Adv. Appl. Prob. (SGSA) **44,** 1–20 (2012)
Printed in Northern Ireland

THE TRANSPARENT DEAD LEAVES MODEL

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Abstract

In this paper we introduce the transparent dead leaves (TDL) random field, a new germ–grain model in which the grains are combined according to a transparency principle. Informally, this model may be seen as the superposition of infinitely many semitransparent objects. It is therefore of interest in view of the modeling of natural images. Properties of this new model are established and a simulation algorithm is proposed. The main contribution of the paper is to establish a central limit theorem, showing that, when varying the transparency of the grain from opacity to total transparency, the TDL model ranges from the dead leaves model to a Gaussian random field.

Keywords: Germ–grain model; dead leaves model; transparency; occlusion; image modeling; texture modeling

2010 Mathematics Subject Classification: Primary 60D05
Secondary 60G60

1. Introduction

In this paper we deal with the stochastic modeling of physical transparency. The main contribution is the introduction and study of a new germ–grain model in which the grains are combined according to a transparency principle. To the best of the authors' knowledge, this type of interaction between grains has not been studied before. Classical interactions between grains include *addition* for shot noise processes [13], [22], *union* for Boolean models [24], [26], *occlusion* for dead leaves models [5], [14], [18], or *multiplication* for compound Poisson cascades [2], [8].

The proposed model, which we call the *transparent dead leaves* (TDL) model, is obtained from a collection of grains (random closed sets) indexed by time, as for the dead leaves model of G. Matheron. We assume that each grain is given a random gray level (intensity). Informally, the TDL model may be seen as the superposition of *transparent* objects associated with the grains. When adding a new grain, new values are obtained as a linear combination of former values and the intensity of the added grain, as illustrated in Figure 1. More precisely, the superposition of a transparent grain X with gray level a on an image (a function $f\colon \mathbb{R}^d \to \mathbb{R}$) results in a new image $\tilde{f}$, defined for each $y \in \mathbb{R}^d$ by

$$\tilde{f}(y) = \begin{cases} \alpha a + (1-\alpha) f(y) & \text{if } y \in X, \\ f(y) & \text{otherwise,} \end{cases} \tag{1}$$

where $\alpha \in (0, 1]$ is a transparency coefficient. The TDL model is defined as the sequential superposition of grains of a marked Poisson point process $\sum_i \delta_{(t_i, x_i, X_i, a_i)}$, with $\sum_i \delta_{(t_i, x_i)}$ a

Received 5 July 2010; revision received 23 August 2011.
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(a) A disc (b) Initial scene (c) Resulting scene

FIGURE 1: Addition of a transparent object. The transparency coefficient of the disc is $\alpha = 0.5$.

homogeneous Poisson point process in $(-\infty, 0) \times \mathbb{R}^d$, and X_i and a_i independent and identically distributed (i.i.d.) random sets and random variables, respectively. In particular, the value of the TDL at each point results from the superposition of infinitely many semitransparent objects.

The main motivation to define such a model originates from the modeling of image formation. Indeed, natural images are obtained from the light emitted by physical objects interacting in various ways. In the case of opaque objects, the main interaction is occlusion. That is, objects hide themselves depending on their respective positions with respect to the eye or the camera. A simple stochastic model for occlusion is given by the dead leaves model, which is therefore useful for the modeling of natural images [7], [11]. When objects are transparent, their interaction may be modeled by (1). This is well known in the field of computer graphics; see [9] where the same principle is used for the creation of synthetic scenes. In this case, transparency is a source of heavy computations, especially in cases where objects are numerous (typically of the order of several thousands), e.g. in the case of grass, fur, smoke, fabrics, etc. The transparency phenomenon may also be encountered in other imaging modality where images are obtained through successive reflection-transmission steps, as in microscopy or ultrasonic imaging. A related nonlinear image formation principle is at work in the field of radiography. In such cases, it is useful to rely on accurate stochastic texture models in order to be able to detect abnormal images. The TDL may be an interesting alternative to Gaussian fields that are traditionally used; see, e.g. [12] and [23]. A last motivation for the TDL is that, as explained in the next paragraph, it is intermediate between the dead leaves model and Gaussian fields, two models that have proven useful for the modeling of natural textures [7], [10].

In this paper, we first define the TDL in Section 2 and give some elementary properties in Section 3, where we also address the problem of simulating the model and show some realizations. The TDL covariance is then computed in Section 4. Eventually, the main result of the paper is stated and proved in Section 5, namely that the normalized TDLs converge, as the transparency coefficient α tends to 0, to a Gaussian random field having the same covariance function as the shot noise associated with the grain X and with intensity 1. Thus, the TDLs with varying transparency coefficient α provide us with a family of models ranging from the dead leaves model to Gaussian fields.

2. Definition of the TDL model

As explained in the introduction, the TDL model is obtained as the superposition of transparent shapes. Formally, it is defined from a marked Poisson point process, in a way similar to the dead leaves model [5]. Let $\mathcal{F}$ denote the set of closed subsets of $\mathbb{R}^d$. On the state space

$$S = (-\infty, 0) \times \mathbb{R}^d \times \mathcal{F} \times \mathbb{R},$$

equipped with its natural product σ-algebra, we define the point process

$$\Phi = \sum_i \delta_{(t_i, x_i, X_i, a_i)}, \tag{2}$$

where

- $\{(t_i, x_i)\}$ is a stationary Poisson point process of intensity 1 in the half-space $(-\infty, 0) \times \mathbb{R}^d$,
- $(X_i)_i$ is a sequence of i.i.d. random closed sets (RACSs) with distribution P_X which is independent of the other random objects,
- $(a_i)_i$ is a sequence of i.i.d. real random variables with distribution P_a which is also independent of the other random objects.

Equivalently, Φ is a Poisson point process with intensity measure $\mu = \lambda \otimes \nu_d \otimes P_X \otimes P_a$, where λ denotes the restriction of the one-dimensional Lebesgue measure to $(-\infty, 0)$ and ν_d denotes the d-dimensional Lebesgue measure on $\mathbb{R}^d$.

Each point (t_i, x_i, X_i, a_i) of the Poisson process Φ is called a *leaf*. Having fixed a transparency coefficient $\alpha \in (0, 1]$, the TDL process f is obtained by sequentially combining the elements of Φ according to (1), which results in the following definition.

Definition 1. (*TDL model.*) The TDL model with transparency coefficient α associated with the Poisson process Φ defined by (2) is the random field $f : \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(y) = \sum_{i \in \mathbb{N}} \mathbf{1}(y \in x_i + X_i)\alpha a_i (1 - \alpha)^{\sum_{j \in \mathbb{N}} \mathbf{1}(t_j \in (t_i, 0) \text{ and } y \in x_j + X_j)}. \tag{3}$$

Let us justify that (3) agrees with the informal description of the TDL model. According to (1), the impact of the leaf (t_i, x_i, X_i, a_i) is to add αa_i and to attenuate the previous contributions by a factor $1-\alpha$. Hence, the contribution of the leaf (t_i, x_i, X_i, a_i) at a point $y \in x_i + X_i$ is αa_i multiplied by $1 - \alpha$ to the number of leaves fallen on the point y after the leaf (t_i, x_i, X_i, a_i), that is, after time $t = t_i$. This number is precisely the exponent of $(1 - \alpha)$ in (3):

$$\sum_{j \in \mathbb{N}} \mathbf{1}(t_j \in (t_i, 0) \text{ and } y \in x_j + X_j).$$

Remark 1. (*Random functional.*) Denoting by $\mathcal{N}(S)$ the set of point processes taking values in the state space S, we remark that the TDL random field $f(y)$ has the form

$$f(y) = \sum_{(t_i, x_i, X_i, a_i) \in \Phi} g(y, (t_i, x_i, X_i, a_i), \Phi),$$

where $g : \mathbb{R}^d \times S \times \mathcal{N}(S) \to \mathbb{R}$ is a measurable function given by (3). Similar random functionals interpreted as the sum of contributions from each point of a (possibly marked) point process $\{x_i\} \subset \mathbb{R}^d$ appear in several contexts in stochastic geometry. In particular, general central limit theorems hold when the intensity of the point process $\{x_i\}$ tends to ∞; see, e.g. [3] and [21]. Note however that our framework is different since the Poisson process has an additional time component t_i, and, consequently, there is always an infinite number of leaves (t_i, x_i, X_i, a_i) influencing the value $f(y)$ (as will be clarified in Proposition 2).

Remark 2. (*Variable transparency.*) For the sake of simplicity, the transparency parameter α is assumed to be the same for all objects. However, one may attach a random transparency α_i to every object in Definition 1 and generalize the results of Sections 3 and 4, as will be briefly commented thereafter.

Since the distribution of the Poisson process Φ is invariant under shifts of the form

$$(t, x, X, a) \mapsto (t, x + y, X, a),$$

the TDL f is a strictly stationary random field.

Before establishing further properties of the TDL random field f, let us introduce some notation and specify several assumptions.

