An Extension of Clairaut's Differential Equation.

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§1. I propose in this paper to investigate the geometrical nature of a certain species of differential equation, which includes, as a particular case, Clairaut's well worn equation

$$y - px = f(p).$$

We may regard any ordinary differential equation

$$\phi(xyp) = 0 \qquad (1)$$

either from a purely analytical or from a geometrical standpoint. We shall only consider, for the moment at least, the latter point of view. Commencing at any point (x_1y_1) in the plane, equation (1) defines a finite number of directions associated with that point. Restricting ourselves to any one of these branches, and moving for an infinitesmal distance along it, to the neighbouring point $(x_1 + \Delta x_1, y_1 + \Delta y_1)$ we are enabled, once more by (1), to determine the gradient of the same branch at this point, and so on by repeating this process we gradually trace out an integral curve of the differential equation. Generally, the whole system of integral curves in the plane may be considered to be exhaustively traced out if we suppose any curve C extending across the plane, and from every point of it, integral curves start off in the directions determined by equation (1) at the point. The gradients of the integral curves at the points of section with C will in general be all different, but it will be possible so to choose C that the tangents to corresponding branches of the integrals where they cross be parallel.

The curve C is then the locus of points of constant gradient on the system defined by (1), and therefore C is merely one, of an infinity of such curves given by

$$\phi(xyc) = 0 \qquad (2)$$

A system such as (2) is termed the "Isoclinal Family" of (1).

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But the curves given by (2) may equally well be regarded as the integral system of some differential equation

$$f(xyp) = 0,$$

which itself has an isoclinal family

$$f(xyc) = 0.$$

The first interesting case arises when this latter system is identical with the integral curves of (1), that is to say, $\phi(xyp) = 0$ and $f(xyp) = 0$ are mutually isoclinal.

In this case the solution of

$$\phi(xyp) = 0 \text{ is } f(xyc) = 0,$$

and of $$f(xyp) = 0 \text{ is } \phi(xyc) = 0.$$

For example, consider the differential equation

$$x^2 - px + y = 0,$$

the solution of which is

$$x^2 + cx - y = 0.$$

The isoclinal family is

$$x^2 - cx + y = 0,$$

and its differential equation

$$x^2 + px - y = 0,$$

a system whose isoclinals are given by

$$x^2 + cx - y = 0.$$

The two families of parabolas

$$x^2 - cx + y = 0$$
$$x^2 + cx - y = 0$$

must therefore be mutually isoclinal.

A more interesting case arises, however, when the isoclinal system is identical with the original curves, that is to say, the system of curves are their own isoclinals, and each member is cut by all the others at the same inclination. $\phi(xyc) = 0$ will now be the complete integral of $\phi(xyp) = 0$.

The gradient at any point (xy) is given by

$$p = -\frac{\phi_x(x\,y\,c)}{\phi_y(x\,y\,c)}$$

where c is to be substituted as a function of x and y from

$$\phi(xyc) = 0.$$

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If $\phi(xyc_1)=0$ be any member of the system, the gradient of all the curves crossing it is given by $p=c_1$.

Hence the condition that $\phi(xyp)=0$ be self isoclinal is that the elimination of c and c_1 between

$$\left.\begin{array}{l} c_1=-\phi_x(xyc)/\phi_y(xyc) \\ \phi(xyc)=0 \\ \phi(xyc_1)=0 \end{array}\right\}\dots\dots\dots\dots\dots\dots\dots(3)$$

should lead to an identity in x and y.

Stated in the general form (3) it becomes excessively difficult to determine the form ϕ must assume to satisfy the required conditions, but there is one case of especial interest which we can treat. If we assume, for the moment, $c=c_1$ the problem becomes, to determine the function $\phi(xyp)=0$, to satisfy the condition

$$\phi_x(xyp)+p\phi_y(xyp)=0\dots\dots\dots\dots\dots\dots\dots(4)$$

Suppose

$$\phi\equiv{}_0\mathrm{A}p^n+{}_1\mathrm{A}p^{n-1}+{}_2\mathrm{A}p^{n-2}+\dots+{}_{n-1}\mathrm{A}p+{}_n\mathrm{A}=0 \qquad (5)$$

where ${}_r\mathrm{A}$ is some function of (xy) to be determined. On substituting the expression (5) in the condition (4), and equating the coefficients of the powers of p to zero, we obtain the equations:—

