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# DETERMINANTAL FORMS FOR SYMPLECTIC AND ORTHOGONAL SCHUR FUNCTIONS

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ABSTRACT. Symplectic and orthogonal Schur functions can be defined combinatorially in a manner similar to the classical Schur functions. This paper demonstrates that they can also be expressed as determinants. These determinants are generated using planar decompositions of tableaux into strips and the equivalence of these determinants to symplectic or orthogonal Schur functions is established by Gessel-Viennot lattice path techniques. Results for rational (also called *composite*) Schur functions are also obtained.

1**. Introduction.** Recent work on the symplectic and orthogonal tableaux and the associated symmetric functions has focused on Robinson-Schensted-type algorithms and Cauchy-type identities. See Berele [3], Sundaram [27] [28], Proctor [19] [20] [21] [22], Okada [17] [18], Benkart and Stroomer [2]. Here we develop determinantal expressions for the characters of the symplectic and orthogonal groups $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ and prove their validity using the techniques of Hamel and Goulden [8]. Some of the determinants generated are symplectic and orthogonal analogues to the Jacobi-Trudi, dual Jacobi-Trudi, and Giambelli determinants defined for the classical Schur functions, and our methods are valid not only for the ordinary symplectic Schur function and so-Schur function, but for skew versions of these as well (defined below). We follow the notation of Macdonald [16] and Sundaram [27].

Let $\lambda$ be a partition of $k$ with at most $l$ parts, *i.e.* $\lambda = (\lambda_1, \ldots, \lambda_l)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ are nonnegative integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = k$ ($\lambda_i$ is the $i$-th part of $\lambda$). The empty partition $\emptyset$ of 0 has no parts. A partition can be represented in the plane by an arrangement of boxes called a *Ferrers diagram*, or simply a *diagram*. This arrangement is top and left justified with $\lambda_i$ boxes in the $i$-th row and we say it has standard shape. Given two partitions, $\lambda$ and $\mu$, we define a Ferrers diagram with *skew shape* $\lambda/\mu$ for $\mu_i \leq \lambda_i$, $i \geq 1$ as an arrangement of boxes where there is a box in row $i$, column $j$ iff $\mu_i < j \leq \lambda_i$. Geometrically, this is the Ferrers diagram of $\lambda$ with the Ferrers diagram of $\mu$ removed from its upper left hand corner. From this point of view, the standard shape $\lambda$ is just the skew shape $\lambda/\mu$ with $\mu = \emptyset$. Define the *content* of a box $\alpha$ in a Ferrers diagram as the quantity $j - i$ where $\alpha$ lies in column $j$ and row $i$ of the Ferrers diagram (referred to as box $(i, j)$ where convenient). Associated with each skew shape is a *conjugate* shape. The conjugate of a skew shape $\lambda/\mu$ is defined to be the skew shape $\lambda'/\mu'$ whose Ferrers

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diagram is the transpose of the Ferrers diagram of $\lambda/\mu$. More explicitly, the number of boxes in the $i$-th row of $\lambda'/\mu'$ is the number of boxes in the $i$-th column of $\lambda/\mu$.

Fix a set of elements $1 < \bar{1} < 2 < \bar{2} \cdots < n < \bar{n}$. The following definition is the skew version of one due to King [13].

DEFINITION 1.1. A symplectic tableau, $\mathrm{SP}_{\lambda/\mu}$, of shape $\lambda/\mu$ is a filling of the Ferrers diagram of $\lambda/\mu$ with the integers $1 < \bar{1} < 2 < \bar{2} \cdots < n < \bar{n}$ such that

1. the entries weakly increase along the rows and strictly increase down the columns,
2. the boxes of content $-i$ contain entries which are greater than or equal to $i + 1$.

We refer to the second condition as the *symplectic condition*. For standard shape tableaux the following condition is usually called the *symplectic condition*:

$$\text{all entries in row } i \text{ are greater than or equal to } i. \tag{1}$$

The condition (1) and condition 2 of Definition 1.1 are easily seen to be equivalent for standard shape since the first box in row $i + 1$ has content $-i$.

The skew symplectic Schur function, $\mathrm{sp}_{\lambda/\mu}(X)$, in the variables, $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$, is given by

$$\mathrm{sp}_{\lambda/\mu}(X) = \sum_{\mathrm{SP}_{\lambda/\mu}} \prod_{\alpha\in \mathrm{SP}_{\lambda/\mu}} x_\alpha^{m(\alpha)} \prod_{\bar{\beta}\in \mathrm{SP}_{\lambda/\mu}} x_\beta^{-m(\bar{\beta})},$$

where the sum is over all tableaux $\mathrm{SP}_{\lambda/\mu}$ of shape $\lambda/\mu$, the first product is over all unbarred integers $\alpha$ in $\mathrm{SP}_{\lambda/\mu}$, the second product is over all barred integers $\bar{\beta}$ in $\mathrm{SP}_{\lambda/\mu}$, and $m(\alpha)$ (resp. $m(\bar{\beta})$) is the multiplicity of $\alpha$ (resp. $\bar{\beta}$) in $\mathrm{SP}_{\lambda/\mu}$, *i.e.* the number of times $\alpha$ (resp. $\bar{\beta}$) appears in a box of the tableau.

There are several equivalent tableau definitions of orthogonal tableaux for $\mathrm{SO}(2n+1)$ (see King [13], Proctor [22], and Koike and Terada [14]). The definition we take is a skew version of the one in Sundaram [27], and is very close to the definition of symplectic tableaux.

DEFINITION 1.2. An so-tableau, $\mathrm{SO}_{\lambda/\mu}$, of shape $\lambda/\mu$ is a filling of the Ferrers diagram of $\lambda/\mu$ with the elements $1 < \bar{1} < 2 < \bar{2} \cdots < n < \bar{n} < \infty$ such that

1. the entries weakly increase along the rows and, when restricted to $1 < \bar{1} < 2 < \bar{2} \cdots < n < \bar{n}$, strictly increase down the columns,
2. the boxes of content $-i$ contain entries which are greater than or equal to $i + 1$,
3. the entries equal to $\infty$ form a shape which is such that no two symbols $\infty$ appear in the same row.

The skew so-Schur function, $\mathrm{so}_{\lambda/\mu}(X)$, in the variables, $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$, is given by

$$\mathrm{so}_{\lambda/\mu}(X) = \sum_{\mathrm{SO}_{\lambda/\mu}} \prod_{\alpha\in \mathrm{SO}_{\lambda/\mu}} x_\alpha^{m(\alpha)} \prod_{\bar{\beta}\in \mathrm{SO}_{\lambda/\mu}} x_\beta^{-m(\bar{\beta})},$$

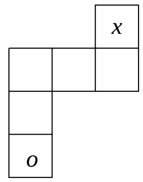


FIGURE 1: Example of a strip.

where the sum is over all tableaux $SO_{\lambda/\mu}$ of shape $\lambda/\mu$, the first product is over all unbarred integers $\alpha$ in $SO_{\lambda/\mu}$, the second product is over all barred integers $\overline{\beta}$ in $SO_{\lambda/\mu}$, and $m(\alpha)$ (resp. $m(\overline{\beta})$) is the multiplicity of $\alpha$ (resp. $\overline{\beta}$) in $SO_{\lambda/\mu}$.

Note that the $\infty$ are in a sense "dummy elements" since they contribute 1 to the weight of the tableau.

Koike and Terada [14] also define skew SP($2n$) and SO($2n+1$) tableaux; however, their definition differs substantially from ours. They restrict the integers that are allowed to appear so that only those greater than the number of parts of $\mu$ are permitted and they use the alternative formulation of the symplectic condition given in (1).

The form of this paper is as follows. Section 2 provides background material from Hamel and Goulden [8], giving the details necessary to define the determinants we generate. Section 3 states and proves two main results, one for symplectic tableaux and one for so-tableaux. Section 4 includes some similar results for rational (also called *composite*) tableaux. As has been pointed out by Stembridge [25], the standard shape symplectic tableaux can be considered to be special cases of standard shape rational tableaux, and hence the results in Section 4 generalize the results in Section 3.