Notation. We define $\beta = 1 - \alpha$, and we respectively denote by X and a a RACS with distribution P_X and a random variable (RV) with distribution P_a which are both independent of all the other random objects. In addition, γ_X denotes the mean geometric covariogram of the RACS X, that is, the function defined by $\gamma_X(\tau) = \mathrm{E}(\nu_d(X \cap (\tau + X)))$, $\tau \in \mathbb{R}^d$ (we refer the reader to [17] and [19] for properties of the mean geometric covariogram).

Assumptions. Throughout the paper, it is assumed that

$$0 < \mathrm{E}(\nu_d(X)) < +\infty.$$

This hypothesis ensures that each point $y \in \mathbb{R}^d$ is covered by a countably infinite number of leaves of Φ, whereas the number of leaves falling on y during a finite time interval $[s_1, s_2]$ is almost surely (a.s.) finite. We also assume that $\mathrm{E}(a^2) < +\infty$.

3. One-dimensional marginal distribution and simulation of the TDL model

3.1. The Poisson process of the leaves intersecting a set

As can be observed from (3), the only leaves which have a contribution to the sum defining $f(y)$ are the leaves (t_i, x_i, X_i, a_i) such that $y \in x_i + X_i$. When considering the restriction of f to a Borel set G, the only leaves of interest are those intersecting G, i.e. the leaves (t_i, x_i, X_i, a_i) such that $x_i + X_i \cap G \neq \varnothing$. The next proposition characterizes the distribution of such leaves, a result to be used further in the paper. We first introduce the following notation. If A and B are two Borel sets then $\check{A} = \{-x : x \in A\}$ and $A \oplus B = \{x + y : x \in A \text{ and } y \in B\}$. Note the equivalence $x + X \cap G \neq \varnothing \Leftrightarrow x \in G \oplus \check{X}$.

Proposition 1. (The Poisson process of the leaves intersecting a Borel set.) *Let $G \subset \mathbb{R}^d$ be a Borel set such that $0 < \mathrm{E}(\nu_d(X \oplus \check{G})) < +\infty$, and let Φ be the Poisson process on $S = (-\infty, 0) \times \mathbb{R}^d \times \mathcal{F} \times \mathbb{R}$ with intensity measure $\mu = \lambda \otimes \nu_d \otimes P_X \otimes P_a$. Denote by Φ^G the point process of the leaves of Φ which intersect G, that is,*

$$\Phi^G = \{(t, x, X, a) \in \Phi : x + X \cap G \neq \varnothing\},$$

and denote by $\mathcal{A}^G \subset \mathbb{R}^d \times \mathcal{F}$ the set $\mathcal{A}^G = \{(x, X) : x + X \cap G \neq \varnothing\}$. Then Φ^G is a Poisson process on S with intensity measure

$$\mu^G = \lambda \otimes (\nu_d \otimes P_X)_{\llcorner \mathcal{A}^G} \otimes P_a.$$

It is an independently marked Poisson process with ground process $\Pi^G = \{t : (t, x, X, a) \in \Phi^G\}$, a homogeneous Poisson process on $(-\infty, 0)$ of intensity $\mathrm{E}(\nu_d(X \oplus \check{G}))$, and with mark distribution

$$\frac{1}{\mathrm{E}(\nu_d(X \oplus \check{G}))} (\nu_d \otimes P_X)_{\llcorner \mathcal{A}^G} \otimes P_a.$$

https://doi.org/10.1239/aap/1331216642 Published online by Cambridge University Press

Proof. The point process Φ^G is the restriction of the Poisson process Φ to the measurable set

$$\{(t, x, X, a) \in (-\infty, 0) \times \mathbb{R}^d \times \mathcal{F} \times \mathbb{R} \colon (x, X) \in \mathcal{A}^G\};$$

thus, Φ^G is a Poisson process and its intensity measure μ^G is the restriction of μ to the above set. As for the interpretation of Φ^G as an independently marked one-dimensional Poisson process, it is based on the factorization of the intensity measure μ^G (see [1, Section 1.8] or [25, Section 3.5]). Indeed, we have

$$0 < \nu_d \otimes P_X(\mathcal{A}^G) = \int_{\mathcal{F}} \int_{\mathbb{R}^d} \mathbf{1}(y \in G \oplus \check{Y}) \nu_d(\mathrm{d}y) P_X(\mathrm{d}Y) = \mathrm{E}(\nu_d(X \oplus \check{G})) < +\infty,$$

and, thus, we can write

$$\mu^G = \mathrm{E}(\nu_d(X \oplus \check{G}))\lambda \otimes \left[\frac{1}{\mathrm{E}(\nu_d(X \oplus \check{G}))}(\nu_d \otimes P_X)_{\llcorner \mathcal{A}^G} \otimes P_a\right],$$

where the measure between the square brackets is a probability distribution.

3.2. One-dimensional marginal distribution

Proposition 2. (One-dimensional marginal distribution.) *Let y be a point in $\mathbb{R}^d$. Then there exists a sequence $(a(y, k))_{k \in \mathbb{N}}$ of i.i.d. RVs with distribution P_a such that*

$$f(y) = \alpha \sum_{k=0}^{+\infty} a(y, k)\beta^k.$$

In particular, we have $\mathrm{E}(f(y)) = \mathrm{E}(a)$ *and* $\mathrm{var}(f(y)) = \alpha\, \mathrm{var}(a)/(2 - \alpha)$.

Informally, $a(y, k)$ is the color of the $(k + 1)$th leaf falling on y; $(a(y, k))_{k \in \mathbb{N}}$ is thus an ordered subfamily of the RV $(a_i)_{i \in \mathbb{N}}$ which depends on y.

Proof of Proposition 2. According to Proposition 1, the point process $\Phi^{\{y\}}$ of the leaves which cover y is an independently marked Poisson process, the ground process of which is a Poisson process on $(-\infty, 0)$ with intensity $0 < \mathrm{E}(\nu_d(X)) < +\infty$. Hence, the falling times of the leaves of $\Phi^{\{y\}}$ are a.s. distinct and we can number (in a measurable way [25, p. 49]) the leaves

$$(t(y, k), x(y, k), X(y, k), a(y, k)), \qquad k \in \mathbb{N},$$

according to an anti-chronological order:

$$0 > t(y, 0) > t(y, 1) > t(y, 2) > \cdots .$$

Proposition 1 also gives the distribution of the marks $(x(y, k), X(y, k), a(y, k))$, and in particular it shows that the RVs $a(y, k)$, $k \in \mathbb{N}$, are i.i.d. with distribution P_a. As already mentioned, the only leaves involved in the sum which defines $f(y)$ are the leaves of $\Phi^{\{y\}}$. Besides, using the above numbering, we have, for all $k \in \mathbb{N}$,

$$\sum_{(t_j, x_j, X_j, a_j) \in \Phi} \mathbf{1}(t_j \in (t(y, k), 0) \text{ and } y \in x_j + X_j) = k.$$

Hence, (3) becomes

$$f(y) = \alpha \sum_{k=0}^{+\infty} a(y, k)\beta^k,$$

and the result follows.

Remark 3. (*Influence of the transparency coefficient* α.) Let us write f_α for the TDL model with transparency coefficient $\alpha \in (0, 1]$. Proposition 2 shows that the expectation of f_α does not depend on α. In contrast, the variance $\operatorname{var}(f_\alpha(y)) = \alpha \operatorname{var}(a)/(2-\alpha)$ decreases as α decreases. Besides, $\operatorname{var}(f_\alpha(y))$ tends to 0 as α tends to 0 (recall that the model is not defined for $\alpha = 0$). However, a central limit theorem for random geometric series [6] shows that, for all $y \in \mathbb{R}^d$, the family of RVs $((f_\alpha(y) - \mathrm{E}(f_\alpha))/\sqrt{\operatorname{var}(f_\alpha)})_\alpha$ converges in distribution to a standard normal distribution as α tends to 0. This pointwise convergence result will be extended in Section 5, where it will be shown that the family of normalized random fields $(y \mapsto (f_\alpha(y) - \mathrm{E}(f_\alpha))/\sqrt{\operatorname{var}(f_\alpha)})_\alpha$ converges in the sense of finite-dimensional distributions.

3.3. Simulation of the TDL model

In this section we draw on Proposition 2 to obtain a simulation algorithm for the restriction of the TDL model f to a finite set $U \subset \mathbb{R}^d$ (e.g. a finite grid of pixels). The algorithm is based on a coupling-from-the-past procedure, as the algorithm developed by Kendall and Thönnes [15] for simulating the dead leaves model (see also [14]). This algorithm consists in sequentially superimposing transparent random objects, but, contrary to the forward procedure described by (1), each new object is placed *below* the former objects. In the case of the dead leaves model, this yields a perfect simulation algorithm. For the TDL model f, simulation is not perfect since the values $f(y)$ are the limits of convergent series. Nevertheless, supposing that the intensities a_i are bounded, we propose, for any precision $\varepsilon > 0$, an algorithm which produces an approximation $\bar{f}$ of f. This approximation satisfies

$$\mathrm{P}\Big(\sup_{y \in U} |f(y) - \bar{f}(y)| \le \varepsilon\Big) = 1,$$

therefore providing a kind of perfect simulation with precision $\varepsilon > 0$.

