, we must have $P_e = N$.

Before proceeding we shall need the following notation and results from **(4)**.

Let $J(a)$ denote the *principal ideal* generated in the semigroup $(R, \circ)$ by a, i.e., $J(a)$ is the smallest ideal of $(R, \circ)$ containing a. For $a \in (R, \circ)$, we let $J_a = \{x \in R: J(x) = J(a)\}$. This is the so-called $\mathfrak{J}$-class of $(R, \circ)$ containing a. A semigroup S is said to be *intra-regular* if $a \in Sa^2S$ for all $a \in S$.

https://doi.org/10.4153/CJM-1968-011-2 Published online by Cambridge University Press

Now, Croisot and, independently, Anderson (see **4**, p. 123), have shown that if S is intra-regular, then every $\mathfrak{J}$-class J_a is a simple semigroup. On the other hand, Rees (see **4**, § 2.7, p. 76) has shown that a simple semigroup containing a primitive idempotent is a union of groups. (An idempotent e is primitive if $f^2 = f$ and $fe = ef = f$ implies $f = e$.) Thus, to show that $(R, \circ)$ is a union of groups, it suffices to show that it is intra-regular and that every $\mathfrak{J}$-class contains a primitive idempotent.

LEMMA 6. *Let e be a principal idempotent in a ring R whose radical is P_e. Then, $eae + P_e \subseteq J(a)$.*

Proof. We recall that since e is principal, $e \circ R \circ e$ is a group with identity e (see preliminaries). Let $a \in R$ and $x \in P_e$, and let $\bar{a} = e \circ \bar{a} \circ e$ and $\bar{x} = e \circ \bar{x} \circ e$ be the inverse of $e \circ a \circ e$ and $e \circ x \circ e$, respectively, in the group $e \circ R \circ e$.

Now set $b = x \circ e - a \circ \bar{a}$ and $c = \bar{a} \circ \bar{x} \circ x - \bar{a} \circ a$. Now using the distributive laws

$$x \circ (y + z - 0) = x \circ y + x \circ z - x \circ 0,$$
$$x \circ (y - z + w) = x \circ y - x \circ z + x \circ w,$$

etc., one may easily calculate that

$$b \circ a \circ c = a - a \circ \bar{a} \circ a + x \circ \bar{x} \circ x.$$

We wish to show that $b \circ a \circ c = eae + x$. To do this, first note that if y is any element of $P_e + e$, then $y = e \circ y + y \circ e - e \circ y \circ e$, since $eP_e e = 0$. Now, $y = a \circ \bar{a} \circ a \in K$ since $\bar{a} \in K$, and

$$K = N \circ e \circ N \subset N + e = P_e + e.$$

Hence, $y = a \circ \bar{a} \circ a \in P_e + e$ and so

$$\begin{aligned} y &= e \circ y + y \circ e - e \circ y \circ e \\ &= e \circ a \circ \bar{a} \circ a + a \circ \bar{a} \circ a \circ e - e \circ a \circ \bar{a} \circ a \circ e \\ &= e \circ a + a \circ e - e \circ a \circ e = a + e - eae. \end{aligned}$$

Similarly, $x \circ \bar{x} \circ x = x + e - exe = x + e$, since $x \in P_e$. Therefore

$$\begin{aligned} b \circ a \circ c &= a - a \circ \bar{a} \circ a + x \circ \bar{x} \circ x \\ &= a - (a + e - eae) + (e + x) = eae + x. \end{aligned}$$

Hence, $eae + x = b \circ a \circ c \in J(a)$.

LEMMA 7. *Let R be a ring containing an idempotent e such that $R = eRe + P_e$. If P_e is the radical of R and if eRe is strongly reglular, then for all $a \in R$*

$$J_a = H + P_e,$$

where H is the maximal subgroup of $(eRe, \circ)$ containing eae.

Proof. We first observe that if ϕ is a homomorphism of a semigroup S onto a semigroup T and if J is a $\mathfrak{J}$-class in S, then $J\phi$ is contained in a $\mathfrak{J}$-class in T. This follows easily from the definitions.