2**. Strips and outside decompositions.** This section gives the tools needed to define classes of determinants equal to the symplectic Schur function and so-Schur functions. The traditional ways of decomposing a tableau to generate a determinant use decompositions by rows (Jacobi-Trudi), columns (dual Jacobi-Trudi) or hooks (Giambelli). We generalize these notions here to allow decompositions by *strips*. The terminology follows that of Hamel and Goulden [8].

DEFINITION 2.1. A *strip* $\theta$ in a skew shape diagram is a skew diagram with an edgewise connected set of boxes that contains no $2 \times 2$ block of boxes.

DEFINITION 2.2. The *starting box* of a strip is the box which is bottommost and leftmost in the strip. The *ending box* of a strip is the box which is topmost and rightmost in the strip.

Figure 1 illustrates these concepts, where the starting box is marked with an $x$ and the ending box is marked with an $o$. We say a box is approached from the left (resp. from below) if either there is a box immediately to its left or the box is on the left perimeter

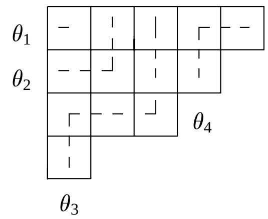


FIGURE 2: Example of an outside decomposition.

of the diagram (resp. there is a box immediately below it or the box is on the bottom perimeter of the diagram).

DEFINITION 2.3. Suppose $\theta_1, \theta_2, \ldots, \theta_m$ are strips in a skew shape diagram of $\lambda/\mu$ and each strip has a starting box on the left or bottom perimeter of the diagram and an ending box on the right or top perimeter of the diagram. Then if the disjoint union of these strips is the skew shape diagram of $\lambda/\mu$, we say the totally ordered set $(\theta_1, \theta_2, \ldots, \theta_m)$ is a *(planar) outside decomposition* of $\lambda/\mu$.

Given a diagram and an outside decomposition of that diagram, then if the diagram is filled with integers to form a symplectic or so-tableau, the portion of the tableau that corresponds to a strip in the outside decomposition forms a *symplectic* or so-*tableau of strip shape*. Hence, given an outside decomposition of a shape, a symplectic or so-tableau of that shape can be thought of as a union of symplectic or so-tableaux of strip shape.

The restrictions of the definition of outside decomposition force the following property:

PROPERTY 2.4. Boxes of the same content are approached from the same direction in their respective strips; that is, they are either all approached from below or all approached from the left.

Figure 2 gives an example of an outside decomposition into four strips: $\theta_1 = 1$, $\theta_2 = 22/1$, $\theta_3 = 3331/22$, $\theta_4 = 21$. In Figure 2 strips $\theta_1$, $\theta_2$ and $\theta_3$ have boxes of content zero approached from the left, while strips $\theta_3$ and $\theta_4$ have boxes of content two approached from below.

To allow for a further level of generality we could include the possibility of *null strips*. These are geometrically empty objects discussed in Hamel and Goulden [8] and correspond to edges rather than boxes in the diagram. Roughly speaking, they correspond to a zero part in a partition or to the case $\lambda_i = \mu_i$ in a skew partition. A full consideration is given in Hamel and Goulden [8]. We have omitted them here to streamline the presentation; however, they are an option and can be included if desired.

In order to define the determinants in the main results we must define an additional operation on strips. This noncommutative operation was first defined in Hamel and Goulden [8].

CASE I. Suppose $\theta_i$ and $\theta_j$ have some boxes with the same content. Slide $\theta_i$ along top-left-to-bottom-right diagonals so that the box of content $k$ in $\theta_i$ is superimposed on the box of content $k$ in $\theta_j$ for all $k \in \mathbf{Z}$. This procedure is well-defined by Proposition 2.4. Define $\theta_i \# \theta_j$ to be the diagram obtained from this superposition by taking all boxes between the ending box of $\theta_i$ and the starting box of $\theta_j$ inclusive.

CASE II. Suppose $\theta_i$ and $\theta_j$ are two disconnected pieces and thus do not have any boxes of the same content. The starting box of one will be to the right and/or above the ending box of the other. To bridge the gap between $\theta_i$ and $\theta_j$, insert boxes from the ending box of one to the starting box of the other so that these inserted boxes follow the approached-from-the-left or approached-from-below arrangement as do other boxes of the same content in the outside decomposition (Property 2.4 ensures the boxes of the same content are arranged in the same way). If there is a content such that there is no box of that content in the diagram (and therefore no determination of the direction from which the box is approached), then arbitrarily choose from which direction boxes of this content should be approached, fix this choice for all boxes of the same content in that particular diagram, and bridge the gap between $\theta_i$ and $\theta_j$ accordingly. Define $\theta_i \# \theta_j$ as in Case I with the following additional conventions: if the ending box of $\theta_i$ is edge connected to the starting box of $\theta_j$, and occurs below or to the left of it, then $\theta_i \# \theta_j = \emptyset$; if the ending box of $\theta_i$ is not edge connected but occurs below or to the left of the starting box of $\theta_j$, $\theta_i \# \theta_j$ is undefined.

Note that $\theta_i \# \theta_i = \theta_i$.

As an example consider again the strips in Figure 2. Then

$$
\begin{array}{ll}
\theta_1 \# \theta_2 = 2 & \theta_2 \# \theta_3 = 331/2 \\
\theta_2 \# \theta_1 = 11 & \theta_3 \# \theta_2 = 222/11 \\
\theta_1 \# \theta_3 = 31 & \theta_2 \# \theta_4 = \emptyset \\
\theta_3 \# \theta_1 = 111 & \theta_4 \# \theta_2 = 3222/111 \\
\theta_1 \# \theta_4 = \text{undefined} & \theta_3 \# \theta_4 = 1 \\
\theta_4 \# \theta_1 = 2111 & \theta_4 \# \theta_3 = 43331/222
\end{array}
$$

In the next section we show how to obtain a determinant from this information.

## 3. The main results.

We now state the two main results of this paper:

THEOREM 3.1. *Let $\lambda/\mu$ be a skew shape partition. Then for any outside decomposition, $(\theta_1, \theta_2, \ldots, \theta_m)$, of $\lambda/\mu$,*

$$\mathrm{sp}_{\lambda/\mu}(X) = \det\left(\mathrm{sp}_{\theta_i \# \theta_j}(X)\right)_{m \times m},$$

*where* $\mathrm{sp}_{\emptyset} = 1$ *and* $\mathrm{sp}_{\text{undefined}} = 0$.

THEOREM 3.2. *Let $\lambda/\mu$ be a skew shape partition. Then for any outside decomposition, $(\theta_1, \theta_2, \ldots, \theta_m)$, of $\lambda/\mu$,*

$$\mathrm{so}_{\lambda/\mu}(X) = \det\left(\mathrm{so}_{\theta_i \# \theta_j}(X)\right)_{m \times m},$$

*where* $\mathrm{so}_{\emptyset} = 1$ *and* $\mathrm{so}_{\text{undefined}} = 0$.