In the remainder of this section we suppose that the colors a_i are a.s. bounded by $A > 0$. The control of the precision is based on the following elementary lemma.

Lemma 1. (Precision associated with the leaves layer.) *Let* $y \in \mathbb{R}^d$*, and let*

$$\bar{f}_n(y) = \alpha \sum_{k=0}^{n-1} a(y, k) \beta^k$$

be the restriction of the sum defining $f(y)$ *to the* n *latest leaves which have fallen on* y*. Then*

$$|f(y) - \bar{f}_n(y)| \le A\beta^n.$$

Lemma 1 shows that to approximate $f(y)$ with a tolerance $\varepsilon > 0$, it is enough to cover the point y with (at least) $N(\varepsilon)$ leaves, where $N(\varepsilon)$ is the smallest integer n such that $A\beta^n \le \varepsilon$, that is, $N(\varepsilon) = \lceil \log(\varepsilon/A)/\log(\beta) \rceil$. The following simulation algorithm relies on this observation.

Algorithm 1. (*Simulation of the TDL model with tolerance* $\varepsilon > 0$.) Let $U \subset \mathbb{R}^d$ be a finite set. Given a precision $\varepsilon > 0$, an approximation $\bar{f}$ of the TDL model f is computed by controlling the number of leaves L at each point.

- *Initialization.* For all $y \in U$, $\bar{f}(y) \leftarrow 0$ and $L(y) \leftarrow 0$.
- *Computation of the required number of leaves.*

$$N(\varepsilon) = \left\lceil \frac{\log(\varepsilon/A)}{\log(\beta)} \right\rceil.$$

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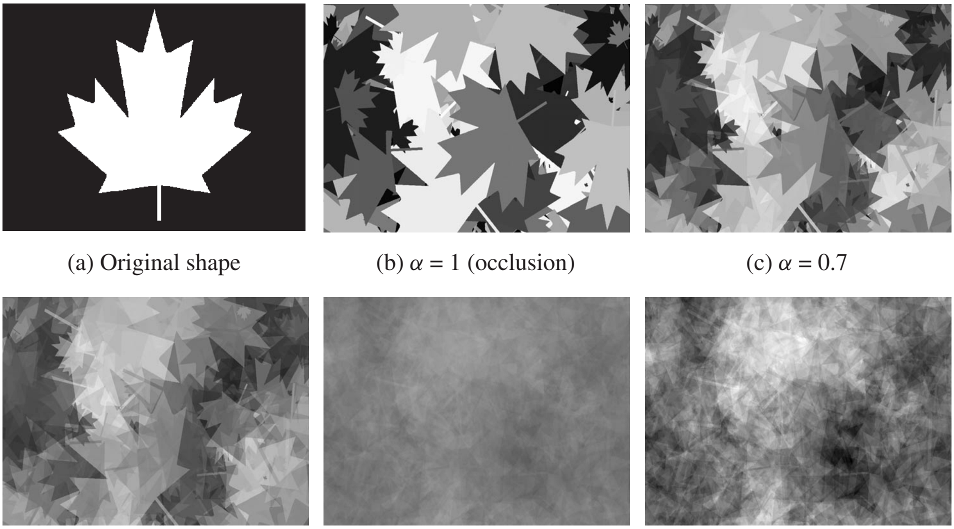

(a) Original shape (b) $\alpha = 1$ (occlusion) (c) $\alpha = 0.7$

(d) $\alpha = 0.4$ (e) $\alpha = 0.05$ (f) $\alpha = 0.05$, enhanced contrast

FIGURE 2: TDL realizations with various transparency coefficients α. The RACSs X_i are all obtained from the original shape (image (a)) by applying a rotation of angle $\theta \sim \text{Unif}(0, 2\pi)$ and a homothety of factor $r \sim \text{Unif}(0, 1)$, and $P_a = \text{Unif}(0, 255)$. For $\alpha = 1$, we obtain a colored dead leaves model. As soon as the leaves are transparent ($\alpha < 1$), one can distinguish several layers of leaves and not only the leaves on top. For $\alpha = 0.05$, the variance of the TDL model is nearly 0 (see Proposition 2). Enhancing the contrast of the image reveals the structure of the image (see (f)).

- *Iteration.* While $(\inf_{y \in U} L(y) < N(\varepsilon))$ add a new leaf.
  - (i) *Draw a leaf* (x, X, a) *hitting* U.
    - (a) Draw $X \sim P_X$.
    - (b) Draw x uniformly in $U \oplus \check{X}$.
    - (c) Draw $a \sim P_a$.
  - (ii) *Add the leaf* (x, X, a) *to* $\bar{f}$. For all $y \in U$, $\bar{f}(y) \leftarrow \bar{f}(y) + \mathbf{1}(y \in x + X)\alpha a \beta^{L(y)}$.
  - (iii) *Update the leaves layer* L. For all $y \in U$, $L(y) \leftarrow L(y) + \mathbf{1}(y \in x + X)$.

Clearly, Algorithm 1 a.s. terminates if every point of U is covered by $N(\varepsilon)$ leaves in an a.s. finite time. This is always the case since U is a finite set and $\mathrm{E}(\nu_d(X)) > 0$.

Several realizations of some TDL models are represented in Figure 2. Note that, as soon as $\alpha < 1$, the TDL random field is not piecewise constant: any region is intersected by the boundaries of some leaves, producing discontinuities.

4. Covariance of the TDL model

This section is devoted to the computation of the covariance of the TDL. A classical way to achieve this would be to use Palm calculus, leading to relatively heavy computations in this case. Instead, we chose an alternative way relying on some memoryless property of the TDL, as explained below.

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The following proposition is an extension of the fact that if $0 > t_0 > t_1 > t_2 > \cdots$ is a homogeneous Poisson process on $(-\infty, 0)$ then the shifted process $0 > t_1 - t_0 > t_2 - t_0 > t_3 - t_0 > \cdots$ is also a Poisson process with the same distribution [16, Chapter 4].

Proposition 3. (Last hitting leaf and the Poisson process preceding the last hit.) *Let F be a locally compact topological space with a countable base [25]. Let Ψ be a Poisson process in $(-\infty, 0) \times F$ with intensity measure of the form $\lambda \otimes \eta$, where λ is the one-dimensional Lebesgue measure on $(-\infty, 0)$ and η is a measure on F. Let $A \subset F$ be an η-measurable set satisfying $0 < \eta(A) < +\infty$. Define*

$$t_0 = \sup\{t_i \mid (t_i, y_i) \in \Psi \cap ((-\infty, 0) \times A)\},$$

y_0 as the a.s. unique $y \in F$ such that $(t_0, y) \in \Psi \cap ((-\infty, 0) \times A)$, and

$$\Psi_{t_0} = \sum_{(t_i, y_i) \in \Psi} \mathbf{1}(t_i < t_0)\delta_{(t_i - t_0, y_i)}.$$

Then

- *t_0, y_0, and Ψ_{t_0} are mutually independent,*
- *$-t_0$ has an exponential distribution with parameter $\eta(A)$,*
- *y_0 has distribution Q_A defined, for all $B \in \mathcal{B}(F)$, by $Q_A(B) = \eta(B \cap A)/\eta(A)$,*
- *Ψ_{t_0} is a Poisson process with intensity measure $\lambda \otimes \eta$, i.e. Ψ_{t_0} has the same distribution as Ψ.*

Proposition 3 will be applied below to the Poisson process Φ of the colored leaves to compute some statistics of the TDL model f. As a first example, let us reobtain the expectation of f by using Proposition 3. Let $y \in \mathbb{R}^d$, and let us denote by (t_0, x_0, X_0, a_0) the leaf which hits y at the maximal time t_0. Then we can decompose $f(y)$ as

$$f(y) = \alpha a_0 + \beta f_{t_0}(y), \tag{4}$$

where f_{t_0} is the TDL model associated with the time-shifted point process Φ_{t_0} and, as before, $\beta = 1 - \alpha$. According to Proposition 3, a_0 has distribution P_a, and both point processes Φ and Φ_{t_0} have the same distribution. Consequently, $f(y)$ and $f_{t_0}(y)$ also have the same distribution, and in particular the same expectation. Hence, the above decomposition of $f(y)$ leads to the equation

$$\mathrm{E}(f(y)) = \alpha\,\mathrm{E}(a) + \beta\,\mathrm{E}(f(y)),$$

which gives $\mathrm{E}(f(y)) = \mathrm{E}(a)$, in accordance with Proposition 2.

The same method is used below to compute the covariance of f. This method will also be applied in Section 5.3 to derive a technical result useful for the central limit theorem of Section 5.