(i) $${}_0\mathrm{A}_y=0$$
$$\therefore\quad {}_0\mathrm{A}=\psi_0(x)$$

(ii) $${}_1\mathrm{A}_y=-{}_0\mathrm{A}_x=-\psi_0'(x)$$
$$\therefore\quad {}_1\mathrm{A}=-y\psi_0'(x)+\psi_1(x)$$

(iii) $${}_2\mathrm{A}_y=-{}_1\mathrm{A}_x=y\psi_0''(x)-\psi_1'(x)$$
$$\therefore\quad {}_2\mathrm{A}=\frac{y^2}{2!}\psi_0''(x)-y\psi_1'(x)+\psi_2(x)$$

etc., etc.

(n) $${}_{n-1}\mathrm{A}_x=-{}_n\mathrm{A}_y.$$
$$\therefore\quad {}_n\mathrm{A}=(-1)^n\left[\frac{y^n}{n!}\psi_0^{(n)}-\frac{y^{n-1}}{(n-1)!}\psi_1^{(n-1)}+\dots+\psi_n\right]$$

($n+1$) $${}_n\mathrm{A}_x=0.$$
$$\therefore\quad \psi_0^{(n)},\ \psi_1^{(n-1)},\ \dots\psi_n \text{ are all constants.}$$

Hence ψ_0 is an arbitrary function of the n^{th} degree, ψ_1 of the $\overline{n-1}^{\text{th}}$ degree, ψ_r of the $\overline{n-r}^{\text{th}}$ degree, and ψ_n is a constant, and the coefficients in (5) are thus completely determined.

Consider in particular the following example which will serve to illustrate the foregoing.

Suppose

$$\psi_0 = x^2,\ \psi_1 = x - 1,\ \psi_2 = 1$$

then the equation formed according to the above is

$$x^2p^2 + (x - 1 - 2xy)p + y^2 - y + 1 = 0,$$

and its solution will therefore be

$$x^2c^2 + (x - 1 - 2xy)c + y^2 - y + 1 = 0.$$

But on solving the proposed equation for y it may be thrown into the form

$$y = px + \frac{1 \pm \sqrt{4p - 3}}{2}$$

which is in the form of Clairaut's equation.

And quite generally it can be shown that every equation of the type (5) already determined must admit of being thrown into Clairaut's form. This may be made at once evident, either by grouping the terms of equation (5), or by regarding the original condition (4) as a partial differential equation with xyp as independent variables, as follows :—

$$\frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\therefore \quad \frac{dx}{1} = \frac{dy}{p} = \frac{dp}{0} = \frac{d\phi}{0},$$

$$\therefore \quad \phi = f(px - y,\ p) = 0$$

where f is arbitrary, or in the more usual form

$$y - px = f(p).$$

It is worthy of remark that the condition (4) affords a valuable test to determine whether any given equation may be thrown directly into Clairaut's form.

Suppose, for example, that

$$\phi \equiv p^3(x^3 - x^2 - 1) - p^2(3x^2y - 2xy + 1) + py^2(3x - 1) - y^3 = 0.$$

Applying the condition (4) we obtain

$$\begin{aligned}\phi_x + p\phi_y &= (3x^2 - 2x)p^3 - (6xy - 2y)p^2 + 3y^2p \\ &\quad + p[-3x^2p^2 + 2xp^2 + 2y(3x - 1)p - 3y^2] \\ &= 0.\end{aligned}$$

Hence the condition is satisfied, and the complete solution is therefore

$$c^3(x^3 - x^2 - 1) - c^2(3x^2y - 2xy + 1) + cy^2(3x - 1) - y^3 = 0.$$

The equation may in fact be represented in the form

$$(xp - y)^3 - p(xp - y)^2 - p^2(p + 1) = 0.$$

We may illustrate this with a further example,

$$\phi \equiv p^3 + p^2x^2 - 2p(1 + x + xy) + (y + 1)^2 = 0.$$

Here $\qquad \phi_x + p\phi_y = 2xp^2 - 2(y + 1)p + 2(y + 1)p - 2xp^2 = 0,$

and the solution is therefore given by

$$c^3 + c^2x^2 - 2c(1 + x + xy) + (y + 1)^2 = 0.$$

As before, the equation may otherwise be written

$$(xp - y)^2 - 2(xp - y) + p^3 - 2p + 1 = 0.$$

If a differential equation, therefore, satisfies the condition $\phi_x + p\phi_y = 0$, whether it be possible to solve for y algebraically or not, and in general it will not, the integral is given by replacing p by c in the original equation.

