Bull. Aust. Math. Soc. **99** (2019), 1–9
doi:10.1017/S0004972718000734

NOTE ON THE NUMBER OF DIVISORS OF REDUCIBLE QUADRATIC POLYNOMIALS

ADRIAN W. DUDEK, ŁUKASZ PAŃKOWSKI✉ and VICTOR SCHARASCHKIN

(Received 15 May 2018; accepted 4 June 2018; first published online 15 August 2018)

Abstract

Lapkova ['On the average number of divisors of reducible quadratic polynomials', *J. Number Theory* **180** (2017), 710–729] uses a Tauberian theorem to derive an asymptotic formula for the divisor sum $\sum_{n\le x} d(n(n+v))$ where v is a fixed integer and $d(n)$ denotes the number of divisors of n. We reprove this result with additional terms in the asymptotic formula, by investigating the relationship between this divisor sum and the well-known sum $\sum_{n\le x} d(n)d(n+v)$.

2010 *Mathematics subject classification*: primary 11A25; secondary 11C08.

Keywords and phrases: divisor function, reducible quadratic polynomials, additive divisor problem.

1. Introduction

The problem of estimating the average number of divisors of a polynomial was first investigated in the middle of the last century. For example, Erdös [5] proved that for every irreducible polynomial $P(n)$ with integer coefficients

$$\sum_{n\le x} d(P(n)) \asymp x\log x,$$

where $d(n)$ counts positive divisors of n. The exact asymptotic formula for the sum $\sum_{n\le x} d(P(n))$ where $P(n)$ is a polynomial of degree greater than two is still unknown and seems to be a very difficult problem. However, the case of irreducible quadratic polynomials of degree two has been thoroughly investigated and it is known that

$$\sum_{n\le x} d(an^2+bn+c) \sim \lambda x\log x \tag{1.1}$$

for any irreducible polynomial ax^2+bx+c with integer coefficients, where λ depends on a, b, c. This is an unpublished result (mentioned by Bellman in [1]) due to Bellman and Shapiro, but the first published proof was given by Scourfield in [16]. For the

The second author was partially supported by the Grant no. 2016/23/D/ST1/01149 from the National Science Centre.

case $a = 1$, $b = 0$, the precise dependence of λ on a, b, c was described by Hooley in [8], and for other cases in the series of papers [12–14] by McKee. In general, λ depends on the class number of the quadratic field defined by P, and hence does not admit a completely elementary description.

Bellman [1] also mentioned that there is an unpublished result due to Bellman and Shapiro that (1.1) holds with $\log x$ replaced by $\log^2 x$ for reducible quadratic polynomials. The first published proof for $a = 1$, $b = 0$ and $c = -1$ was given by the first author [4], who proved that $\sum_{n\le x} d(n^2 - 1) \sim (6/\pi^2)x\log^2 x$. The approach is essentially based on the precise description of the function $\rho_{a,b,c}(n)$ (in the case where $a = 1$, $b = 0$ and $c = -1$) which denotes the number of solutions of the congruence $ax^2 + bx + c \equiv 0 \bmod n$ in $\mathbb{Z}_n$, and the fact, inspired by the approach suggested by Bellman in [1], that the left-hand side of (1.1) can be written as

$$2x\sum_{n\le x}\frac{\rho_{a,b,c}(n)}{n} + O\Big(\sum_{n\le x}\rho_{a,b,c}(n)\Big).$$

Very recently, this approach was extended by Lapkova in [11] for the polynomial $P(x) = (x - b)(x - c)$ with $b < c$. Since in this case the polynomial is reducible over $\mathbb{Z}$, it is reducible modulo p for every p and so $\rho_{a,b,c}(p^n) = 2$ for almost all p. As shown by Lapkova, in this case the constant λ in (1.1) does not depend on the coefficients of the given polynomial, and

$$\sum_{n\le x} d((n-b)(n-c)) \sim \frac{6}{\pi^2}x\log^2 x. \tag{1.2}$$

Moreover, by using a different method, Lapkova extended the recent result of Cipu and Trudgian [2] concerning the case $-b = c = 1$ and gave the explicit upper bound for the left-hand side of (1.2) with $-b = c = 4^s$, $s \ge 0$, where the fastest growing term is exactly $(6/\pi^2)x\log^2 x$ and agrees with (1.2). The explicit upper bounds for these kinds of sums with $b = -c$ are important in searching for $D(c^2)$-m-tuples, namely sets of positive integers $\{a_1, \ldots, a_m\}$ such that $a_ia_j + c^2$ is a perfect square for all i, j with $1 \le i < j \le m$.