We recall that $\gamma_X(\tau) = \mathrm{E}(\nu_d(X \cap (\tau + X)))$ is the mean covariogram of X. In addition, we denote the covariance of f by

$$\operatorname{cov}(f)(\tau) = \operatorname{cov}(f(y), f(y+\tau)) = \mathrm{E}((f(y) - \mathrm{E}(a))(f(y+\tau) - \mathrm{E}(a))).$$

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Proposition 4. (Covariance of the TDL model.) *The TDL model f is a square-integrable stationary random field and its covariance is given by*

$$\operatorname{cov}(f)(\tau) = \frac{\alpha\gamma_X(\tau)}{2\,\mathrm{E}(\nu_d(X)) - \alpha\gamma_X(\tau)}\operatorname{var}(a), \qquad \tau \in \mathbb{R}^d.$$

Proof. Let y and z be such that $z - y = \tau$, and write $m = \mathrm{E}(a) = \mathrm{E}(f)$ as a shorthand notation. We have to compute $\operatorname{cov}(f)(\tau) = \mathrm{E}((f(y) - m)(f(z) - m))$.

Denote by (t_0, x_0, X_0, a_0) the last leaf which hits y or z at the maximal time t_0, and let Φ_{t_0} be the corresponding time-shifted Poisson process. According to Proposition 3, (x_0, X_0, a_0) is independent of Φ_{t_0}. In addition, $\Phi_{t_0} \overset{\mathrm{D}}{=} \Phi$, and, consequently, noting that f_{t_0} is the TDL associated with Φ_{t_0}, $(f_{t_0}(y), f_{t_0}(z)) \overset{\mathrm{D}}{=} (f(y), f(z))$. Proposition 3 also shows that a_0 has distribution P_a. As for the distribution of (x_0, X_0), a straightforward computation shows that

$$\nu_d \otimes P_X(\{(x, X),\ \{y, z\} \cap x + X \neq \varnothing\}) = \mathrm{E}(\nu_d(X \oplus \{-y, -z\})) = 2\gamma_X(0) - \gamma_X(\tau)$$

and

$$\nu_d \otimes P_X(\{(x, X),\ \{y, z\} \subset x + X\}) = \mathrm{E}(\nu_d(-y + X \cap -z + X)) = \gamma_X(\tau).$$

Hence, we have

$$\mathrm{P}(\{y, z\} \subset x_0 + X_0) = \frac{\nu_d \otimes P_X(\{(x, X),\ \{y, z\} \subset x + X\})}{\nu_d \otimes P_X(\{(x, X),\ \{y, z\} \cap x + X \neq \varnothing\})} = \frac{\gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)},$$

and, by symmetry,

$$\mathrm{P}(y \in x_0 + X_0 \text{ and } z \notin x_0 + X_0) = \mathrm{P}(z \in x_0 + X_0 \text{ and } y \notin x_0 + X_0) = \frac{\gamma_X(0) - \gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)}.$$

Conditioning with respect to the coverage of the last leaf (t_0, x_0, X_0, a_0), we have

$$\begin{aligned}
&\mathrm{E}((f(y) - m)(f(z) - m)) \\
&\quad = \mathrm{E}((f(y) - m)(f(z) - m) \mid \{y, z\} \subset x_0 + X_0)\frac{\gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)} \\
&\qquad + \mathrm{E}((f(y) - m)(f(z) - m) \mid y \in x_0 + X_0 \text{ and } z \notin x_0 + X_0)\frac{\gamma_X(0) - \gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)} \\
&\qquad + \mathrm{E}((f(y) - m)(f(z) - m) \mid z \in x_0 + X_0 \text{ and } y \notin x_0 + X_0)\frac{\gamma_X(0) - \gamma_X(\tau)}{2\gamma_X(0) - \gamma_X(\tau)}.
\end{aligned}$$

By symmetry, it is clear that the two last terms of the above sum are equal. On the event $\{\{y, z\} \subset x_0 + X_0\}$ we have

$$f(y) - m = \alpha(a_0 - m) + \beta(f_{t_0}(y) - m) \quad \text{and} \quad f(z) - m = \alpha(a_0 - m) + \beta(f_{t_0}(z) - m),$$

so that

$$\begin{aligned}
(f(y) - m)(f(z) - m) = \alpha^2(a_0 - m)^2 &+ \beta^2(f_{t_0}(y) - m)(f_{t_0}(z) - m) \\
&+ \alpha\beta(a_0 - m)((f_{t_0}(y) - m) + (f_{t_0}(z) - m)).
\end{aligned}$$

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By Proposition 3, a_0, (x_0, X_0), and $(f_{t_0}(y), f_{t_0}(z))$ are mutually independent; hence,

$$\begin{aligned}
&\mathrm{E}((f(y)-m)(f(z)-m) \mid \{y,z\} \subset x_0 + X_0) \\
&\quad = \alpha^2 \,\mathrm{E}((a_0-m)^2) + \beta^2 \,\mathrm{E}((f_{t_0}(y)-m)(f_{t_0}(z)-m)) \\
&\quad = \alpha^2 \operatorname{var}(a) + \beta^2 \operatorname{cov}(f(y), f(z)).
\end{aligned}$$

On the event $\{y \in x_0 + X_0 \text{ and } z \notin x_0 + X_0\}$ we have

$$f(y) - m = \alpha(a_0 - m) + \beta(f_{t_0}(y) - m) \quad \text{and} \quad f(z) - m = f_{t_0}(z) - m.$$

Hence, by the same arguments,

$$\mathrm{E}((f(y)-m)(f(z)-m) \mid y \in x_0 + X_0 \text{ and } z \notin x_0 + X_0) = \beta \operatorname{cov}(f(y), f(z)).$$

Replacing the terms in the decomposition of $\mathrm{E}((f(y)-m)(f(z)-m))$ leads to an equation involving the covariance $\operatorname{cov}(f(y), f(z))$, the values $\gamma_X(0)$ and $\gamma_X(\tau)$ of the mean covariogram of X, and the variance $\operatorname{var}(a)$. Simplifying this equation we obtain the enunciated formula.

Remark 4. (*Variable transparency and the second-order property.*) The technique used in this section enables us to generalize second-order formulae to the case where the transparency parameter α is assumed to be different for each object, that is, when it is assumed that each object X_i is assigned a transparency α_i distributed as a random variable α and independent of other objects. First, it is straightforward to show that in this case we still have $\mathrm{E}(f(y)) = \mathrm{E}(a)$. Then, a simple application of (4) yields $\operatorname{var} f(y) = \mathrm{E}(\alpha^2)\operatorname{var}(a)(2\,\mathrm{E}(\alpha) - \mathrm{E}(\alpha^2))^{-1}$. Observe that a direct computation starting from the definition of f would be much more arduous. Eventually, applying the same technique, we can show that the covariance of the model with variable transparency satisfies, for $\tau \in \mathbb{R}^d$,

$$\operatorname{cov}(f)(\tau) = \frac{\mathrm{E}(\alpha^2)\gamma_X(\tau)}{2\,\mathrm{E}(\alpha)\,\mathrm{E}(\nu_d(X)) - \mathrm{E}(\alpha^2)\gamma_X(\tau)} \operatorname{var}(a).$$

5. Gaussian convergence as the objects become fully transparent

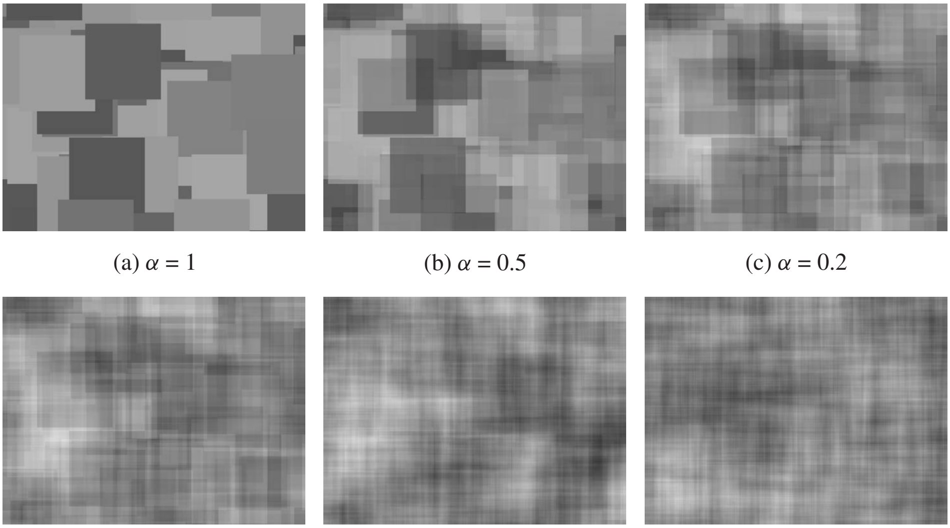
Recall that the TDL model with transparency coefficient α is denoted by f_α.

Theorem 1. (Normal convergence of the TDL model.) *Suppose that* $\operatorname{var}(a) > 0$. *Then, as the transparency coefficient* α *tends to* 0, *the family of random fields* $((f_\alpha - \mathrm{E}(f_\alpha))/\sqrt{\operatorname{var}(f_\alpha)})_\alpha$ *converges in the sense of finite-dimensional distributions to a stationary Gaussian random field with covariance function*

$$C(\tau) = \frac{\gamma_X(\tau)}{\mathrm{E}(\nu_d(X))} = \frac{\gamma_X(\tau)}{\gamma_X(0)}.$$

Before proving Theorem 1, we illustrate in Figure 3 the normal convergence of the normalized family of RVs $((f_\alpha - \mathrm{E}(f_\alpha))/\sqrt{\operatorname{var}(f_\alpha)})_\alpha$. The five first images of Figure 3 are normalized TDL realizations obtained from the same random colored leaves but with various transparency coefficients α. The last image is a realization of the limit Gaussian random field given by Theorem 1. Observe that this Gaussian field is also the limit of the normalized shot noise associated with X when the intensity of germs tends to ∞ [13].