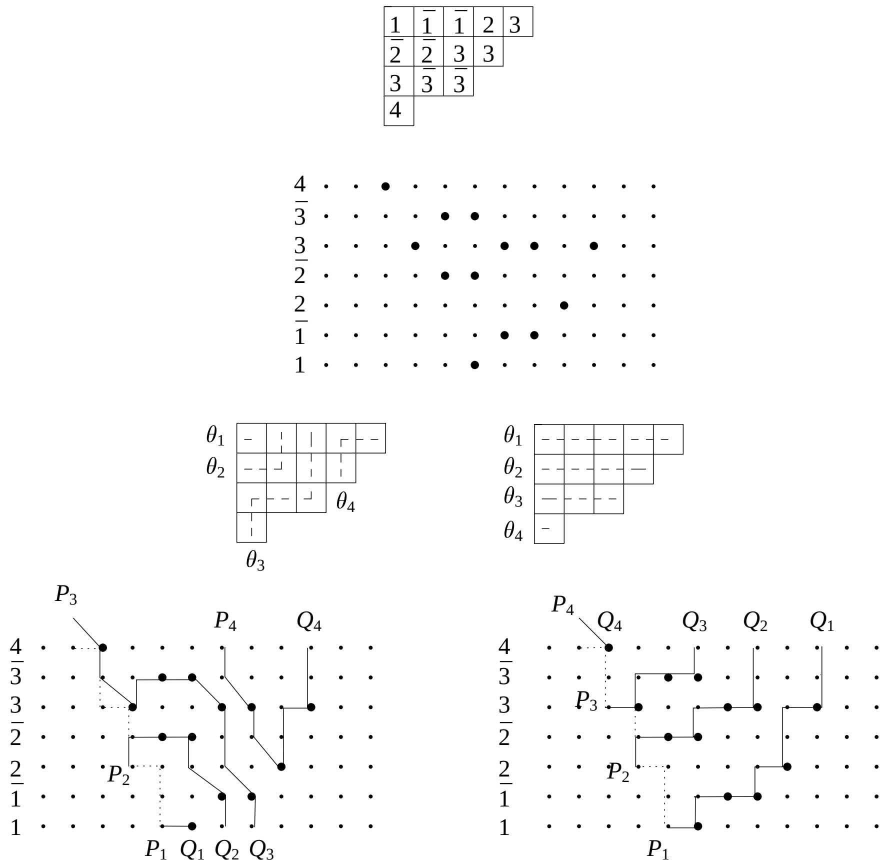

FIGURE 3: Two outside decompositions and the corresponding lattice paths.

For the diagram and outside decomposition in Figure 2, Theorem 3.1 gives the following determinant:

$$\det\begin{pmatrix} sp_1 & sp_2 & sp_{31} & 0 \\ sp_{11} & sp_{22/1} & sp_{331/2} & 1 \\ sp_{111} & sp_{222/11} & sp_{3331/22} & sp_{3331} \\ sp_{2111} & sp_1 & sp_{43331/22} & sp_{21} \end{pmatrix}$$

Under the same conditions, Theorem 3.2 gives the following determinant:

$$\det\begin{pmatrix} so_1 & so_2 & so_{31} & 0 \\ so_{11} & so_{22/1} & so_{331/2} & 1 \\ so_{111} & so_{222/11} & so_{3331/22} & so_{3331} \\ so_{2111} & so_1 & so_{43331/22} & so_{21} \end{pmatrix}$$

In Hamel and Goulden [8], the Gessel-Viennot lattice path procedure is used to construct a bijection establishing a family of determinantal results for Schur functions. We now show that this procedure extends easily to a bijection for symplectic Schur functions. We refer the reader to Hamel and Goulden [8] where certain essential details of the proof have been verified.

The proof hinges on an application of Theorem 1.2 of Stembridge [26]. This theorem is a generalization of the Gessel-Viennot procedure to acyclic digraphs. Given an acyclic digraph, let $w(e)$ be the weight function defined on the edge $e$, let $w(P) = \prod_{e\in P} w(e)$ be the weight of a path in the digraph, let $\mathcal{P}(u_a, v_b)$ be the set of paths from $u_a$ to $v_b$, let $\mathcal{P}(u, v)$ (resp. $\mathcal{P}_0(u, v)$) denote the set of $r$-tuples (resp. nonintersecting $r$-tuples) of paths $(P_1, \ldots, P_r)$ with $u = u_1, \ldots, u_r$ a set of starting points and $v = v_1, \ldots, v_r$ a set of ending points and such that $P_i \in \mathcal{P}(u_i, v_i)$, and let $\mathrm{GF}[\mathcal{P}(u, v)] = \sum_{P\in\mathcal{P}} w(P)$ be the generating function for these paths according to the weight. The theorem states that if $u = (u_1, \ldots, u_r)$ and $v = (v_1, \ldots, v_r)$ are two $r$-tuples of vertices in an acyclic digraph, and if the only non-intersecting $r$-tuples of paths from $u$ to some permutation of $v$ connect $u_i$ to $v_i$ for $i = 1, \ldots, r$, then

$$\mathrm{GF}[\mathcal{P}_0(u, v)] = \det\big(\mathrm{GF}[\mathcal{P}(u_i, v_j)]\big).$$

PROOF OF THEOREM 3.1. : Let the $y$-axis be labeled by $1, \bar{1}, 2, \bar{2}, \ldots$. Before describing the paths we need some guidelines to permissible steps and path restrictions. There are four types of permissible steps: up-vertical steps that *increase* the $y$-coordinate by 1; down-vertical steps that *decrease* the $y$-coordinate by 1; right-horizontal (referred to simply as horizontal) steps that increase the $x$-coordinate by 1; and down-diagonal (referred to simply as diagonal) steps that increase the $x$-coordinate by 1 *and* decrease the $y$-coordinate by 1. We specify some additional restrictions: a down-vertical step must not precede an up-vertical step, an up-vertical step must not precede a down-vertical step, a down-vertical step must not precede a horizontal step, and an up-vertical step must not precede a diagonal step. Because of the symplectic condition, we require an additional restriction not present in Hamel and Goulden [8], a *left boundary* in the form of a "backwards lattice path" from $(0, 1)$ to $(0, \bar{1})$ to $(0, 2)$ to $(-1, 2)$ to $(-1, \bar{2})$ to $(-1, 3)$ to $(-2, 3)$ to $(-2, \bar{3})$, *etc.* See Figure 3 where this boundary is indicated by a dotted line. A path may touch but not cross the left boundary. This boundary may be interpreted as representing a "phantom" zeroth column in the symplectic tableau, a column containing $1, 2, 3, 4, \ldots$. We also require that all steps between lines $x = c$ and $x = c+1$ for all $c \in \mathbf{Z}$ are either all horizontal or all diagonal. The determination of whether these steps are horizontal or diagonal is made by the outside decomposition in the following manner. If boxes of content $d$ are approached from the left, then steps between $x = d$ and $x = d + 1$ must be horizontal; if the boxes of content $d$ are approached from below, then steps between $x = d$ and $x = d + 1$ must be diagonal. We are now ready to construct paths corresponding to strips.

Consider an outside decomposition $(\theta_1, \ldots, \theta_m)$ of $\lambda/\mu$. We will construct a nonintersecting $m$-tuple of lattice paths that corresponds to a symplectic tableau of shape $\lambda/\mu$

with the outside decomposition $(\theta_1, \ldots, \theta_m)$, such that the $i$-th path corresponds to the $i$-th strip and begins at $P_i$ and ends at $Q_i$, $i = 1, \ldots, m$ as described now. Fix points $P_i = \bigl(t-s, -(t-s)+1\bigr)$ if strip $i$ has starting box on left perimeter in box $(s,t)$ of the diagram and if $t - s \leq 0$ (*i.e.* $P_i$ is on the left boundary), or $P_i = (t-s, 1)$ if strip $i$ has starting box on left perimeter in box $(s,t)$ of the diagram and if $t - s > 0$, or $P_i = (t-s, \infty)$ if strip $i$ has starting box on the bottom perimeter in box $(s,t)$ of the diagram ($P_i = (t-s, \infty)$ if both), $i = 1, \ldots, m$. Fix points $Q_i = (v-u+1, 1)$ if strip $i$ has ending box on the top perimeter in box $(u,v)$ of the diagram, or $Q_i = (v-u+1, \infty)$ if strip $i$ has ending box on the right perimeter in box $(u,v)$ of the diagram ($Q_i = (v-u+1, \infty)$ if both), $i = 1, \ldots, m$.