Clairaut's equation, however, represents a system of straight lines, and it might be asked in what sense such a system can be regarded as self-isoclinal, since any one member is cut by all the others in general at various angles.

But the assumption $c = c_1$ in the conditions (3), by which we arrived at this form, is in reality equivalent to the statement that every member of the system of curves is met by itself at a constant inclination, that is to say, the inclination of the curve at all points of it is constant, and consequently it represents a straight line. Each member is an isoclinal for itself.

§2. But after all, the assumption $c = c_1$ would appear to be merely a very particular case of the general problem, while from the geometrical point of view there is every reason to suppose that forms of ϕ may exist other than Clairaut's, which represent a a system of self-isoclinals.

The question may be considerably simplified by the following consideration, which gives an insight into the problem so far as to indicate how such equations may arise analytically.

Suppose, if possible, that

$$\phi(xyp) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is solved for p in terms of x and y, and expressed finally in the form

$$\chi + p\psi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where χ and ψ are irrational functions of x and y, containing many ambiguities as regards sign. The rationalization of (6) will give rise to (1).

Consider the conditions which must be imposed on χ and ψ in order that

$$\chi + c\psi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

may be the solution of (6), in which case on rationalization a form of ϕ will be evolved of the required type.

On differentiating (7) we obtain

$$\chi_x + p\chi_y + c(\psi_x + p\psi_y) = 0,$$

and eliminating p and c by means of (6) and (7) the condition is obtained

$$\chi\psi(\chi_y + \psi_x) = \chi^2\psi_y + \psi^2\chi_x$$

or

$$(\chi/\psi)\frac{\partial}{\partial y}(\chi/\psi) - \frac{\partial}{\partial x}(\chi/\psi) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Regarding (8) as a partial differential equation, the solution is evidently

$$y + x\chi/\psi = f(-\chi/\psi)$$

where f is an arbitrary function.

But from equation (6)

$$p = -\chi/\psi,$$

hence

$$y - px = f(p)$$

must be the differential equation sought, and this is of Clairaut's form once more. But if we glance back to equations (6) and (7) the reason is at once evident, for we have chosen χ and ψ in the former identical with those in the latter, whereas these functions were capable of numerous ambiguous forms. We have in fact chosen one of the branches of (1) at any point and made it isoclinal to itself everywhere, thus giving rise, as already seen earlier in the paper, to Clairaut's equation, whereas what ought to be done, is to choose two branches of (1) passing through any point, and make these isoclinal everywhere in the plane. Therein lies the true extension of Clairaut's equation.

Expressed analytically, if

$$\chi_1 + p\psi_1 = 0 \quad \text{.............................(10)}$$

$$\chi_2 + p\psi_2 = 0 \quad \text{.............................(11)}$$

be two equations such as (6), where χ_1 and χ_2, ψ_1 and ψ_2 differ only as regards the signs of the irrational parts involved, then (10) and (11) on rationalization will lead to the same equation. It follows immediately that if the solution of (10) be

$$\chi_2 + c\psi_2 = 0,$$

then the solution of the rationalized equation

$$\phi(xyp) = 0$$

is

$$\phi(xyc) = 0.$$

For example, consider the case where ϕ is of the second degree in p, so that we may write

$$p = \chi + \sqrt{\psi},$$

and suppose that the solution of this is

$$c = \chi - \sqrt{\psi}.$$

Differentiating the latter and inserting the value of p from the former, we obtain the equation

$$2(\chi_x + \chi_y\chi)\sqrt{\psi} + 2\chi_y\psi = \psi_x + \psi_y\chi + \psi_y\sqrt{\psi} \quad \text{......... (12)}$$

In order that p may appear in the final equation to the second degree, it will be sufficient to determine χ and ψ to satisfy these conditions

$$\left.\begin{aligned} 2(\chi_x + \chi_y\chi) &= \psi_y \\ \psi_x + \psi_y\chi &= 2\psi\chi_y \end{aligned}\right\} \quad \text{.................... (13)}$$

As an illustration, suppose $\chi_y = 0$; then

$$\psi = -c^2x^2 - dx + 2cy + f,$$

$$\chi = cx + d/2c,$$

where c, d, and f are constants.