The remainder of this section is devoted to the proof of Theorem 1. The proof consists in showing that the finite moments of the normalized TDL random fields converge to the



(a) $\alpha = 1$ (b) $\alpha = 0.5$ (c) $\alpha = 0.2$

(d) $\alpha = 0.1$ (e) $\alpha = 0.01$ (f) Limit Gaussian random field

FIGURE 3: From colored dead leaves to Gaussian random fields. Visual illustration of the normal convergence of the normalized TDL random fields $((f_\alpha - \mathrm{E}(f_\alpha))/\sqrt{\mathrm{var}(f_\alpha)})_\alpha$ (see Theorem 1). As α decreases to 0, the normalized TDL realizations look more and more similar to the Gaussian texture (see (f)).

corresponding moments of the limit Gaussian random field. As for the computation of the covariance (see Section 4), this convergence is established by conditioning with respect to the coverage of the last leaf hitting the considered set of points (see below for details).

5.1. Some classical results of probability theory

This section gathers two classical theoretical results needed to prove Theorem 1.

5.1.1. *Moments and convergence in distribution.*

Proposition 5. (Moments and convergence in distribution.) *Let (f_n) be a sequence of random fields having finite moments of all orders, and let f_G be a Gaussian random field. If, for all $p \in \mathbb{N}$ and all (not necessarily distinct) $y_1, \dots, y_p \in \mathbb{R}^d$,*

$$\lim_{n\to+\infty} \mathrm{E}\left(\prod_{j=1}^{p} f_n(y_j)\right) = \mathrm{E}\left(\prod_{j=1}^{p} f_G(y_j)\right),$$

then (f_n) converges to f_G in the sense of finite-dimensional distributions.

5.1.2. *A recurrence relation for the moments of a multivariate normal distribution.* Explicit expressions for the moments of a multivariate normal distribution are given by Isserlis' theorem, which we recall below (see, e.g. [20] and the references therein).

Theorem 2. (Isserlis' theorem.) *Let $Y_1, \dots, Y_{2N+1}$, $N \geq 1$, be normalized (i.e.* $\mathrm{E}(Y_i) = 0$ *and* $\mathrm{var}(Y_i) = \mathrm{E}(Y_i^2) = 1$*), jointly Gaussian RVs. Then*

$$\mathrm{E}(Y_1 Y_2 \cdots Y_{2N}) = \sum \prod \mathrm{E}(Y_i Y_j) = \sum \prod \mathrm{cov}(Y_i, Y_j)$$

and

$$\mathrm{E}(Y_1 Y_2 \cdots Y_{2N+1}) = 0,$$

where the notation $\sum \prod$ *means summation over all distinct ways of partitioning the set* $\{Y_1, \ldots, Y_{2N}\}$ *into* N *pairs* $\{Y_i, Y_j\}$ *and taking the product of the* N *terms*

$$\mathrm{E}(Y_i Y_j) = \mathrm{cov}(Y_i, Y_j).$$

From Isserlis' theorem we deduce a recurrence relation for the moments of a multivariate normal distribution.

Proposition 6. (A recurrence relation for the moments of a multivariate normal distribution.) *Let* $Y = (Y_1, \ldots, Y_p)$*,* $p \geq 2$*, be a normalized Gaussian vector. Then,*

$$\mathrm{E}\left(\prod_{j=1}^{p} Y_j\right) = \frac{2}{p} \sum_{\{j,k\} \subset \{1,\ldots,p\}} \mathrm{cov}(Y_j, Y_k)\, \mathrm{E}\left(\prod_{l \in \{1,\ldots,p\} \setminus \{j,k\}} Y_l\right).$$

Proof. If $p \geq 2$ is odd then, by Isserlis' theorem, the above formula is trivial. Hence, in the following we suppose that p is even. First, let $j \in \{1, \ldots, p\}$. Factorizing with all the pairs containing j in Isserlis' identity, we obtain

$$\mathrm{E}\left(\prod_{j=1}^{p} Y_j\right) = \sum_{\substack{k=1 \\ k \neq j}}^{p} \mathrm{cov}(Y_j, Y_k)\, \mathrm{E}\left(\prod_{l \in \{1,\ldots,p\} \setminus \{j,k\}} Y_l\right).$$

The above identity is valid for all $j \in \{1, \ldots, p\}$. Summing these p identities gives

$$\mathrm{E}\left(\prod_{j=1}^{p} Y_j\right) = \frac{1}{p} \sum_{j=1}^{p} \sum_{\substack{k=1 \\ k \neq j}}^{p} \mathrm{cov}(Y_j, Y_k)\, \mathrm{E}\left(\prod_{l \in \{1,\ldots,p\} \setminus \{j,k\}} Y_l\right).$$

Now note that in this double sum over (j, k), the terms $\mathrm{cov}(Y_j, Y_k)\, \mathrm{E}(\prod_l Y_l)$ depend only on the pair $\{j, k\}$, not on the order. Hence, the above expression simplifies to

$$\mathrm{E}\left(\prod_{j=1}^{p} Y_j\right) = \frac{2}{p} \sum_{\{j,k\} \subset \{1,\ldots,p\}} \mathrm{cov}(Y_j, Y_k)\, \mathrm{E}\left(\prod_{l \in \{1,\ldots,p\} \setminus \{j,k\}} Y_l\right).$$

5.2. Notation and plan of the proof of Theorem 1

Let s be a real number such that $0 < s < \frac{1}{6}$ (this choice for s will become clear later). For all $\alpha \in (0, 1]$, we define the truncation operator

$$T_\alpha(b) = \begin{cases} b & \text{if } b \in [-\alpha^{-s}, \alpha^{-s}], \\ \alpha^{-s} & \text{if } b > \alpha^{-s}, \\ -\alpha^{-s} & \text{if } b < -\alpha^{-s}. \end{cases}$$

For all $\alpha \in (0, 1]$, f_α denotes the TDL model with transparency coefficient α and

$$g_\alpha(y) = \frac{f_\alpha(y) - \mathrm{E}(a)}{\sqrt{\mathrm{var}(f_\alpha)}}$$

denotes its normalization. For all $\alpha \in (0, 1]$, f_α^T denotes the TDL model with transparency coefficient α associated with the Poisson process

$$\Phi^T = \{(t_i, x_i, X_i, T_\alpha(a_i)), (t_i, x_i, X_i, a_i) \in \Phi\},$$

that is, the TDL model obtained by truncating the colors a_i of the leaves of Φ. We have

$$\mathrm{E}(f_\alpha^T) = \mathrm{E}(T_\alpha(a)) \quad \text{and} \quad \mathrm{var}(f_\alpha^T) = \frac{\alpha}{2-\alpha}\,\mathrm{var}(T_\alpha(a)).$$

As for the TDL f_α, we define

$$g_\alpha^T(y) = \frac{f_\alpha^T(y) - \mathrm{E}(T_\alpha(a))}{\sqrt{\mathrm{var}(f_\alpha^T)}}.$$

Thanks to the truncation, f_α^T is bounded by α^{-s}. In particular, for all $\alpha \in (0, 1]$, f_α^T and g_α^T have finite moments of all orders.

We will denote by f_G a centered stationary Gaussian random field with covariance function $C\colon \tau \mapsto \gamma_X(\tau)/\gamma_X(0)$.

The proof of Theorem 1 is divided into two steps.

1. We show that the normalized TDL with truncated colors g_α^T converges in distribution to f_G by the method of moments. More precisely, the sufficient condition of Proposition 5 will be shown to be true by induction on the number of points p.
2. We show that the family $g_\alpha - g_\alpha^T$ converges to 0 in L^2.

By Slutsky's theorem (see, e.g. [4]), these two steps ensure that g_α converges in distribution to f_G.

5.3. Normal convergence of the normalized TDL having truncated colors

With the above notation, by Proposition 5, it is enough to show the following lemma.

Lemma 2. (Convergence of moments.) *For all $p \in \mathbb{N}$ and all (not necessarily distinct) $y_1, \dots, y_p \in \mathbb{R}^d$,*

$$\lim_{\alpha \to 0} \mathrm{E}\bigg(\prod_{j=1}^{p} g_\alpha^T(y_j)\bigg) = \mathrm{E}\bigg(\prod_{j=1}^{p} f_G(y_j)\bigg).$$

We will show this lemma by induction on p. First note that, by the definition of $g_\alpha^T(y_j)$, the statement is true for $p = 0$ and $p = 1$.

For the proof by induction, we now consider an integer $p \geq 2$ and p points $y_1, \dots, y_p$ of $\mathbb{R}^d$, and we suppose that the convergence of moments holds for all moments of order $k < p$.