For strip $\theta_j$ construct a path starting at $P_j$ (called the *starting point*) and ending at $Q_j$ (called the *ending point*) as follows: if a box containing $i$ (resp. $\bar{i}$) and at coordinates $(a,b)$ in the diagram is approached from the left in the strip, put a horizontal step from $(b-a, i)$ to $(b-a+1, i)$ (resp. $(b-a, \bar{i})$ to $(b-a+1, \bar{i})$); if a box containing $i$ (resp. $\bar{i}$) and at coordinates $(a,b)$ in the diagram is approached from below in the strip, put a diagonal step from $(b-a, \bar{i})$ to $(b-a+1, i)$ (resp. $(b-a, i+1)$ to $(b-a+1, \bar{i})$). Notice that the physical locations of the termination points of the steps are independent of the outside decomposition and depend only on the contents of the boxes. See Figure 3 in which first the ending points of steps are shown alone and then complete paths for two different outside decompositions are shown. Note that no two paths can have the same starting and/or ending points, since that would imply two boxes of the same content on the same section of perimeter. Connect these nonvertical steps with vertical steps. It is routine to verify that there is a unique way of doing this.

We must verify that an intersecting $m$-tuple of lattice paths does not correspond to a symplectic tableau. This follows from the column strictness and row weakness conditions on the symplectic tableau and also from the fact strips are themselves skew diagrams. The argument is a case-by-case analysis which follows exactly as in Hamel and Goulden [8]. For full details we refer the reader to that paper. However, we do construct a sample case now. The other cases are similar.

Suppose we have an intersecting $m$-tuple of lattice paths, and suppose a horizontal step at height $a$ in path $i$ intersects an up-vertical step in path $j$. Suppose further that path $j$ has a step at height $d$ (necessarily horizontal) before the up-vertical steps and a step at height $e$ (necessarily horizontal) after the up-vertical steps. We show by contradiction it is not possible for this configuration to correspond to a tableau.

Suppose on the contrary that it did. The content of the box containing $e$ is one more than the content of the box containing $a$, and $e \geq a$, so by column strictness and row weakness, the box containing $e$ is right of and below (or beside) the box containing $a$. The content of the box containing $d$ is the same as the content of the box containing $a$, and $d < a$, so again by column strictness and row weakness, the box containing $d$ is above and to the left of the box containing $a$. But the box containing $e$ and the box containing $d$ are in the same strip, yet located on different sides of the box containing $a$. This provides a contradiction.

The construction described above for producing paths given symplectic tableaux is reversible, and now we verify that a nonintersecting $m$-tuple of lattice paths obeying the path conditions corresponds to a symplectic tableau and an outside decomposition where each path in the nonintersecting $m$-tuple gives rise to a symplectic tableau of strip shape. The choice of the starting and ending points and the restrictions on the steps ensure that the $m$-tuple corresponds to a diagram of the required shape, but we must show that the entries in the tableau obey the column strictness and row weakness rules and also the symplectic condition. We begin by ensuring that a lattice path that starts at $P_j$ and ends at $Q_i$ corresponds to the strips $\theta_i \# \theta_j$. (Note that the situation in which $P_j$ is to the right of $Q_i$ and hence no lattice path is possible, corresponds to $\theta_i \# \theta_j$ being undefined). The proof follows exactly as in Hamel and Goulden [8]. Begin with the empty partition. If the lattice path has no nonvertical steps, then $\theta_i \# \theta_j$ is empty as we would expect. Otherwise, at iteration $k$, if the $k$-th nonvertical step from the left in the lattice path is horizontal ending at $(i, j)$, then place a box containing $j$ in the symplectic tableau to the right of the previous box; if it is diagonal ending at $(i, j)$, then place a box containing $j$ in the symplectic tableau on top of the previous box. The fact that a down-vertical step does not precede a horizontal step ensures that a horizontal step is at a height higher than or the same as the step just before it. This means the entries in a row of the symplectic tableau are weakly increasing. The fact that an up-vertical step does not precede a diagonal step ensures that a diagonal step ends at a height strictly lower than the step just before it. This means the entries in a column of the symplectic tableau are strictly increasing. Since the symplectic tableau is built by placing boxes always to the right or on top, we know the shape is a strip. Moreover, since the starting and ending points come from $\theta_j$ and $\theta_i$, since boxes of the same content correspond to the same type of step, and since the # operation is based on boxes of the same content, we know the strip is $\theta_i \# \theta_j$.

Now let $T(l, j)$ denote the entry in box $(l, j)$ of the symplectic tableau. We claim $T(l, j) < T(l+1, j)$ and $T(l, j) \leq T(l, j+1)$. These inequalities are obvious if the boxes in question are in the same strip. Suppose they are not. Then the first claim follows from the fact that the paths are nonintersecting. To see this, suppose that the step starting at line $x = c$ in path $i$ starts at height $t$. If this step is horizontal, $T(l, j) = t$ (resp. $\overline{t}$), and the step starting at line $x = c - 1$ but in path $i + 1$ must end at height $\overline{t}$ (resp. $t + 1$) or higher to avoid intersection, implying $T(l + 1, j) \geq \overline{t}$ (resp. $t + 1$). If this step is diagonal, then the box $(l+1, j)$ must be in the same strip as $(l, j)$, and so column strictness is guaranteed by the conditions internal to a path. The second claim follows again by the fact that the paths are nonintersecting. To see this, suppose that the step starting at line $x = c$ in path $i$ starts at height $t$ (resp. $\overline{t}$). If this step is horizontal, $T(l, j) = t$ (resp. $\overline{t}$), and the step starting at line $x = c + 1$ but in path $i + 1$ must start at height $\overline{t}$ (resp. $t + 1$) or higher, implying $T(l, j + 1) \geq t$ (resp. $\overline{t}$). If the step is diagonal, $T(l, j) = \overline{t - 1}$ (resp. $t$), and the step starting at line $x = c+1$ in path $i+1$ must start at height $t$ (resp. $\overline{t}$) or higher, implying $T(l, j + 1) \geq \overline{t - 1}$ (resp. $t$).

We must verify that both the individual strips and the entire tableau are symplectic. In both cases this follows from the left boundary and from the content-based nature both of the symplectic condition and the lattice path environment. The left boundary effectively implies that a step between lines $x = -c$ and $x = -c + 1$ must occur at a height of $c + 1$ or higher, *i.e.* the corresponding box of content $-c$ must contain an integer greater than or equal to $c + 1$—the symplectic condition. This makes the situation clear for entire tableaux and also for individual strips, if we make this additional proviso: strips $\theta_i$ and $\theta_j$ for all $i$ and $j$ are to be considered as retaining their original contents (and passing them on to $\theta_i \# \theta_j$) and are not to be reinitialized with content 0 for the upper left hand corner, so that

$$\begin{array}{|c|}\hline 1 \\ \hline \bar{1} \\ \hline \end{array}$$

can still satisfy the symplectic condition if the contents of the boxes are, say, 2 and 1 respectively and not 0 and $-1$. The same content-intact provision has also been used previously in the case of factorial Schur functions, symmetric functions whose variables are modified by content. See Hamel [9].

For each horizontal or diagonal step that ends at $(i, j)$, we choose a weight of $x_j$. For each horizontal or diagonal step that ends at $(i, \bar{j})$, we choose a weight of $x_j^{-1}$. For each up-vertical or down-vertical step, regardless of position, we choose a weight of one. Since there is a one-to-one correspondence between lattice paths and symplectic skew tableaux whose shape is a strip, the generating function for these lattice paths is the symplectic Schur function for the shape of a strip.