In this paper, we will derive a more precise asymptotic formula for the sum $\sum_{n\le x} d((n - b)(n - c))$ and prove the following result. We use the standard notation, that is, γ denotes the Euler–Mascheroni constant, $\zeta(s)$ denotes the Riemann zeta-function, $\Lambda = \Lambda_1$ and μ respectively denote the classical von Mangoldt and Möbius multiplicative functions and

$$\Lambda_k(n) = \sum_{d\mid n}\mu(d)\Big(\log\frac{n}{d}\Big)^k.$$

THEOREM 1.1. *For every positive integer v and every $\varepsilon > 0$,*

$$\sum_{n\le x} d(n(n+v)) = \frac{6}{\pi^2}x(\log^2 x + A_1(v)\log x + A_2(v)) + O(x^{2/3+\varepsilon})$$

where

https://doi.org/10.1017/S0004972718000734 Published online by Cambridge University Press

$$A_1(v) = 4\gamma - 2 - 4\frac{\zeta'}{\zeta}(2) - 2\sum_{e|v}\frac{\Lambda(e)}{e}$$

$$A_2(v) = \Big(2\gamma - 1 - 2\frac{\zeta'}{\zeta}(2)\Big)^2 + 1 - 4\frac{\zeta''}{\zeta}(2) + 4\Big(\frac{\zeta'}{\zeta}(2)\Big)^2 - \Big(4\gamma - 2 - 4\frac{\zeta'}{\zeta}(2)\Big)\sum_{e|v}\frac{\Lambda(e)}{e} + 2\sum_{e|v}\frac{\Lambda(e)\log e}{e} + \sum_{e|v}\frac{\Lambda_2(e)}{e}.$$

To prove Theorem 1.1, we shall follow Hooley's suggestion (see [8]) to find the relation between the sums $\sum_{n\le x} d(n(n+v))$ and $\sum_{n\le x} d(n)d(n+v)$. The latter sum is well investigated and the problem of finding its asymptotic behaviour is known as the binary additive divisor problem. It was first studied by Ingham in relation to the fourth moment of the Riemann zeta function in [9]. Subsequently, in [10], Ingham proved that

$$\sum_{n\le x} d(n)d(n+v) = \frac{6}{\pi^2}\sigma_{-1}(v)x\log^2 x + O(x\log x),$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Ingham's result was improved by Estermann [6], who showed that

$$\sum_{n\le x} d(n)d(n+v) = \frac{6}{\pi^2}\sigma_{-1}(v)x(\log^2 x + c_1(v)\log x + c_2(v)) + O(x^{11/12}\log^3 x), \quad (1.3)$$

where

$$c_1(v) = 4\gamma - 2 - 4\frac{\zeta'}{\zeta}(2) - 4\frac{\sigma_{-1}^{(1)}}{\sigma_{-1}}(v)$$

$$c_2(v) = \Big(2\gamma - 1 - 2\frac{\zeta'}{\zeta}(2)\Big)^2 + 1 - 4\frac{\zeta''}{\zeta}(2) + 4\Big(\frac{\zeta'}{\zeta}(2)\Big)^2 - 2\Big(4\gamma - 2 - 4\frac{\zeta'}{\zeta}(2)\Big)\frac{\sigma_{-1}^{(1)}}{\sigma_{-1}}(v) + 4\frac{\sigma_{-1}^{(2)}}{\sigma_{-1}}(v)$$

and $\sigma_\alpha^{(k)}(n) = \sum_{d|n} d^\alpha \log^k d$. The best estimate for the error term in (1.3) is due to Deshouillers and Iwaniec [3], who showed that the error term is $O(x^{2/3+\varepsilon})$ for every $\varepsilon > 0$. This error term, through the method of this paper, appears in Theorem 1.1, because the proof of this theorem relies essentially on the following crucial lemma.