5.3.1. *Decomposition of the multivariate characteristic function by conditioning with respect to the coverage of the last hitting leaf.* We consider the random vector

$$(g_\alpha^T(y_1), \dots, g_\alpha^T(y_p)) = \bigg(\frac{f_\alpha^T(y_1) - \mathrm{E}(T_\alpha(a))}{\sigma_\alpha^T}, \dots, \frac{f_\alpha^T(y_p) - \mathrm{E}(T_\alpha(a))}{\sigma_\alpha^T}\bigg),$$

where $\sigma_\alpha^T = \sqrt{\mathrm{var}(f_\alpha^T)}$. We denote by $\phi_\alpha(t_1, \dots, t_p)$ the multivariate characteristic function of this random vector, that is,

$$\phi_\alpha(t_1, \dots, t_p) = \mathrm{E}(\exp[\mathrm{i}(t_1 g_\alpha^T(y_1) + \dots + t_p g_\alpha^T(y_p))]).$$

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We denote by ψ_α the characteristic function of the random variable $T_\alpha(a) - \mathrm{E}(T_\alpha(a))$, where a follows the color distribution P_a, that is,

$$\psi_\alpha(t) = \mathrm{E}(\exp[\mathrm{i}t(T_\alpha(a) - \mathrm{E}(T_\alpha(a)))]).$$

In addition, we introduce the shorthand notation $\mathcal{Y}$ for the set $\mathcal{Y} = \{y_1, \dots, y_p\}$.

In what follows we apply Proposition 3 when considering the leaves of Φ^T which hit the set $\mathcal{Y}$. Hence, let $(t_0, x_0, X_0, T_\alpha(a_0))$ denote the last leaf covering at least one point of $\mathcal{Y}$, and denote by g^T_{α,t_0} the corresponding time-shifted random field. Then, for all $y_j \in \mathcal{Y}$, we have the decomposition

$$g^T_\alpha(y_j) = \begin{cases} \alpha \dfrac{T_\alpha(a_0) - \mathrm{E}(T_\alpha(a))}{\sigma^T_\alpha} + \beta g^T_{\alpha,t_0}(y_j) & \text{if } y_j \in x_0 + X_0, \\ g_{\alpha,t_0}(y_j) & \text{otherwise,} \end{cases}$$

which can also be written as

$$g^T_\alpha(y_j) = \alpha\, \mathbf{1}(y_j \in x_0 + X_0) \frac{T_\alpha(a_0) - \mathrm{E}(T_\alpha(a))}{\sigma^T_\alpha} + \beta^{\mathbf{1}(y_j \in x_0 + X_0)} g^T_{\alpha,t_0}(y_j).$$

Besides, by Proposition 3, g^T_{α,t_0}, (x_0, X_0), and a_0 are mutually independent.

To obtain a decomposition of the characteristic function ϕ_α, we will condition with respect to the coverage of the last leaf $x_0 + X_0$. Hence, for all subsets $\mathcal{X} \subset \mathcal{Y}$, $\mathcal{X} \neq \varnothing$, let us denote by $A_\mathcal{X} \subset \Omega$ the event

$$A_\mathcal{X} = \{(x_0 + X_0) \cap \mathcal{Y} = \mathcal{X}\}$$

and

$$p_\mathcal{X} = \mathrm{P}(A_\mathcal{X}).$$

The events $A_\mathcal{X}$, $\mathcal{X} \neq \varnothing$, form a partition of the probability space Ω, and, in particular,

$$\sum_{\mathcal{X} \subset \mathcal{Y},\, \mathcal{X} \neq \varnothing} p_\mathcal{X} = 1.$$

Note that on the event $A_\mathcal{X}$, the above decomposition of $g^T_\alpha(y_j)$ becomes

$$g^T_\alpha(y_j) = \alpha\, \mathbf{1}(y_j \in \mathcal{X}) \frac{T_\alpha(a_0) - \mathrm{E}(T_\alpha(a))}{\sigma^T_\alpha} + \beta^{\mathbf{1}(y_j \in \mathcal{X})} g^T_{\alpha,t_0}(y_j).$$

Hence, using the mutual independence of the different random variables,

$$\begin{aligned} \phi_\alpha(t_1, \dots, t_p) &= \mathrm{E}(\exp[\mathrm{i}(t_1 g^T_\alpha(y_1) + \dots + t_p g^T_\alpha(y_p))]) \\ &= \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} \mathrm{E}(\exp[\mathrm{i}(t_1 g^T_\alpha(y_1) + \dots + t_p g^T_\alpha(y_p))] \mid A_\mathcal{X}) p_\mathcal{X} \\ &= \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} \psi_\alpha\bigg(\frac{\alpha}{\sigma^T_\alpha} \sum_{j=1}^p \mathbf{1}(y_j \in \mathcal{X}) t_j\bigg) \phi_\alpha(\beta^{\mathbf{1}(y_1 \in \mathcal{X})} t_1, \dots, \beta^{\mathbf{1}(y_p \in \mathcal{X})} t_p) p_\mathcal{X}. \end{aligned} \tag{5}$$

The next step of the proof consists in differentiating the above decomposition of the multivariate characteristic function in order to obtain a recurrence relation for the moments of $(g^T_\alpha(y_1), \dots, g^T_\alpha(y_p))$.

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5.3.2. *A recurrence relation for the moments of* g_α^T. We compute below the partial derivative $\partial^p \phi_\alpha(t_1, \ldots, t_p)/\partial t_1 \cdots \partial t_p$ of the characteristic function ϕ_α to obtain an expression for the moment $\mathrm{E}(\prod_{j=1}^p g_\alpha^T(y_j))$. Starting from (5), to compute $\partial^p \phi_\alpha(t_1, \ldots, t_p)/\partial t_1 \cdots \partial t_p$, we need to differentiate, with respect to each variable t_j, functions of the form

$$F_{\mathcal{X}}(t_1, \ldots, t_p) = \psi_\alpha\bigg(\frac{\alpha}{\sigma_\alpha^T}\sum_{j=1}^p \mathbf{1}(y_j \in \mathcal{X})t_j\bigg)\phi_\alpha(\beta^{\mathbf{1}(y_1\in\mathcal{X})}t_1, \ldots, \beta^{\mathbf{1}(y_p\in\mathcal{X})}t_p).$$

First let us introduce some notation. In what follows, for every subset $\mathcal{I} = \{i_1, \ldots, i_k\} \subset \{1, \ldots, p\}$, we write $\#\mathcal{I} = k$ for the cardinality of $\mathcal{I}$, and

$$\frac{\partial^k f}{\partial t_{\mathcal{I}}}(t_1, \ldots, t_p) = \frac{\partial^k f}{\partial t_{i_1}\partial t_{i_1}\cdots\partial t_{i_k}}(t_1, \ldots, t_p).$$

Let $\mathcal{I}^{\mathrm{c}}$ denote the complementary set of indices $\mathcal{I}^{\mathrm{c}} = \{1, \ldots, p\} \setminus \mathcal{I}$. With this notation,

$$\begin{aligned}
&\frac{\partial^p F_{\mathcal{X}}}{\partial t_1\cdots\partial t_p}(t_1, \ldots, t_p)\\
&\quad= \sum_{k=0}^p \sum_{\substack{\mathcal{I}\subset\{1,\ldots,p\}\\ \#\mathcal{I}=k}} \frac{\partial^k}{\partial t_{\mathcal{I}}}\bigg[\psi_\alpha\bigg(\frac{\alpha}{\sigma_\alpha^T}\sum_{j=1}^p \mathbf{1}(y_j\in\mathcal{X})t_j\bigg)\bigg]\\
&\qquad\qquad \times \frac{\partial^{p-k}}{\partial t_{\mathcal{I}^{\mathrm{c}}}}[\phi_\alpha(\beta^{\mathbf{1}(y_1\in\mathcal{X})}t_1, \ldots, \beta^{\mathbf{1}(y_p\in\mathcal{X})}t_p)]\\
&\quad= \sum_{k=0}^p \bigg(\frac{\alpha}{\sigma_\alpha^T}\bigg)^k \sum_{\substack{\mathcal{I}\subset\{1,\ldots,p\}\\ \#\mathcal{I}=k}} \bigg(\prod_{i\in\mathcal{I}}\mathbf{1}(y_i\in\mathcal{X})\bigg)\psi_\alpha^{(k)}\bigg(\frac{\alpha}{\sigma_\alpha^T}\sum_{j=1}^p \mathbf{1}(y_j\in\mathcal{X})t_j\bigg)\\
&\qquad\qquad \times \bigg(\prod_{i\in\mathcal{I}^{\mathrm{c}}}\beta^{\mathbf{1}(y_i\in\mathcal{X})}\bigg)\frac{\partial^{p-k}}{\partial t_{\mathcal{I}^{\mathrm{c}}}}\phi_\alpha(\beta^{\mathbf{1}(y_1\in\mathcal{X})}t_1, \ldots, \beta^{\mathbf{1}(y_p\in\mathcal{X})}t_p).
\end{aligned}$$