The proof now follows by the well-known Gessel-Viennot lattice path procedure. To obtain the full generality we require, we invoke the broader result of Stembridge [26, Theorem 1.2]. To do this we must insure that the only $m$-tuples of nonintersecting paths from starting points $P_1, \ldots, P_m$ to ending points $Q_1, \ldots, Q_m$ must connect $P_i$ to $Q_i$ for $i = 1, \ldots, m$; however, this is routine. Note that the introduction of a left boundary does not interfere with the intersecting/nonintersecting properties of the lattice paths. As has been demonstrated in Stembridge [26], the underlying structure does not have to be a lattice at all, but may be as general a structure as an acyclic digraph. Note additionally that although Stembridge does not impose conditions on which steps may follow each other (as we do in this proof), his theorem is still applicable for the step restrictions actually serve to define the digraph in which Stembridge's theorem is set (*i.e.* in the digraph, vertices are all integer lattice points, and edges are defined as follows: if boxes of content $d$ are approached from the left, there are up-vertical edges from $(d, k)$ to $(d, \bar{k})$ and from $(d, \bar{k})$ to $(d, k + 1)$, and there are horizontal edges from $(d, k)$ to $(d + 1, k)$ and $(d, \bar{k})$ to $(d + 1, \bar{k})$ for all $k$ and $\bar{k}$; if boxes of content $d$ are approached from below there are down-vertical edges from $(d, k)$ to $(d, \overline{k-1})$ and $(d, \bar{k})$ to $(d, k)$, and diagonal edges from $(d, \bar{k})$ to $(d + 1, k)$ and $(d, k)$ to $(d + 1, \overline{k-1})$). ∎

We now present two corollaries to Theorem 3.1. One is an identity involving a determinantal form which has appeared previously in the literature; the other is a version of Theorem 3.1 for odd symplectic groups.

The literature contains some determinantal forms for $\mathrm{Sp}(2n)$, although the subject does not appear to be as well-developed as for the classical Schur functions. There

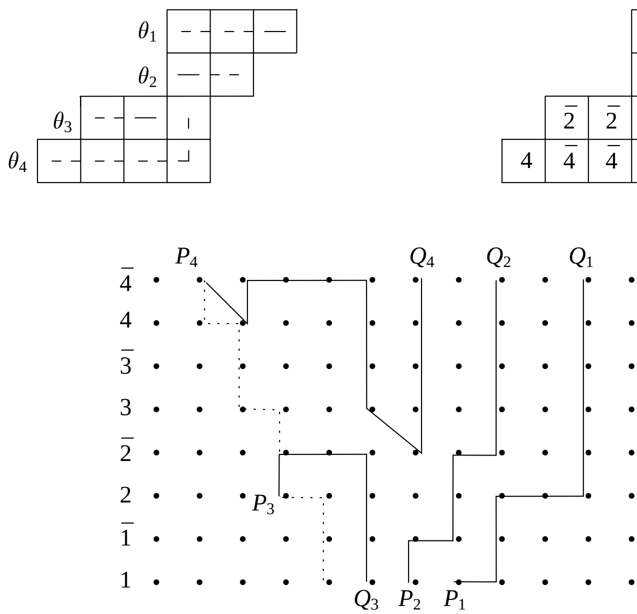


FIGURE 4: An outside decomposition of a skew symplectic tableau and the associated 4-tuple of lattice paths.

are bideterminantal forms dating back to Weyl [29] and Littlewood [15] and also more recent results due to Proctor [22]. Determinantal results in which each matrix element is expressed as a difference of symmetric functions can also be found in King [12], El Samra and King [5], Koike and Terada [14], Sagan [23], Stembridge [26] and Proctor [21] [22]. In addition El Samra and King [5] give a determinant which is a special case of Theorem 3.1 above for an outside decomposition into hooks (a Giambelli-type result, see Macdonald [16, p. 30]).

COROLLARY 3.3 (EL SAMRA AND KING [5]). *Let $\lambda$ be a partition. Then*

$$\mathrm{sp}_\lambda = \det\bigl(\mathrm{sp}_{\lambda_i - i+1, 1^{\lambda'_j - j}}(X)\bigr).$$

PROOF. Theorem 3.1 with outside decomposition $\theta_1 = \lambda_1$, $\theta_2 = 1^{\lambda'_1 - 1}$, $\theta_3 = \lambda_2 - 1$, $\theta_4 = 1^{\lambda'_2 - 2}, \ldots, \theta_{2r-1} = \lambda_r - r + 1$, $\theta_{2r} = 1^{\lambda'_r - r}$ where there are $r$ boxes on the main diagonal of $\lambda$ (*i.e.* $r$ boxes of content 0). ∎

A second corollary to Theorem 3.1 concerns the odd symplectic groups as defined by Proctor [19]. The odd symplectic tableaux are an easy generalization of the symplectic tableaux we defined in Section 1. We generalize Proctor's original definition to skew shape.

DEFINITION 3.4. An odd symplectic tableau, $\mathrm{SPO}_{\lambda/\mu}$, of shape $\lambda/\mu$ is a filling of the Ferrers diagram of $\lambda/\mu$ with the integers $1 < \bar{1} < 2 < \bar{2} \cdots < n < \bar{n} < n+1$ such that

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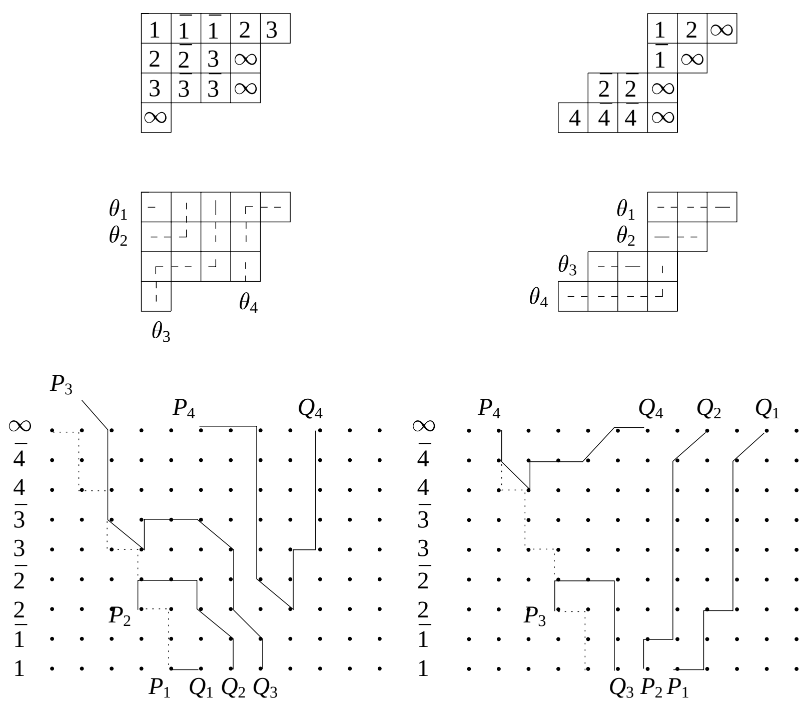

FIGURE 5: Outside decompositions of two so-tableaux and the corresponding paths.

1. the entries are weakly increasing along the rows and strictly increasing down the columns,
2. the boxes of content $-i$ contain entries which are greater than or equal to $i+1$.

We can define a Schur-type function for these tableaux. The odd skew symplectic Schur function, $\mathrm{spo}_{\lambda/\mu}(X)$, in the variables, $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}, x_{n+1}$, is given by

$$\mathrm{spo}_{\lambda/\mu}(X) = \sum_{\mathrm{SPO}_{\lambda/\mu}} \prod_{\alpha \in \mathrm{SPO}_{\lambda/\mu}} x_\alpha^{m(\alpha)} \prod_{\bar{\beta} \in \mathrm{SPO}_{\lambda/\mu}} x_\beta^{-m(\bar{\beta})},$$

where the sum is over all odd symplectic tableaux $\mathrm{SPO}_{\lambda/\mu}$ of shape $\lambda/\mu$, the first product is over all unbarred integers $\alpha$ in $\mathrm{SPO}_{\lambda/\mu}$, the second product is over all barred integers $\bar{\beta}$ in $\mathrm{SPO}_{\lambda/\mu}$, and $m(\alpha)$ (resp. $m(\bar{\beta})$) is the multiplicity of $\alpha$ (resp. $\bar{\beta}$) in $\mathrm{SPO}_{\lambda/\mu}$.

