FORUM

Error Distributions in Navigation

J. B. Parker

GRADUATING error distributions by families of curves has a long and distinguished history. In seeking a class of frequency distributions which graduate navigational data, Anderson and Ellis[1] are motivated by the well-known shortcomings of the normal distribution which so often fails to do justice to the data in the tails of the distribution. They generalize the one parameter (σ) zero-mean gaussian family to a two parameter (α, β) family which is in fact the Pearson Type VII class.[2] They then observe that this class graduates published navigational distributions very well.

Leaving for the moment the debatable point of whether it is desirable to make the attempt to graduate all navigational distributions by means of a two-parameter family, and if so, what for, there are at least three desiderata against which a suggested family of frequency distributions may be judged:

(1) Does the family fit existing data well?
(2) Is the structure of the family explicable on mathematical or logical grounds?
(3) Is the family easy to handle in practice?

Anderson and Ellis show that the answer in question (1) is an affirmative one and justify the structure by showing that their model can be constructed by assuming that the observations can be regarded as members of a whole series of gaussian distributions, of different σ, the σ in fact following the distribution described at the foot of p. 436 of reference 1. They note that the general form of $h(\sigma)$is intuitively satisfying. There are of course other general forms which are also intuitively satisfying, in particular the form

$$h^*(\sigma) = \frac{2\beta^\alpha}{\Gamma(\alpha)}\sigma^{2\alpha-1}\exp(-\beta\sigma^2)$$

this leading, however, to a somewhat less tractable form for the authors' $p(x)$ (p. 437). So a fair answer to question (2) is an affirmative one, with the proviso that other structures are equally admissible. The basic idea of superposing, on the one-parameter gaussian family, a prior probability distribution of σ – the authors' $h(\sigma)$ is not a novel idea, being found, for example, on p. 527 of reference 3, where my $g(\sigma)$ is exactly analogous to Anderson and Ellis's $h(\sigma)$. A plethora of graduating schemes have in fact been suggested in previous articles in the *Journal*, one of the earliest being the double gaussian of Durst (ref. 4, p. 44); and there are other relevant articles.[5-7]

In discussing question (3) we are faced with a number of unanswered questions. First, given a mass of data how easy is it to calculate the appropriate α and β? In fact it is not very easy. Next, what are the errors associated with these calculated α and β? Both these questions can be addressed by conventional statistical techniques (for example α and β can be fitted by the method of maximum likelihood which also gives large sample estimators of their errors), but the equations are not straightforward ones. Their solution, however, is easy using modern computers.

Summing up, therefore, the family of curves suggested by Anderson and Ellis is reasonably attractive, but it is by no means clear that equally attractive systems could not have been built up by equally admissible procedures. A good feature of their system is that the gaussian distribution can be accommodated within it as a special case, as the authors remark on p. 439. The structure of their system however does not permit the accommodation of the exponential distribution described on p. 434, and this is a drawback in view of the prominence given to this particular distribution in Section 6 of the paper.

As a statistician I would prefer to graduate raw data for myself before making inferences, rather than accept that these should be graduated for me by means of a two-parameter family. The conclusion, which I believe is implicit in the paper, that navigational distributions can be nicely pigeon-holed by setting numbers to two parameters, α and β, is not of really great value unless clear advice is given about what to do with α and β once you've got them. As an old-fashioned sceptic, I would prefer to be given the data straight, possibly in the form of a table or a histogram. This would leave me free to fit an α and β if I chose, but to do something else if I preferred to reject the model in this instance. In applications where the data are really voluminous, there is a lot to be said for estimating upper percentage points (provided these are not too extreme) from the observations themselves without doing any graduating at all; depending on the problem, it may be possible further to calculate pessimistic values for these upper percentage points, an excellent example being given by Reich.[8]

Turning to a fresh point, it seems that the authors, in Section 6 of ref. 1, are hinting at the possibility of using an exponential distribution to model the frequency distribution of *position line errors*. This leads to repercussions which are difficult to reconcile with common sense. First, the arguments used to explain the deficiencies of the gaussian distribution in Section 5 of ref. 1, valid where many instruments and observers of different accuracies are involved, do not apply in an environment where a single navigator is obtaining two or three position lines during a relatively brief time interval using a single instrument (perhaps a sextant). There are no logical grounds therefore for favouring a negative exponential rather than a gaussian pattern in this application.[3] Students on advanced air navigation courses during the war may remember being told to work out thirty or forty ground shots and to plot the errors, getting a bell-shaped curve. While the whole mass of data (all students, all sextants, and all bodies) would not be expected to be gaussian, those for a single pupil would be expected to be not too far removed from normality and generally this was the case. This is the situation of para. 4 of ref. 1, not para. 5.

Secondly it is in fact not difficult to rewrite position finding theory[9,10] against the backcloth of negative exponential rather than gaussian error assumptions and some years ago I started to do this as an exercise. One change is that the usual error ellipse associated with a two-line fix[9] becomes a diamond; another is that the 'best' point to choose for a three-line fix is not in general some point in the middle of the cocked hat but one of the vertices (actually, in the case of equal position line errors, the one opposite the largest side). These conclusions are so repugnant to the common sense of the practical navigator that I abandoned my exercise. I do not in fact believe that a single observer's position line error is better approximated by an exponential rather than a gaussian distribution.

The first lesson of Anderson and Ellis's paper, an old one, but one that it is valuable to reassert strongly at all times, is that the gaussian distribution is inappropriate in a great many cases that occur in navigational practice.[11] It is the attempt to replace this distribution by a particular two-parameter family that I find irksome and restrictive. If the proposed form is put forward simply as an aid to understanding the structure of navigational distributions that do not confirm to the gaussian law, the size of the parameter α indicating the closeness or otherwise of the distribution to normal (large α—near normal; low α—much longer tailed, see Fig. 7 of ref. 1), it may help people interpret the nature of their data. But for general purposes I would favour, in navigational work, replacing the gaussian distribution, where it is patently inadequate, not by a multi-parameter family of distributions, however elegant, but by an open mind. Let the data speak for themselves, rather than subject them to a two-parameter strait jacket!

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Gaussian Logarithms and Navigation

D. H. Sadler

THE following comments on Captain C. H. Cotter's note (this *Journal*, Vol. 24, page 569) on the use of addition and subtraction logarithms in navigation may be of interest. Captain C. Carić was not the first to introduce gaussian logarithms to navigators, since their use was advocated, at least to Portuguese navigators, in

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