Summing over all subsets $\mathcal{X}$, we have the identity

$$\begin{aligned}
&\frac{\partial^p \phi_\alpha}{\partial t_1\cdots\partial t_p}(t_1, \ldots, t_p)\\
&\quad= \sum_{k=0}^p \bigg(\frac{\alpha}{\sigma_\alpha^T}\bigg)^k \sum_{\substack{\mathcal{I}\subset\{1,\ldots,p\}\\ \#\mathcal{I}=k}} \sum_{\substack{\mathcal{X}\subset\mathcal{Y}\\ \mathcal{X}\neq\varnothing}} \bigg(\prod_{i\in\mathcal{I}}\mathbf{1}(y_i\in\mathcal{X})\bigg)\psi_\alpha^{(k)}\bigg(\frac{\alpha}{\sigma_\alpha^T}\sum_{j=1}^p \mathbf{1}(y_j\in\mathcal{X})t_j\bigg)\\
&\qquad\qquad \times \bigg(\prod_{i\in\mathcal{I}^{\mathrm{c}}}\beta^{\mathbf{1}(y_i\in\mathcal{X})}\bigg)\\
&\qquad\qquad \times \frac{\partial^{p-k}}{\partial t_{\mathcal{I}^{\mathrm{c}}}}\phi_\alpha(\beta^{\mathbf{1}(y_1\in\mathcal{X})}t_1, \ldots, \beta^{\mathbf{1}(y_p\in\mathcal{X})}t_p)p_{\mathcal{X}}.
\end{aligned}$$

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Evaluating at $(t_1, \dots, t_p) = (0, \dots, 0)$, we obtain

$$
\begin{aligned}
&\frac{\partial^p \phi_\alpha}{\partial t_1 \cdots \partial t_p}(0, \dots, 0) \\
&\quad = \sum_{k=0}^{p} \bigg(\frac{\alpha}{\sigma_\alpha^T}\bigg)^k \sum_{\substack{\mathcal{I} \subset \{1,\dots,p\} \\ \#\mathcal{I}=k}} \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} \bigg(\prod_{i \in \mathcal{I}} \mathbf{1}(y_i \in \mathcal{X})\bigg) \psi_\alpha^{(k)}(0) \bigg(\prod_{i \in \mathcal{I}^{\mathrm{c}}} \beta^{\mathbf{1}(y_i \in \mathcal{X})}\bigg) \\
&\qquad\qquad \times \frac{\partial^{p-k}}{\partial t_{\mathcal{I}^{\mathrm{c}}}} \phi_\alpha(0, \dots, 0) p_{\mathcal{X}} \\
&\quad = \sum_{k=0}^{p} \bigg(\frac{\alpha}{\sigma_\alpha^T}\bigg)^k \psi_\alpha^{(k)}(0) \sum_{\substack{\mathcal{I} \subset \{1,\dots,p\} \\ \#\mathcal{I}=k}} \frac{\partial^{p-k}}{\partial t_{\mathcal{I}^{\mathrm{c}}}} \phi_\alpha(0, \dots, 0) \\
&\qquad\qquad \times \bigg(\sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} \bigg(\prod_{i \in \mathcal{I}} \mathbf{1}(y_i \in \mathcal{X})\bigg) \bigg(\prod_{i \in \mathcal{I}^{\mathrm{c}}} \beta^{\mathbf{1}(y_i \in \mathcal{X})}\bigg) p_{\mathcal{X}}\bigg).
\end{aligned}
$$

In the above sum, note that, for $k = 0$, $\mathcal{I} = \varnothing$ and, thus, all the terms are proportional to $\partial^p \phi_\alpha(0, \dots, 0)/\partial t_1 \cdots \partial t_p$. Besides, since $T_\alpha(a) - \mathrm{E}(T_\alpha(a))$ is centered, $\psi_\alpha^{(1)}(0) = 0$, and, thus, for $k = 1$, all the terms are 0. Hence, we have the following equation:

$$
\begin{aligned}
&\frac{\partial^p \phi_\alpha}{\partial t_1 \cdots \partial t_p}(0, \dots, 0) \bigg(1 - \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} \bigg(\prod_{j=1}^{p} \beta^{\mathbf{1}(y_j \in \mathcal{X})}\bigg) p_{\mathcal{X}}\bigg) \\
&\quad = \sum_{k=2}^{p} \bigg(\frac{\alpha}{\sigma_\alpha^T}\bigg)^k \psi_\alpha^{(k)}(0) \sum_{\substack{\mathcal{I} \subset \{1,\dots,p\} \\ \#\mathcal{I}=k}} \frac{\partial^{p-k}}{\partial t_{\mathcal{I}^{\mathrm{c}}}} \phi_\alpha(0, \dots, 0) \\
&\qquad\qquad \times \bigg(\sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} \bigg(\prod_{i \in \mathcal{I}} \mathbf{1}(y_i \in \mathcal{X})\bigg) \bigg(\prod_{i \in \mathcal{I}^{\mathrm{c}}} \beta^{\mathbf{1}(y_i \in \mathcal{X})}\bigg) p_{\mathcal{X}}\bigg). \qquad (6)
\end{aligned}
$$

5.3.3. *Recurrence relation for the limit of the moments.* The next step of the proof consists in dividing by α and letting α tend to 0 in (6). First, recalling that $\beta = 1 - \alpha$, and using the fact that $\sum p_{\mathcal{X}} = 1$, we have

$$
1 - \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} \bigg(\prod_{j=1}^{p} \beta^{\mathbf{1}(y_j \in \mathcal{X})}\bigg) p_{\mathcal{X}} = \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} p_{\mathcal{X}} - \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} \beta^{\#\mathcal{X}} p_{\mathcal{X}} = \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} (1 - (1-\alpha)^{\#\mathcal{X}}) p_{\mathcal{X}}.
$$

Hence,

$$
\lim_{\alpha \to 0} \frac{1}{\alpha} \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} (1 - (1-\alpha)^{\#\mathcal{X}}) p_{\mathcal{X}} = \sum_{\substack{\mathcal{X} \subset \mathcal{Y} \\ \mathcal{X} \neq \varnothing}} (\#\mathcal{X}) p_{\mathcal{X}},
$$

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and, by the definition of $p_{\mathcal{X}}$,

$$\begin{aligned}\sum_{\substack{\mathcal{X}\subset\mathcal{Y}\\ \mathcal{X}\neq\varnothing}} (\#\mathcal{X})p_{\mathcal{X}} &= \mathrm{E}(\#((x_0+X_0)\cap\mathcal{Y}))\\ &= \mathrm{E}\bigg(\sum_{j=1}^{p}\mathbf{1}(y_j\in x_0+X_0)\bigg)\\ &= p\frac{\mathrm{E}(\nu_d(X))}{\mathrm{E}(\nu_d(\mathcal{Y}\oplus\check{X}))}\\ &\neq 0.\end{aligned}$$

Let us now turn to the limit of the right-hand side of (6) when dividing by α and letting α tend to 0. First let us show that all the terms for which $k\geq 3$ will tend to 0. By induction, for all $k\geq 2$, the terms $\partial^{p-k}\phi_\alpha(0,\dots,0)/\partial t_{\mathcal{I}^{\mathrm{c}}}$ have a finite limit when α tends to 0. Besides, for all $k\geq 3$, $|\psi_\alpha^{(k)}(0)|=|\mathrm{i}^k\,\mathrm{E}((T_\alpha(a)-\mathrm{E}(T_\alpha(a)))^k)|\leq \mathrm{E}(|T_\alpha(a)-\mathrm{E}(T_\alpha(a))|^k)\leq 2^k\alpha^{-sk}$ and

$$\sigma_\alpha^T=\sqrt{\frac{\alpha}{2-\alpha}\operatorname{var}(T_\alpha(a))}\underset{\alpha\to 0}{\sim}\sqrt{\frac{\operatorname{var}(a)}{2}}\alpha^{1/2},$$

where $u(\alpha)\sim_{\alpha\to 0}v(\alpha)$ means that $u(\alpha)/v(\alpha)$ tends to 1 as α tends to 0. Using the classic notation $u(\alpha)=\mathcal{O}_{\alpha\to 0}\,v(\alpha)$ (meaning that there exists some constant Γ such that $|u(\alpha)|\leq\Gamma v(\alpha)$ in the neighborhood of 0), we observe that, for all $k\geq 3$,

$$\begin{aligned}&\frac{1}{\alpha}\bigg(\frac{\alpha}{\sigma_\alpha^T}\bigg)^k\psi_\alpha^{(k)}(0)\sum_{\substack{\mathcal{I}\subset\{1,\dots,p\}\\ \#\mathcal{I}=k}}\frac{\partial^{p-k}}{\partial t_{\mathcal{I}^{\mathrm{c}}}}\phi_\alpha(0,\dots,0)\bigg(\sum_{\substack{\mathcal{X}\subset\mathcal{Y}\\ \mathcal{X}\neq\varnothing}}\bigg(\prod_{i\in\mathcal{I}}\mathbf{1}(y_i\in\mathcal{X})\bigg)\bigg(\prod_{i\in\mathcal{I}^{\mathrm{c}}}\beta^{\mathbf{1}(y_i\in\mathcal{X})}\bigg)p_{\mathcal{X}}\bigg)\\ &\quad=\underset{\alpha\to 0}{\mathcal{O}}(\alpha^{k/2-sk-1}).\end{aligned}$$

But, since $s<\frac{1}{6}$, the above exponent $\frac{1}{2}k-sk-1$ is positive for all $k\geq 3$. Hence, all the terms for which $k\geq 3$ tend to 0.