COROLLARY 3.5. *Let $\lambda/\mu$ be a skew shape partition. Then for any outside decomposition, $(\theta_1, \theta_2, \ldots, \theta_m)$, of $\lambda/\mu$,*

$$\mathrm{spo}_{\lambda/\mu}(X) = \det\bigl(\mathrm{spo}_{\theta_i \# \theta_j}(X)\bigr)_{m\times m},$$

*where* $\mathrm{spo}_{\emptyset} = 1$ *and* $\mathrm{spo}_{\text{undefined}} = 0$.

PROOF. Use the same lattice path set-up as for Theorem 3.1 with the $y$-axis labeled $1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}, n+1$. Then Stembridge's generalization [26] of the Gessel-Viennot lattice path argument provides the proof, as in Theorem 3.1. ∎

The proof of Theorem 3.2 is quite similar to that of Theorem 3.1. Since an so-tableau consists of a symplectic tableau adjoined to a (possible discontinuous) strip filled with $\infty$'s, the only difference between the proofs of Theorem 3.2 and Theorem 3.1 will be accounting for the presence of $\infty$. The $\infty$ has special characteristics which distinguish it from the integers filling the so-tableau. It is permitted to appear more than once in the same column but not more than once in each row. This can be translated as "$\infty$ is inserted row strictly and column weakly." Hence an so-tableau has two types of entries, those inserted row weakly and column strictly, and those inserted row strictly and column weakly. But this is precisely the arrangement for *supersymmetric tableaux*, tableaux which contain $1, 2, \ldots$ forming a row weak, column strict "inside shape," and $1', 2', \ldots$ forming a row strict, column weak "outside shape." These tableaux can be weighted by $x_i$ for each entry $i$ and $y_i$ for each entry $i'$, and supersymmetric Schur functions can be defined using this weighting. Results similar to Theorems 3.1 and 3.2 but for supersymmetric Schur functions can also be obtained in two different ways, either indirectly by replacing $\{x_1, x_2, \ldots\}$ by $\{x_1, x_2, \ldots, y_1, y_2, \ldots\}$ in the main result of Hamel and Goulden [8] and applying $\omega_y$ where $\omega_y$ is the operator $\omega_y s_\lambda(Y) = s_{\lambda'}(Y)$, or directly using lattice paths as has been done in Hamel [10].

PROOF OF THEOREM 3.2:. Label the $y$-axis with $1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}, \infty$. We call heights corresponding to any one of $1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}$ *integer levels*. Fix a left boundary as in Theorem 3.1. Define lattice paths with five types of permissible steps—the four as in Theorem 3.1, and up-diagonal steps from height $\overline{n}$ to height $\infty$ that increase the $x$-coordinate by 1 and increase the $y$-coordinate by 1. We distinguish between horizontal steps at integer levels and horizontal steps at $\infty$. The steps are subject to same restrictions as in Theorem 3.1 plus the following additional restrictions: an up-vertical step must not precede a horizontal step at a $\infty$ level, and a down-vertical step must not precede an up-diagonal step. We also require that all steps between lines $x = c$ and $x = c + 1$ for all $c$ are either 1) horizontal at $\infty$ or down-diagonal, or 2) horizontal at integer levels or up-diagonal. The determination of whether the steps are of type 1) or 2) is made by the outside decomposition: if boxes of content $d$ are approached from the left, then steps between $x = d$ and $x = d+1$ must be of type 2); if the boxes of content $d$ are approached from below, then steps between $x = d$ and $x = d + 1$ must be of type 1). Fix starting points and ending points as in Theorem 3.1 with the adjustment that the $y$-coordinate of the highest points is $\infty + 1$ instead of $\infty$ (this is so there is no conflict with the $\infty$ used here as a symbol). Given an so-tableau of shape $\lambda/\mu$ with an outside decomposition, we can construct an $m$-tuple of nonintersecting lattice paths. For each strip construct a path as follows: if a box contains $i$ or $\overline{i}$, place a step as in the proof of Theorem 3.1. If a box contains $\infty$, is at coordinates $(a, b)$ in the diagram, and is approached from the left in the strip, put an up-diagonal step from $(a-b, \overline{n})$ to $(a-b+1, \infty)$; if it is approached from

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below, put a horizontal step from $(a-b,\infty)$ to $(a-b+1,\infty)$. Connect these steps with vertical steps. It is routine to verify that there is a unique way of doing this.

We verify that an intersecting $m$-tuple of lattice paths does not correspond to an so-tableau. This can be verified by a case-by-case analysis as in Theorem 3.1. Precise details on cases for lattice paths with the five distinct types of steps used here can be found in Hamel [10] where decomposition results for the Schur $Q$-functions and supersymmetric functions are proved.

The construction described above for generating paths given so-tableaux is reversible, and now we verify that a nonintersecting $m$-tuple of lattice paths obeying these conditions corresponds to an so-tableau with the given outside decomposition. We begin by ensuring that a lattice path that starts at $P_j$ and ends at $Q_i$ corresponds to the strip $\theta_i \# \theta_j$. The proof is as follows. Begin with the empty partition. At iteration $k$, if the $k$-th nonvertical step from the left is horizontal or down-diagonal, proceed as in Theorem 3.1. If it is horizontal ending at $(i,\infty)$, then place a box containing $\infty$ on top of the previous box. If it is up-diagonal ending at $(i,\infty)$, then place a box containing $\infty$ in the so-tableau beside the previous box. As in the proof of Theorem 3.1, the path restrictions ensure the entries in a row of the so-tableau are weakly increasing, and integer entries in a column of the so-tableau are strictly increasing. Since the so-tableau is built by placing boxes always to the right or on top, we know the shape is a strip. Moreover, since the starting and ending points come from $\theta_j$ and $\theta_i$, since boxes of the same content correspond to the same type of step, and since the # operation is based on boxes of the content, we know the strip is $\theta_i \# \theta_j$.

Let $T(l,j)$ denote the entry in box $(l,j)$ of the so-tableau. The inequalities $T(l,j) \leq T(l,j+1)$ and $T(l,j) < T(l+1,j)$ for $T(l,j)$ integer (row weakness and column strictness) follow from the arguments in the proof of Theorem 3.1 and from the fact $\infty$ is greater than $1,\bar{1},2,\bar{2},\ldots,n,\bar{n}$. Now consider the case where $T(l,j)$ is $\infty$. We must show $T(l,j+1)$ does not exist and $T(l,j)=T(l+1,j)$. These assertions are obvious if the boxes in question are in the same strip, so suppose they are not. Consider $T(l,j+1)$. Suppose the step in path $i$ starting at line $x=c$ and representing $T(l,j)$ ends at height $\infty$. Then the step starting at line $x=c+1$ but in path $i+1$ must start at a height higher than $\infty$ to avoid intersection. This is impossible and hence $T(l,j+1)$ does not exist. Consider now $T(l+1,j)$. Suppose again the step in path $i$ starting at line $x=c$ and representing $T(l,j)$ ends at height $\infty$. If this step is horizontal, the step starting at line $x=c-1$ but in path $i+1$ must end at a height higher than $\infty$ to avoid intersection, implying $T(l+1,j)$ does not exist. If this step is up-diagonal, the step starting at the line $x=c-1$ but in path $i+1$ must end at height $\infty$ and $T(l+1,j)=\infty$.

The verification that the symplectic condition is satisfied for these so-tableaux follows from the same argument as in Theorem 3.1.