Now since P_e is an ideal of R, $x \to exe$ is a homomorphism of R onto eRe, and therefore a (semigroup) homomorphism of $(R, \circ)$ onto $(eRe, \circ)$. Since eRe has an identity, $(eRe, \circ)$ is isomorphic to $eRe, \cdot)$. Therefore $(eRe, \circ)$ is a union of groups. Since idempotents are in the centre of eRe, the same is true of $(eRe, \circ)$. This implies by **(4**, Th. 1.17, p. 28**)** that $(eRe, \circ)$ is an inverse semigroup. Since it is also a union of groups, every $\mathfrak{J}$-class is a (maximal) group **(**see **4**, § 4.2, p. 126**)**. It therefore follows from the previous paragraph that $eJ_a e$ lies in the maximal subgroup H of $(eRe, \circ)$ containing eae. This implies that $J_a \subseteq H + P_e$.

On the other hand, from Lemma 6 we have immediately that $J(b) = J(ebe)$ for all $b \in R$. If $ebe \in H$, then one easily sees that $J(eae) = J(ebe)$. Hence if $b \in H + P_e$, then $J(b) = J(ebe) = J(eae) = J(a)$. Therefore $b \in J_a$, implying $H + P_e \subseteq J_a$.

LEMMA 8. *If everything is as in Lemma* 7, *then every* $\mathfrak{J}$*-class of* $(R, \circ)$ *contains a primitive idempotent.*

Proof. Let $a \in R$. By Lemma 7, $J_a = H + P_e$ where H is a subgroup of $(eRe, \circ)$ containing eae. J_a contains at least one idempotent, namely, the identity f of H. We claim that f is primitive. Since $H \subseteq eRe$, we have that $f = ef = fe$. Hence $fP_e f = 0$, since $eP_e e = 0$. Now let g be any idempotent in J_a such that $g \circ f = f \circ g = g$ or, equivalently, $gf = fg = f$. Then $g = h + p$, where $h \in H$ and $p \in P_e$. Since P_e is an ideal and $R = eRe \oplus P_e$,

$$h + p = (h + p)^2 = h^2 + hp + ph + p^2$$

implies $h^2 = h$. Therefore $h \circ h = h \in H$, and we must have $h = f$. Now from $gf = f$ we have that $0 = gf - f = (f + p)f - f = pf$. Similarly, $fp = 0$. From $g^2 = g$ we get $f + fp + pf + p^2 = f + p$, and since $fp = pf = 0$, $p = p^2$. But p lies in P_e, the radical, which contains no non-zero idempotents; thus, $p = 0$ and $g = f$.

LEMMA 9. *If everything is as in Lemmas* 7 *and* 8, *then* $(R, \circ)$ *is intra-regular.*

Proof. Since P_e is an ideal, and $eP_e e = 0$, we have $eabe = eaebe$ for all $a, b \in R$. It follows that $e(a \circ b)e = (eae) \circ (ebe)$. As pointed out in the proof of Lemma 7, $(eRe, \circ)$ is a union of groups. Let c be the inverse of eae in some subgroup of $(eRe, \circ)$. Since $c \in eRe$, we have $c = ece$. If $b = c \circ a \circ a$,

$$ebe = e(c \circ a \circ a)e = c \circ eae \circ eae = eae.$$

Therefore, by Lemma 6 there exist $x, y \in R$ such that

$$a = x \circ b \circ y = (x \circ c) \circ a \circ a \circ (y).$$

Hence $a \in R \circ a \circ a \circ R$, which was to be shown.

Now, by Lemmas 8 and 9 and the comments preceding Lemma 6, a ring R which is a splitting extension of its radical by a strongly regular subring eRe for $e = e^2 \in R$, is a generalized radical ring. This completes the proof of Theorem B.

References

1. A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Coll. Publ., *24* (1939).

2. W. E. Clark and J. Lewin, *On minimal ideals in the circle composition semigroup of a ring* (to appear in Publ. Math. Debrecen).

3. A. H. Clifford, *Semigroups containing minimal ideals*, Amer. J. Math., *70* (1948), 521–526.

4. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Math Surveys, No. 7 (Amer. Math. Soc., 1961).

5. M. P. Drazin, *Rings with central idempotent or nilpotent elements*, Proc. Edinburgh Math. Soc., *9* (1958), 157–165.

6. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Coll. Publ., *37* (1956).

7. J. Luh, *A note on strongly regular rings*, Proc. Japan Acad., *40*, No. 2 (1964), 74–75.

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