A NOTE ON COMBINATIONS

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We call k integers $x_1 < x_2 < \ldots < x_k$ chosen from $\{1, 2, \ldots, n\}$ a <u>k-choice (combination) from n</u>. With $1, 2, \ldots, n$ arranged in a circle, so that 1 and n are consecutive, we have a <u>circular</u> k-choice from n. A <u>part</u> of a k-choice from n is a sequence of consecutive integers not contained in a longer one. Let $\bar{A}_r(n,k;w)$ denote the number of circular k-choices from n with exactly r parts all $\leq w$. Of course $\bar{A}(n,k;w) = \sum_{r=1}^{w} \bar{A}_r(n,k;w)$ is the number of circular k-choices from n with all parts $\leq w$. In this note we prove that

$$(1) \qquad \bar{A}_r(n,k;w) = \frac{n}{n-k}\binom{n-k}{r}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{k-iw-1}{r-1}, \qquad 0 < k < n,$$

and deduce an expression for $\bar{A}(n,k;w)$, the numbers mentioned in [1, p. 593].

To establish (1) observe that circular k-choices from n can be conveniently represented by $n-k$ symbols 0 (one for each integer not in the k-choice) and k symbols 1 (one for each integer in the k-choice) arranged in a circle, with one of the symbols marked (by a* say) corresponding to the integer 1 (rising order being clockwise). For example, for $n = 8$,

1	0
0* 0	0 1
0 0	0 1*
1 1	1 1
1	0
represents	represents
2, 5, 6, 7	1, 2, 4, 8

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We find the arrangements representing the choices counted in $\bar{A}_r(n,k;w)$ as follows. Array n-k symbols 0 in a circle, forming n-k cells (the spaces between); label the cells so that they are distinguishable. Choose r of them in $\binom{n-k}{r}$ ways. The k symbols 1 may be distributed into the r chosen cells, with none empty, in $C(k,r;w)$ ways, where $C(k,r;w)$ is the number of r-compositions of k with all parts $\leq w$. We now mark one of the n symbols with a*, obtaining $n\binom{n-k}{r} C(k,r;w)$ configurations. Removing the labels from the cells, the configurations fall into sets of n-k each which are the same by rotation. These

$$\frac{n}{n-k}\binom{n-k}{r} C(k,r;w)$$

arrangements represent the k-choices from n with r parts all $\leq w$. Since [4, p. 124] $C(k,r;w)$ is the coefficient of x^k in $(x+x^2+\ldots+x^w)^r$, it easily follows that $C(k,r;w) = \sum_{i=0}^{r} (-1)^i \binom{r}{i}\binom{k-iw-1}{r-1}$, and hence (1). Furthermore

$$\begin{aligned}\bar{A}(n,k;w) &= \frac{n}{n-k} \sum_{i=0}^{r} (-1)^i \sum_{r=1}^{k-iw-1} \binom{n-k}{r}\binom{r}{i}\binom{k-iw-1}{r-1} \\ &= \frac{n}{n-k} \sum_{i=0}^{r} (-1)^i \binom{n-k}{i} \sum_{r=1}^{k-iw-1} \binom{n-k-i}{r-i}\binom{k-iw-1}{r-1} \\ &= \frac{n}{n-k} \sum_{i=0}^{r} (-1)^i \binom{n-k}{i}\binom{n-iw-i-1}{n-k-1}.\end{aligned}$$

A similar argument in the "straight line case yields

$$(3)\qquad A_r(n,k;w) = \binom{n-k+1}{r} \sum_{i=0}^{r} (-1)^i \binom{r}{i}\binom{k-iw-1}{r-1}.$$

Summing (3) over r yields [1]

$$A(n,k;w) = \sum_{i=0}^{r} (-1)^i \binom{n-iw-i}{n-k}\binom{n-k+1}{i}$$

If we agree to let $A(n,k;w) = 0$ when $n < 0$ or $k < 0$ or $n < k$, the following recurrence relation holds for all values of n, k, w except $n = k = w + 1$:

$$(4) \qquad A_r(n,k;w) = A_r(n-1,k;w) + A_{r-1}(n-2,k-1;w) + A_r(n-1,k-1;w) - A_r(n-2,k-1;w) - A_r(n-w-2,k-w-1;w) .$$

For sufficiently large values of w (say $w = n$), (1), (3) and (4) reduce respectively to relations (5), (3) and (7) given in [2]. (The proof of (5) in [2] is incorrect, because of an unfortunate error, although the formula is correct.)

Calling a pair of consecutive integers i, $i+1$ a <u>succession</u>, we see that a k-choice from n with exactly r parts contains exactly $k-r$ successions. Hence the number of straight line or circular k-choices from n containing s successions and having all parts $\leq w$ is respectively $A_{k-s}(n,k;w)$ or $\bar{A}_{k-s}(n,k;w)$. With w large, the former reduces to a theorem of Riordan [3].

REFERENCES

1. M. Abramson, Restricted choices. Canad. Math. Bull. 8 (1965) 585-600.

2. M. Abramson and W. Moser, Combinations, successions and the n-kings problem. Math. Mag.

3. J. Riordan, Permutations without 3-sequences. Bull. Amer. Math. Soc. 51 (1945) 745-48.

4. ______________, An Introduction to Combinatorial Analysis, New York, 1958.

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