For each horizontal or diagonal step that ends at $(i,j)$, we choose a weight of $x_j$. For each horizontal or diagonal step that ends at $(i,\bar{j})$, we choose a weight of $x_j^{-1}$. For each horizontal or diagonal step that ends at $(i,\infty)$, choose a weight of one. For each up-vertical or down-vertical step, regardless of position, we choose a weight of one. Since

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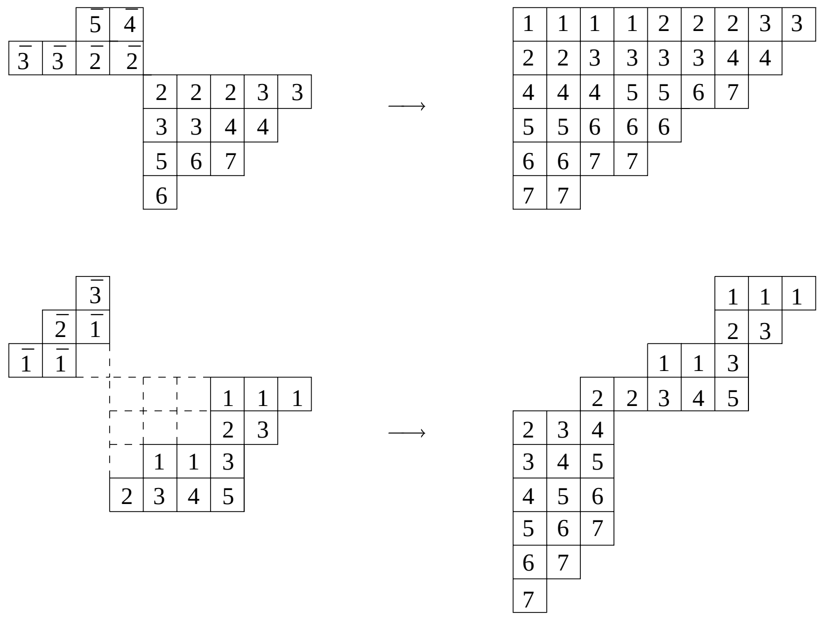

FIGURE 6: Example of rational tableaux and their complements.

there is a one-to-one correspondence between lattice paths and skew so-tableaux whose shape is a strip, the generating function for these lattice paths is the so-Schur function for the shape of a strip.

The proof now follows as in Theorem 3.1 by Stembridge's generalization of Gessel-Viennot [26, Theorem 1.2]. ■

4. **Rational tableaux.** This final section gives determinantal results for rational Schur functions (also called *composite Schur functions*). The tableaux underlying these functions are *rational tableaux* defined originally for standard shape by King [11]. We take a modified version due to Stembridge [25]. First, however, we define a new type of shape. A Ferrers diagram of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$ is defined as follows. Take the Ferrers diagram of $\nu/\rho$ and reflect it first about a vertical axis along its left perimeter and then about a horizontal axis along its top perimeter. Place it to the left of the Ferrers diagram of $\lambda/\mu$ such that the content zero boxes form a continuous diagonal. See the diagrams on the left side of Figure 6.

$$
\begin{array}{cccccccccc}
 & & & & \overline{8} & & & & & \\
 & & \overline{8} & \overline{8} & \overline{7} & & & & & \\
 & \overline{5} & \overline{4} & \overline{4} & \overline{4} & & & & & \\
\overline{4} & \overline{3} & \overline{3} & \overline{2} & \overline{2} & & & & & \\
 & & & & & 2 & 2 & 3 & 3 & 4 \\
 & & & & & 4 & 4 & 4 & 5 & \\
 & & & & & 7 & 8 & 8 & & \\
 & & & & & 8 & & & &
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccc}
1 & 1 & \overline{1} & \overline{1} & 2 \\
2 & 2 & 2 & \overline{2} & \\
\overline{3} & 4 & 4 & & \\
4 & & & &
\end{array}
$$

FIGURE 7: Transformation from rational to symplectic tableau.

DEFINITION 4.1. A rational tableau, $T_{\overline{\nu}/\overline{\rho};\lambda/\mu}$, of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$, where we let $T_{\overline{\nu}/\overline{\rho}}$ denote the $\overline{\nu}/\overline{\rho}$ portion and $T_{\lambda/\mu}$ denote the $\lambda/\mu$ portion and where we let $T_{\overline{\nu}/\overline{\rho}}(i,j)$ (resp. $T_{\lambda/\mu}(i,j)$) denote the entry in box $(i,j)$ of $T_{\overline{\nu}/\overline{\rho}}$ (resp. $T_{\lambda/\mu}$), is a filling of the Ferrers diagram of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$ such that

1. $T_{\overline{\nu}/\overline{\rho}}$ is filled with integers from $\overline{1} < \overline{2} < \cdots < \overline{n}$.
2. $T_{\lambda/\mu}$ is filled with integers from $1 < 2 < \cdots < n$.
3. The entries in $T_{\overline{\nu}/\overline{\rho}}$ strictly decrease in the columns and weakly decrease in the rows.
4. The entries in $T_{\lambda/\mu}$ strictly increase in the columns and weakly increase in the rows.
5. $|\{\overline{j} : T_{\overline{\nu}/\overline{\rho}}(j,1) \leq \overline{i}| + |\{j : T_{\lambda/\mu}(j,1) \leq i\}| \leq i$ for $1 \leq i \leq n$.

The skew rational Schur function, $s_{\overline{\nu}/\overline{\rho};\lambda/\mu}(X)$, in the variables, $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$, is given by

$$
s_{\overline{\nu}/\overline{\rho};\lambda/\mu}(X) = \sum_{T_{\overline{\nu}/\overline{\rho};\lambda/\mu}} \prod_{\alpha \in T_{\lambda/\mu}} x_\alpha^{m(\alpha)} \prod_{\overline{\beta} \in T_{\overline{\nu}/\overline{\rho}}} x_\beta^{-m(\overline{\beta})},
$$

where the sum is over all tableaux $T_{\overline{\nu}/\overline{\rho};\lambda/\mu}$ of shape $\overline{\nu}/\overline{\rho};\lambda/\mu$, the first product is over all unbarred integers $\alpha$ in $T_{\lambda/\mu}$, the second product is over all barred integers $\overline{\beta}$ in $T_{\overline{\nu}/\overline{\rho}}$, and $m(\alpha)$ (resp. $m(\overline{\beta})$) is the multiplicity of $\alpha$ (resp. $\overline{\beta}$) in $T_{\overline{\nu}/\overline{\rho};\lambda/\mu}$.

As mentioned in Section 1, the standard shape symplectic tableaux are actually a special case of the standard shape rational tableaux. This correspondence has been outlined by Stembridge [25] and proceeds as follows. Let $T_{\overline{\lambda};\lambda}$ be a standard shape rational tableau such that, if we ignore the bars on elements in $T_{\overline{\lambda}}$ and rotate $\overline{\lambda}$ to the same orientation as $\lambda$, then $T_{\overline{\lambda}} = T_\lambda$, and such that the largest entry is less than or equal to $2n+1$. Let $T'_\lambda$ be the tableau obtained from $T_\lambda$ by replacing $2 < 3 < \cdots < 2n < 2n+1$ by $1 < \overline{1} < \cdots < n < \overline{n}$ (note $T_\lambda$ will not contain 1 because of restriction 5 in

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Definition 4.1). Then $T'_\lambda$ satisfies the symplectic condition and is a symplectic tableau. See Figure 7.

Rational tableaux are also related to ordinary tableaux, and a rational tableau gives rise to an ordinary tableau in the following manner. Take the complement in $1, 2, \ldots, n$ of the entries in each column of $T_{\overline{\nu}/\overline{\rho}}$ and place these complements to the left of the columns in $T_{\lambda/\mu}$ such that the resulting diagram has shape $\gamma/\alpha$, where it is most natural to define $\gamma/\alpha$ in terms of columns:

$$\gamma'_i = \begin{cases} n - \nu'_{\nu_1-i+1} + \mu'_1 + \rho'_1 - \rho'_{\nu_1-i+1} & 1 \le i \le \nu_1 \\ \lambda'_{i-\nu_1} & \nu_1 + 1 \le i \le \nu_1 + \lambda_1 \end{cases}$$
$$\alpha'_i = \begin{cases} \mu'_1 + \rho'_1 - \rho'_{\nu_1-i+1} & 1 \le i \le \nu_1 \\ \mu'_{i-\nu_1} & \nu_1 + 1 \le i \le \nu_1 + \lambda_1. \end{cases}$$

Call the tableau of shape $\gamma/\alpha$ the *complement* of the tableau of shape $\overline{\nu}/\overline{\rho}; \lambda/\mu$. See Figure 6.