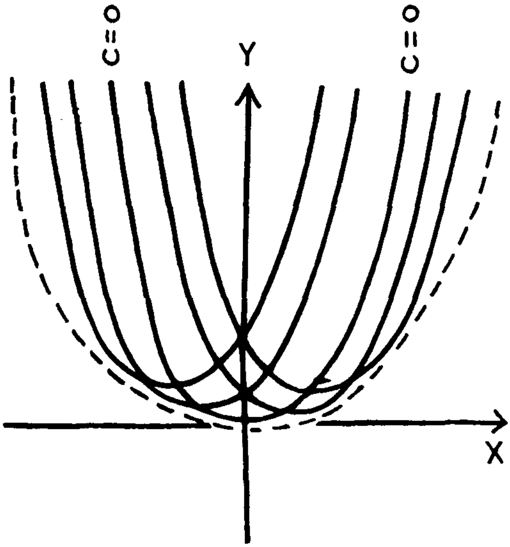
When $\quad c = 1 \quad d = 0 \quad f = 0$

$$p = x + \sqrt{2y - x^2},$$

that is, $\quad p^2 - px + 2x^2 - 2y = 0,$

the solution of which is $\quad c^2 - 2cx + 2x^2 - 2y = 0.$

The last equation represents a system of parabolas, any member of which is cut by all the others at the same inclination. (See fig.)



Suppose $\chi_x = 0$ in (13), then

$$\psi = (cy + d)^2 + ae^{2cx},$$

$$\chi = cy + d.$$

Taking $c = 1$, $d = 0$, $a = 1$, we get the equation

$$p = y + \sqrt{y^2 + e^{2x}},$$

that is, $$p^2 - 2py - e^{2x} = 0,$$

and its solution is $$c^2 - 2cy - e^{2x} = 0,$$

a system possessing a curious property, to which we shall return. In the same manner, by making certain assumptions as regards ψ or χ, numerous other examples may be constructed illustrating the above.

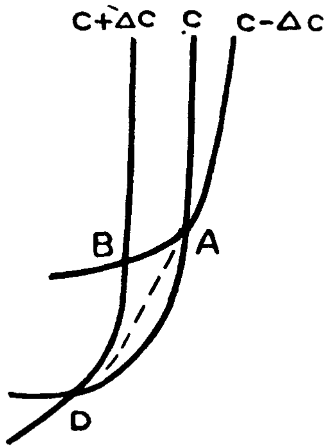
There is a special property of Clairaut's differential equation which appears at first sight not to be necessarily shared by the generalized equation as we view it, and that is, the fact that every system of Clairaut's type possesses a singular solution as envelope locus.

Now, in discussing the question geometrically, it was tacitly assumed that every member of the self-isoclinal system was cut by every other, and therefore by its consecutive, but this is not necessarily true. For example, in the case above quoted,

$$c^2 - 2cy - e^{2x} = 0,$$

no two members, for which c has the same sign, intersect in real points.

If we restrict ourselves to such families of curves where every member meets every other, then it can easily be shown that every such system must have an envelope locus. For, consider the the behaviour of three such consecutive curves $c - \triangle c$, c, $c + \triangle c$, at their points of intersection, A, B, C. (See fig.)



The gradient at A of c is $c - \triangle c$, the gradient at D of $c + \triangle c$ is c, and therefore the gradient of AD lies between $c - \triangle c$ and $c + \triangle c$, and hence in the limit DA will be an elemental tangent of inclination c, and this can be repeated continuously in the two directions $\overrightarrow{DA}$ and $\overrightarrow{AD}$, and thus we trace out an envelope locus, the gradient at any point of which is equal to the constant determining the particular member of the isoclinals which touches the envelope at the point in question.

The equation

$$p^2 - 2px + 2x^2 - 2y = 0,$$

representing a system of self-isoclinals, has, for example, the singular solution

$$x^2 - 2y = 0\ ;$$

while in the case of

$$p^2 - 2py - e^{2x} = 0,$$

any two consecutive members of the system meet in an imaginary point having contact there with the imaginary envelope

$$y^2 + e^{2x} = 0.$$