Lemma 1.2. *For every $v > 0$,*

$$\sum_{n\le x} d(n)d(n+v) = \sum_{e|v}\sum_{n\le x/e} d(n(n+v/e)) \quad (1.4)$$

and, in consequence,

$$\sum_{n\le x} d(n(n+v)) = \sum_{e|v}\mu(e)\sum_{n\le x/e} d(n)d(n+v/e). \quad (1.5)$$

https://doi.org/10.1017/S0004972718000734 Published online by Cambridge University Press

REMARK 1.3. Thus, as noted, our proof relies on results from the binary additive divisor problem. Similar problems about the self-correlations of arithmetic functions (such as the left-hand side of (1.4)) can be very difficult. For the Möbius function, Chowla conjectured $\sum_{n\le x}\mu(n+a_1)\cdots\mu(n+a_t)=o(x)$. The case $t=1$ is already equivalent to the Prime Number Theorem. Larger t values are related to the recent Möbius disjointness conjecture of Sarnak. For the von Mangoldt function, the asymptotic formula $\sum_{n\le x}\Lambda(n)\Lambda(n+2)\sim Ax$ is essentially the twin prime conjecture.

REMARK 1.4. There is no serious obstacle to make the implied constant in (1.3) explicit. One can easily obtain the following explicit upper bound

$$\sum_{n\le x} d(n(n+v)) \le \frac{6}{\pi^2}x(\log^2 x + A_1(v)\log x + A_2(v)) + A_3(v)x^{11/12}\log^3 x,$$

where the constants A_1 and A_2 are defined as before and the constant A_3 can be explicitly computed from Estermann's proof of (1.3). The above inequality improves known upper bounds in the sense that it holds for general polynomials and, what is more important, the first three leading terms agree with our asymptotic formula in Theorem 1.1, whereas known results (see [2, Lemma 5.2], [11, Theorem 3 and Corollary 4]) give an explicit upper bound where only the first leading term agrees with the asymptotic formula.

2. The proof of Lemma 1.2

First, notice that for every multiplicative function $f(n)$ and for all integers a, b,

$$f(a)f(b) = f(\gcd(a,b))f(\mathrm{lcm}(a,b)).$$

To see this, it suffices to consider the case when $a = p^\alpha$, $b = p^\beta$ and then use the fact that $\{\min(\alpha,\beta),\max(\alpha,\beta)\} = \{\alpha,\beta\}$.

Now let us assume that $f(n)$ is a multiplicative function such that, for every prime p and every positive integer n,

$$f(p^{n+1}) = f(p)f(p^n) - f(p^{n-1}). \tag{2.1}$$

We now prove by induction on α that for such a multiplicative function,

$$\sum_{m=0}^{\beta} f(p^{\alpha+\beta-2m}) = f(p^\alpha)f(p^\beta) \tag{2.2}$$

for all integers α,β with $0\le\beta\le\alpha$. If $\alpha=0$, then $\beta=0$ and (2.2) holds trivially. If $\alpha=1$, then $\beta=0$ or $\beta=1$. In the former case, (2.2) holds trivially, whereas the latter case needs (2.1) with $n=2$.

Now, let us assume that (2.2) holds for $\alpha\le A$ and all nonnegative integers $\beta\le\alpha$, and consider $\alpha=A+1$. If $\beta\le A-1$, then our assertion is implied by using (2.1) twice along with our inductive hypothesis. So it remains to consider the cases $(\alpha,\beta)=(A+1,A)$ and $(\alpha,\beta)=(A+1,A+1)$. In the first case, it suffices to write the left-hand

https://doi.org/10.1017/S0004972718000734 Published online by Cambridge University Press

side of (2.2) as $f(p) + \sum_{m=0}^{A-1} f(p^{2A+1-2m})$ and apply (2.1). In the second case, we firstly apply (2.1) for $n = 1$ to write the left-hand side of (2.2) as $f(p)^2 + \sum_{m=0}^{A-1} f(p^{2A+2-2m})$, and then apply (2.1) again for $n = 2A + 1$.