Now, for $k=2$, we have $\psi_\alpha^{(2)}(0)=\mathrm{i}^2\operatorname{var}(T_\alpha(a))$. Besides, by induction,

$$\lim_{\alpha\to 0}\frac{\partial^{p-2}}{\partial t_{\{j_1,j_2\}^{\mathrm{c}}}}\phi_\alpha(0,\dots,0)=(\mathrm{i})^{p-2}\,\mathrm{E}\bigg(\prod_{l\in\{1,\dots,p\}\setminus\{j_1,j_2\}}^{p}f_G(y_l)\bigg).$$

Hence, considering subsets $\mathcal{I}$ of $\{1,\dots,p\}$ with two elements,

$$\begin{aligned}&\lim_{\alpha\to 0}\frac{1}{\alpha}\bigg(\frac{\alpha}{\sigma_\alpha^T}\bigg)^2\psi_\alpha^{(2)}(0)\sum_{\substack{\mathcal{I}\subset\{1,\dots,p\}\\ \#\mathcal{I}=2}}\frac{\partial^{p-2}}{\partial t_{\mathcal{I}^{\mathrm{c}}}}\phi_\alpha(0,\dots,0)\\ &\qquad\times\bigg(\sum_{\substack{\mathcal{X}\subset\mathcal{Y}\\ \mathcal{X}\neq\varnothing}}\bigg(\prod_{i\in\mathcal{I}}\mathbf{1}(y_i\in\mathcal{X})\bigg)\bigg(\prod_{i\in\mathcal{I}^{\mathrm{c}}}\beta^{\mathbf{1}(y_i\in\mathcal{X})}\bigg)p_{\mathcal{X}}\bigg)\\ &\quad=(\mathrm{i})^p 2\sum_{\{j_1,j_2\}\subset\{1,\dots,p\}}\mathrm{E}\bigg(\prod_{l\in\{1,\dots,p\}\setminus\{j_1,j_2\}}^{p}f_G(y_l)\bigg)\bigg(\sum_{\substack{\mathcal{X}\subset\mathcal{Y}\\ \mathcal{X}\neq\varnothing}}\mathbf{1}(y_{j_1}\in\mathcal{X})\,\mathbf{1}(y_{j_2}\in\mathcal{X})p_{\mathcal{X}}\bigg).\end{aligned}$$

In addition, note that

$$\sum_{\substack{\mathcal{X}\subset\mathcal{Y}\\ \mathcal{X}\neq\varnothing}} \mathbf{1}(y_{j_1}\in\mathcal{X})\,\mathbf{1}(y_{j_2}\in\mathcal{X})\,p_{\mathcal{X}} = \mathrm{P}(\{y_{j_1},y_{j_2}\}\subset x_0+X_0) = \frac{\gamma_X(y_{j_1}-y_{j_2})}{\mathrm{E}(\nu_d(\mathcal{Y}\oplus\check{X}))}.$$

Returning to (6), we see that $\partial^p\phi_\alpha(0,\dots,0)/\partial t_1\cdots\partial t_p$ admits a finite limit when α tends to 0. Writing $\mathrm{i}^p L$ the value of this finite limit, that is,

$$L = \lim_{\alpha\to 0}\mathrm{E}\biggl(\prod_{j=1}^p g_\alpha^T(y_j)\biggr),$$

we have the expression

$$\begin{aligned} L &= \frac{\mathrm{E}(\nu_d(\mathcal{Y}\oplus\check{X}))}{p\,\mathrm{E}(\nu_d(X))}2\sum_{\{j_1,j_2\}\subset\{1,\dots,p\}}\mathrm{E}\biggl(\prod_{l\in\{1,\dots,p\}\setminus\{j_1,j_2\}}^{p} f_G(y_l)\biggr)\frac{\gamma_X(y_{j_1}-y_{j_2})}{\mathrm{E}(\nu_d(\mathcal{Y}\oplus\check{X}))} \\ &= \frac{2}{p}\sum_{\{j_1,j_2\}\subset\{1,\dots,p\}}\frac{\gamma_X(y_{j_1}-y_{j_2})}{\mathrm{E}(\nu_d(X))}\mathrm{E}\biggl(\prod_{l\in\{1,\dots,p\}\setminus\{j_1,j_2\}}^{p} f_G(y_l)\biggr). \end{aligned}$$

This is exactly the recursive formula for the moments of a Gaussian vector given in Proposition 6. Hence,

$$L = \lim_{\alpha\to 0}\mathrm{E}\biggl(\prod_{j=1}^p g_\alpha^T(y_j)\biggr) = \mathrm{E}\biggl(\prod_{j=1}^p f_G(y_j)\biggr),$$

which completes the proof of Lemma 2.

5.4. Convergence in L^2 of the difference of the normalized random fields

To conclude the proof of Theorem 1, we require the following lemma.

Lemma 3. (Convergence to 0 in L^2 of $g_\alpha - g_\alpha^T$.) *Let g_α and g_α^T respectively be the normalized TDL model and the normalized TDL model with truncated colors. Then, for all $y\in\mathbb{R}^d$,*

$$g_\alpha(y) - g_\alpha^T(y) \xrightarrow[\alpha\to 0]{L^2} 0.$$

Proof. Since $a\in L^2$, and, for all $b\in\mathbb{R}$, $|T_\alpha(b)|\le|b|$ and $\lim_{\alpha\to 0}T_\alpha(b)=b$, by dominated convergence,

$$\lim_{\alpha\to 0}\mathrm{var}(a - T_\alpha(a)) = 0$$

and, in particular,

$$\lim_{\alpha\to 0}\mathrm{var}(T_\alpha(a)) = \mathrm{var}(a).$$

Let $y\in\mathbb{R}^d$. Recall that

$$\mathrm{var}(f_\alpha) = \frac{\alpha}{2-\alpha}\mathrm{var}(a) \quad\text{and}\quad \mathrm{var}(f_\alpha^T) = \frac{\alpha}{2-\alpha}\mathrm{var}(T_\alpha(a)).$$

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We have

$$\begin{aligned} g_\alpha(y) - g_\alpha^T(y) &= \frac{f_\alpha(y) - \mathrm{E}(a)}{\sqrt{\operatorname{var}(f_\alpha)}} - \frac{f_\alpha^T(y) - \mathrm{E}(T_\alpha(a))}{\sqrt{\operatorname{var}(f_\alpha^T)}} \\ &= \frac{f_\alpha(y) - \mathrm{E}(a)}{\sqrt{\operatorname{var}(f_\alpha)}} - \frac{f_\alpha^T(y) - \mathrm{E}(T_\alpha(a))}{\sqrt{\operatorname{var}(f_\alpha)}} + \frac{f_\alpha^T(y) - \mathrm{E}(T_\alpha(a))}{\sqrt{\operatorname{var}(f_\alpha)}} \\ &\quad - \frac{f_\alpha^T(y) - \mathrm{E}(T_\alpha(a))}{\sqrt{\operatorname{var}(f_\alpha^T)}} \\ &= \underbrace{\frac{f_\alpha(y) - f_\alpha^T(y) - \mathrm{E}(a - T_\alpha(a))}{\sqrt{\operatorname{var}(f_\alpha)}}}_{I_1(\alpha)} + \underbrace{\left(\frac{\sqrt{\operatorname{var}(f_\alpha^T)}}{\sqrt{\operatorname{var}(f_\alpha)}} - 1\right) g_\alpha^T(y)}_{I_2(\alpha)}. \end{aligned}$$

Note that the numerator of $I_1(\alpha)$ is a TDL model with color distribution $a - T_\alpha(a) - \mathrm{E}(a - T_\alpha(a))$. Hence, we have

$$\mathrm{E}(I_1(\alpha)^2) = \frac{\alpha/\operatorname{var}(a - T_\alpha(a))/(2-\alpha)}{\alpha \operatorname{var}(a)/(2-\alpha)} = \frac{\operatorname{var}(a - T_\alpha(a))}{\operatorname{var}(a)} \xrightarrow[\alpha\to 0]{} 0.$$

In addition,

$$\mathrm{E}(I_2(\alpha)^2) = \left(\frac{\sqrt{\operatorname{var}(f_\alpha^T)}}{\sqrt{\operatorname{var}(f_\alpha)}} - 1\right)^2 = \left(\frac{\sqrt{\operatorname{var}(T_\alpha(a))}}{\sqrt{\operatorname{var}(a)}} - 1\right)^2 \xrightarrow[\alpha\to 0]{} 0.$$

Hence, $g_\alpha(y) - g_\alpha^T(y)$ is the sum of two RVs which tends to 0 in L^2. This completes the proof.

Acknowledgements

This work was supported by ANR MATAIM. The authors are grateful to Professors Jean-Michel Morel and François Roueff for their insightful suggestions.

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