It is then obvious that, for $n \ge \lambda'_1 + \nu'_1$,

$$(2) \qquad (x_1 x_2 \cdots x_n)^{\nu_1} s_{\overline{\nu}/\overline{\rho};\lambda/\mu}(X) = s_{\gamma/\alpha}(X),$$

where $s_{\gamma/\alpha}(X)$ is the ordinary skew Schur function of shape $\gamma/\alpha$ as defined in Macdonald [16] or Sagan [24].

A special case of (2) is

$$(3) \qquad (x_1 \cdots x_n) s_{\overline{1^k}}(X) = s_{1^{n-k}}(X).$$

We can use (2) and (3) and the main result of Hamel and Goulden [8] to prove a determinantal result, Theorem 4.3, for rational Schur functions. However, this result does not apply to all outside decompositions, but only to those having the form described in the next definition (see Figure 8). The restriction to such outside decompositions is a restriction necessitated by a transformation performed in the proof, and it is likely that a more general form of Theorem 4.3—one as general as Theorems 3.1 and 3.2—can be proved.

DEFINITION 4.2. A *columns-first outside decomposition* $(\theta_1, \ldots, \theta_m)$ of shape $\overline{\nu}/\overline{\rho}$; $\lambda/\mu$ is an outside decomposition such that $\theta_1 = \nu'_{\nu_1} - \rho'_{\nu_1}, \theta_2 = \nu'_{\nu_1-1} - \rho'_{\nu_1-1}, \ldots, \theta_{\nu_1} = \nu'_1 - \rho'_1$ and such that $(\theta_{\nu_1+1}, \ldots, \theta_m)$ is an outside decomposition of $\lambda/\mu$ where the only strip allowed to start in the first column of $\lambda/\mu$ is the strip starting in the $(\lambda'_1, 1)$ box.

Given any columns-first outside decomposition of $\overline{\nu}/\overline{\rho}; \lambda/\mu$ there is a related outside decomposition, $\theta^c$, of the complement, $\gamma/\alpha$, where strips $\theta^c_i = 1^n/\theta_i$ for $1 \le i \le \nu_1$ and $\theta^c_i = \theta_i$ for $\nu_1 + 1 \le i \le m$.

THEOREM 4.3. *Let $\overline{\nu}/\overline{\rho}; \lambda/\mu$ be a shape. Then for any columns-first outside decomposition $(\theta_1, \ldots, \theta_m)$ of $\overline{\nu}/\overline{\rho}; \lambda/\mu$ and corresponding outside decomposition $(\theta^c_1, \ldots, \theta^c_m)$*

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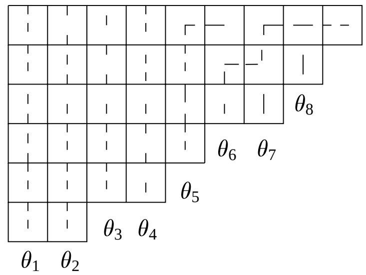


FIGURE 8: Example of a columns-first outside decomposition.

*of* $\gamma/\alpha$,

$$s_{\bar{\nu}/\bar{\rho};\lambda/\mu}(X) = \det\begin{pmatrix} \Big(s_{\overline{1^n/(\theta_i^c\,\#\,\theta_j^c)}}(X)\Big)_{1\le i\le\nu_1;1\le j\le\nu_1} & \vdots & \Big(s_{\theta_i^c\,\#\,\theta_j^c}(X)\Big)_{1\le i\le\nu_1,\nu_1+1\le j\le m} \\ \cdots & \vdots & \cdots \\ \Big(s_{\overline{1^n/(\theta_i^c\,\#\,\theta_j^c)'_1};(\theta_i^c\,\#\,\theta_j^c)/(\theta_i^c\,\#\,\theta_j^c)'_1}(X)\Big)_{\nu_1+1\le i\le m,1\le j\le\nu_1} & \vdots & \Big(s_{\theta_i\,\#\,\theta_j}(X)\Big)_{\nu_1+1\le i\le m,\nu_1+1\le j\le m} \end{pmatrix}$$

*where* $(\theta_i^c\,\#\,\theta_j^c)'_1$ *is the first column of* $\theta_i^c\,\#\,\theta_j^c$.

PROOF. From Theorem 3.1 of Hamel and Goulden [8], the following identity holds:

$$s_{\gamma/\alpha}(X) = \det\Big(s_{\theta_i\,\#\,\theta_j}(X)\Big). \tag{4}$$

Apply (2) to the left hand side of (4); apply (3) to each of the first $\nu_1$ columns on the right hand side of (4). The result follows. ∎

The determinants (for $n = 7$) corresponding to the outside decomposition in Figure 8 are as follows (note $s_{1^k} = 0$ for $k > n$).

$$s_{987542}(X) = \det\begin{pmatrix} s_{111111} & s_{11111} & s_{111} & s_{11} & 1 & 0 & 0 & 0 \\ s_{1111111} & s_{111111} & s_{1111} & s_{111} & s_1 & 0 & 0 & 0 \\ 0 & s_{1111111} & s_{11111} & s_{1111} & s_{11} & 1 & 0 & 0 \\ 0 & 0 & s_{111111} & s_{11111} & s_{111} & s_1 & 1 & 0 \\ 0 & 0 & s_{2111111} & s_{211111} & s_{2111} & s_{21} & s_2 & 0 \\ 0 & 0 & s_{42111111/1} & s_{4211111/1} & s_{42111/1} & s_{421/1} & s_{42/1} & s_3 \\ 0 & 0 & s_{1111111} & s_{111111} & s_{1111} & s_{11} & s_1 & 0 \\ 0 & 0 & 0 & 0 & s_{22111/1} & s_{221/1} & s_{22/1} & s_1 \end{pmatrix}$$

$$s_{\overline{42};5431}(X) = \det \begin{pmatrix} s_{\bar{1}} & s_{\overline{11}} & s_{\overline{1111}} & s_{\overline{11111}} & 1 & 0 & 0 & 0 \\ 1 & s_{\bar{1}} & s_{\overline{111}} & s_{\overline{1111}} & s_1 & 0 & 0 & 0 \\ 0 & 1 & s_{\overline{11}} & s_{\overline{111}} & s_{11} & 1 & 0 & 0 \\ 0 & 0 & s_{\bar{1}} & s_{\overline{11}} & s_{111} & s_1 & 1 & 0 \\ 0 & 0 & s_{\emptyset;1} & s_{\bar{1};1} & s_{2111} & s_{21} & s_2 & 0 \\ 0 & 0 & s_{\emptyset;31} & s_{\bar{1};31} & s_{42111/1} & s_{421/1} & s_{42/1} & s_3 \\ 0 & 0 & 1 & s_{\bar{1}} & s_{1111} & s_{11} & s_1 & 0 \\ 0 & 0 & 0 & 0 & s_{22111/1} & s_{221/1} & s_{22/1} & s_1 \end{pmatrix}$$

The following corollary due to Balankentin and Bars [1] for standard shape has been proved by Cummins and King [4] using the same complementing transformation technique as in the proof of Theorem 4.3.

COROLLARY 4.4. *Let $\bar{\nu};\lambda$ be a shape. Then*

$$s_{\bar{\nu};\lambda}(X)$$
$$= \det\Big(\big(e_{\nu'_{\nu_1-j+1}-j+i}(X)\big)_{1\le i\le \nu_1+\lambda_1,1\le j\le \nu_1} \vdots \big(e_{\lambda'_{j-\nu_1}-j+\nu_1+i}(X)\big)_{1\le i\le \nu_1+\lambda,\nu_1+1\le j\le \nu_1+\lambda_1}\Big),$$

*where e is the elementary symmetric function (i.e. $e_k = s_{1^k}$).*

PROOF. Theorem 4.3 with outside decomposition $\theta_i = \nu'_i$ for $1 \le i \le \nu_1$ and $\theta_i = \lambda'_{\nu_1-i}$ for $\nu_1 + 1 \le i \le \lambda_1 + \nu_1$. ∎

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