Next, let us observe that for every multiplicative function satisfying (2.1),

$$f(a)f(b) = \sum_{e|\gcd(a,b)} f\Big(\frac{ab}{e^2}\Big).$$

Indeed, the above equation holds trivially when $\gcd(a, b) = 1$, so let us assume that $\gcd(a, b) = \prod_{j=1}^{k} p_j^{\alpha_j}$ for some positive integers α_j, and $ab = q \prod_{j=1}^{k} p_j^{\beta_j}$ for some $\beta_j \geq 2\alpha_j$ and some integer q coprime with the p_j. Then (2.2) gives

$$\begin{aligned}
\sum_{e|\gcd(a,b)} f\Big(\frac{ab}{e^2}\Big) &= f(q) \sum_{\substack{(a_1,\ldots,a_k)\in\mathbb{Z} \\ 0\leq a_j\leq\alpha_j}} \prod_{j=1}^{k} f(p_j^{\beta_j-2a_j}) = f(q) \prod_{j=1}^{k} \sum_{a_j=0}^{\alpha_j} f(p_j^{\beta_j-2a_j}) \\
&= f(q) \prod_{j=1}^{k} f(p_j^{\alpha_j}) f(p_j^{\beta_j-\alpha_j}) = f(\gcd(a, b)) f(\mathrm{lcm}(a, b)) \\
&= f(a)f(b).
\end{aligned}$$

Since the multiplicative function $d(n)$ satisfies (2.1), one obtains the following lemma.

Lemma 2.1. *Let v be a positive integer. Then*

$$d(n)d(n + v) = \sum_{e|\gcd(n,v)} d\Big(\frac{n(n + v)}{e^2}\Big).$$

Now we are ready to prove (1.4). Lemma 2.1 implies (1.4), since

$$\begin{aligned}
\sum_{f|v} \sum_{n\leq \frac{x}{f}} d\Big(n\Big(n + \frac{v}{f}\Big)\Big) &= \sum_{f|v} \sum_{\substack{n\leq x \\ f|n}} d\Big(\frac{n(n + v)}{f^2}\Big) \\
&= \sum_{e|v} \sum_{\substack{n\leq x \\ \gcd(n,v)=e}} \sum_{f|e} d\Big(\frac{n(n + v)}{f^2}\Big) \\
&= \sum_{e|v} \sum_{\substack{n\leq x \\ \gcd(n,v)=e}} d(n)d(n + v) \\
&= \sum_{n\leq x} d(n)d(n + v).
\end{aligned}$$

https://doi.org/10.1017/S0004972718000734 Published online by Cambridge University Press

On the other hand, one can easily deduce from (1.4) that

$$\begin{aligned}\sum_{e|v}\mu(e)\sum_{n\le x/e}d(n)d(n+v/e) &= \sum_{e|v}\mu(e)\sum_{f|v/e}\sum_{n\le x/ef}d(n(n+v/ef))\\ &= \sum_{e'|v}\sum_{n\le x/e'}d(n(n+v/e'))\sum_{e|e'}\mu(e)\\ &= \sum_{n\le x}d(n(n+v)).\end{aligned}$$

REMARK 2.2. The crucial property of a multiplicative function $f(n)$ for the above reasoning is that it satisfies (2.1). In the literature there are many well-known multiplicative functions satisfying the similar identity

$$f(p^{n+1}) = f(p)f(p^n) - g(p)f(p^{n-1}) \tag{2.3}$$

for a suitable completely multiplicative function g. Obviously, from our point of view, the case $g \equiv 0$ is not interesting as it implies that f is completely multiplicative, so let us assume that $g \not\equiv 0$. Then, for example, σ_α satisfies the above identity with $g(p) = p^\alpha$. Moreover, it was noticed by Ramanujan, and proved by Mordell [15], that Ramanujan's τ function satisfies this identity with $g(p) = p^{11}$, and more generally, (2.3) is true for normalised eigenforms of weight $2k$ with $g(p) = p^{2k-1}$.

Using a similar argument as above, one can easily show that for every multiplicative function $f(n)$ satisfying (2.3),

$$\sum_{n\le x} f(n)f(n+v) = \sum_{e|v} g(e)\sum_{n\le x/e} f(n(n+v/e))$$

and, in consequence, since every nonzero completely multiplicative function g is inverse to μg with respect to the Dirichlet convolution,

$$\sum_{n\le x} f(n(n+v)) = \sum_{e|v}\mu(e)g(e)\sum_{n\le x/e} f(n)f(n+v/e).$$

Hence, the asymptotic behaviour of $\sum_{n\le x} f(n(n+v))$ can be deduced from the behaviour of $\sum_{n\le x} f(n)f(n+v)$ and vice-versa. For example, one can easily deduce from [7] that for $\alpha > 0$,

$$\sum_{n\le x}\sigma_\alpha(n(n+v)) = \frac{1}{2\alpha+1}\frac{\zeta(\alpha+1)^2}{\zeta(2\alpha+2)}x^{2\alpha+1}\sum_{d|v}d^{-2\alpha-1}\sum_{e|d}\mu(e)e^\alpha + O(x^\omega\log^c x),$$

where $\omega = 2\alpha + 1 - \min(\alpha, 1)$ and $c = \begin{cases}0, & \alpha > 1,\\ 1, & \alpha < 1,\\ 2, & \alpha = 1.\end{cases}$

https://doi.org/10.1017/S0004972718000734 Published online by Cambridge University Press

3. The proof of Theorem 1.1

First let us note that

$$\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}^{(k)}\Big(\frac{v}{e}\Big)=\sum_{d\mid v}\frac{\Lambda_k(d)}{d} \tag{3.1}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}^{(k)}\Big(\frac{v}{e}\Big)(\log e)^{n-k}=\sum_{d\mid v}\frac{(\log d)^n}{d}\sum_{e\mid d}\mu(e)=\begin{cases}1, & n=0,\\ 0, & n\geq 1.\end{cases} \tag{3.2}$$

Note that (1.3) together with (1.5) and (3.2) for $n=0$ gives

$$\sum_{n\leq x}d(n(n+v))\sim\frac{6}{\pi^2}\sum_{e\mid v}\mu(e)\sigma_{-1}\Big(\frac{v}{e}\Big)\frac{x}{e}\log^2 x=\frac{6}{\pi^2}x\log^2 x.$$

Next, combining (1.3) with (1.5) yields

$$\begin{aligned}A_1(v)&=\frac{6}{\pi^2}\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}\Big(\frac{v}{e}\Big)\Big(c_1\Big(\frac{v}{e}\Big)-2\log e\Big)\\&=\frac{6}{\pi^2}\Big(4\gamma-2-4\frac{\zeta'}{\zeta}(2)-4\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}^{(1)}\Big(\frac{v}{e}\Big)-2\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}\Big(\frac{v}{e}\Big)\log e\Big).\end{aligned}$$

Thus, (3.1) and (3.2) for $n=1$ give

$$\begin{aligned}A_1(v)&=\frac{6}{\pi^2}\Big(4\gamma-2-4\frac{\zeta'}{\zeta}(2)-2\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}^{(1)}\Big(\frac{v}{e}\Big)\Big)\\&=\frac{6}{\pi^2}\Big(4\gamma-2-4\frac{\zeta'}{\zeta}(2)-2\sum_{e\mid v}\frac{\Lambda(e)}{e}\Big).\end{aligned}$$

Similarly, one can compute $A_2(v)$. First let us note that

$$\begin{aligned}A_2(v)&=\frac{6}{\pi^2}\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}\Big(\frac{v}{e}\Big)\Big(\log^2 e-c_1\Big(\frac{v}{e}\Big)\log e+c_2\Big(\frac{v}{e}\Big)\Big)\\&=\frac{6}{\pi^2}\Big(\Big(2\gamma-1-2\frac{\zeta'}{\zeta}(2)\Big)^2+1-4\frac{\zeta''}{\zeta}(2)+4\Big(\frac{\zeta'}{\zeta}(2)\Big)^2\\&\quad+\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}\Big(\frac{v}{e}\Big)\log^2 e+4\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}^{(1)}\Big(\frac{v}{e}\Big)\log e+4\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}^{(2)}\Big(\frac{v}{e}\Big)\\&\quad-\Big(4\gamma-2-4\frac{\zeta'}{\zeta}(2)\Big)\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}\Big(\frac{v}{e}\Big)\log e\\&\quad-2\Big(4\gamma-2-4\frac{\zeta'}{\zeta}(2)\Big)\sum_{e\mid v}\frac{\mu(e)}{e}\sigma_{-1}^{(1)}\Big(\frac{v}{e}\Big)\Big).\end{aligned}$$

https://doi.org/10.1017/S0004972718000734 Published online by Cambridge University Press

Then, as in the case of $A_1(v)$, we see that the last two summands give

$$-\Big(4\gamma - 2 - 4\frac{\zeta'}{\zeta}(2)\Big)\sum_{e|v}\frac{\Lambda(e)}{e}.$$

Finally, (3.2) for $n = 2$ together with (3.1) and the fact that

$$\sum_{e|v}\frac{\mu(e)}{e}\sigma_{-1}\Big(\frac{v}{e}\Big)\log^2 e + \sum_{e|v}\frac{\mu(e)}{e}\sigma_{-1}^{(1)}\Big(\frac{v}{e}\Big)\log e = -\sum_{e|v}\frac{\Lambda(e)\log e}{e}$$

completes the proof.

References

[1] R. Bellman, 'Ramanujan sums and the average value of arithmetic functions', *Duke Math. J.* **17** (1950), 159–168.

[2] M. Cipu and T. Trudgian, 'Searching for Diophantine quintuples', *Acta Arith.* **173**(4) (2016), 365–382.

[3] J.-M. Deshouillers and H. Iwaniec, 'An additive divisor problem', *J. Lond. Math. Soc. (2)* **26**(2) (1982), 1–14.

[4] A. W. Dudek, 'On the number of divisors of $n^2 - 1$', *Bull. Aust. Math. Soc.* **93**(2) (2016), 194–198.

[5] P. Erdös, 'On the sum $\sum_{k=1}^{x} d(f(k))$', *J. Lond. Math. Soc. (2)* **27** (1952), 7–15.

[6] T. Estermann, 'Über die darstellung einer zahl als differenz von zwei produkten', *J. reine angew. Math.* **164** (1931), 173–182.

[7] H. Halberstam, 'Four asymptotic formulae in the theory of numbers', *J. Lond. Math. Soc.* **24** (1949), 13–21.

[8] C. Hooley, 'On the number of divisors of quadratic polynomials', *Acta Math.* **110** (1963), 97–114.

[9] A. E. Ingham, 'Mean-value theorems in the theory of the Riemann zeta function', *Proc. Lond. Math. Soc. (2)* **27** (1926), 273–300.

[10] A. E. Ingham, 'Some asymptotic formulae in the theory of numbers', *J. Lond. Math. Soc.* **2** (1927), 202–208.

[11] K. Lapkova, 'On the average number of divisors of reducible quadratic polynomials', *J. Number Theory* **180** (2017), 710–729.

[12] J. McKee, 'On the average number of divisors of quadratic polynomials', *Math. Proc. Cambridge Philos. Soc.* **117** (1995), 389–392.

[13] J. McKee, 'A note on the number of divisors of quadratic polynomials', in: *Sieve Methods, Exponential Sums, and their Applications in Number Theory*, London Mathematical Society Lecture Note Series, 237 (Cambridge University Press, Cambridge, 1997), 275–281.

[14] J. McKee, 'The average number of divisors of an irreducible quadratic polynomial', *Math. Proc. Cambridge Philos. Soc.* **126**(1) (1999), 17–22.

[15] L. J. Mordell, 'On Mr. Ramanujan's empirical expansions of modular functions', *Proc. Cambridge Philos. Soc.* **19** (1917), 117–124.

[16] E. J. Scourfield, 'The divisors of a quadratic polynomial', *Proc. Glasg. Math. Soc.* **5** (1961), 8–20.

ADRIAN W. DUDEK, Cronulla NSW 2230, Australia
e-mail: awdudek@gmail.com

ŁUKASZ PAŃKOWSKI, Faculty of Mathematics and Computer Science,
Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
e-mail: lpan@amu.edu.pl

https://doi.org/10.1017/S0004972718000734 Published online by Cambridge University Press

VICTOR SCHARASCHKIN, Department of Mathematics,
University of Queensland, St Lucia, QLD 4072, Australia
e-mail: vscharaschkin@gmail.com