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EXTENSION OF RIESZ HOMOMORPHISMS. III

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Abstract

In this paper we prove an analogue of the separable version of Nachbin's characterization of injective Banach spaces in the setting of Banach lattices. The mappings involved are continuous Riesz homomorphisms defined on ideals of separable Banach lattices which can be extended to Riesz homomorphisms on the whole Banach lattice. We discuss applications to simultaneous extension operators and to extension of continuous mappings between certain topological spaces.

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Introduction

In connection with extension theorems of the Hahn-Banach type, Nachbin came across a property which is nowadays called the σ-interpolation property. A little later, Cohen [6] proved the following result.

THEOREM A. *$C(Y)$ is a Riesz space with the σ-interpolation property if and only if for every two separable Banach spaces $B_1 \supset B_2$ and every operator from B_2 into $C(Y)$, there exists a norm preserving extension from B_1 into $C(Y)$.*

A somewhat similar result was obtained by Lindenstrauss in [10]. Though not obvious, it is a fact that the word 'operator' in Cohen's theorem can be replaced by 'positive operator' and 'extension' by 'positive extension' if one changes 'Banach spaces' to 'Banach lattices'. For Riesz homomorphisms it is natural to

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investigate the situation in which B_2 is a special type of sublattice of B_1. Indeed in [5] we proved the following result.

THEOREM B. *For every two separable Banach lattices $B_1 \supset B_2$ such that B_2 is majorizing in B_1 and for every continuous Riesz homomorphism from B_2 into $C(Y)$ there exists a Riesz homomorphic (not necessarily norm preserving) extension if and only if $C(Y)$ has the σ-interpolation property and Y is totally disconnected.*

In this paper, we will prove yet another extension theorem concerning spaces with the σ-interpolation property of which the nature will be very different from the above Theorems A and B.

THEOREM C. *For every two separable Banach lattices $B_1 \supset B_2$ such that B_2 is an ideal in B_1 and every continuous Riesz homomorphism from B_2 into $C(Y)$ there exists a Riesz homomorphic extension from B_1 into $C(Y)$ if and only if $C(Y)$ has the σ-interpolation property.*

Contrary to Theorems A and B, it seems that Theorem C is not rooted in the line of Hahn-Banach theorems. Indeed, our approach has its origins in Tietze's extension theorem rather than in the Hahn-Banach theorem. It is therefore not surprising that Theorem C has applications to topological problems.

It is for its topological consquences and results about simultaneous extension operators that we singled out Theorem C from a rather lengthy report [2].

I. Simultaneous extension operators

In this section Y will always be a compact F-space (see [7]). $C_b(X)$ for a topological space X, denotes the space of all bounded continuous functions on X. For a subset X of Y we define the restriction operator R_X: $C(Y) \to C_b(X)$ by $R_X(f) = f_{|X}$ for all $f \in C(Y)$. We will call a bounded linear operator T: $C_b(X) \to C(Y)$ an *extensor* if it is a Riesz homomorphism and $R_X \circ T = \mathrm{Id}_{C_b(X)}$. Also, if H is a Riesz subspace of $C_b(X)$, a bounded linear operator T: $H \to C(Y)$ is called an extensor if it is a Riesz homomorphism and $R_X \circ T = \mathrm{Id}_H$. The definition of an F-space merely states that for every cozero-set $U \subset Y$, the restriction operator R_U is surjective and if $H \subset C_b(U)$ is not finite dimensional this is not very helpful in finding an extensor T: $H \to C(Y)$. The following can easily be derived, for instance from Theorem 21.13 in [14].

PROPOSITION 1. *The following are equivalent.*

(1) *For every cozero-set $U \subset Y$, there exists an extensor $C_b(U) \to C(Y)$ which sends 1_U to 1_Y.*

(2) *There exists a retraction $Y \to \overline{U}$ for every cozero-set U in Y.*

From certain points of view Proposition 1 is not very satisfactory. For instance, what F-spaces do have property (2) of Proposition 1? It easily follows that Y has to be totally disconnected. Basic disconnectedness is a sufficient condition, but not a necessary condition. (Negrepontis in [12] shows that $\beta\mathbf{N} \setminus \mathbf{N}$ satisfies (2) of Proposition 1.) We will show that, as long as one only considers extensors from separable Riesz subspaces of $C_b(U)$ for cozero-sets $U \subset Y$, the picture becomes more transparent. The technique which we will use is essentially due to Arens [1] and is based on the following theorem (see [1]).

THEOREM 2. *Let A be a closed subset of a fully normal space X. Let f be continuous on A with values in a convex subset K of a locally convex linear topological space that is a complete metric space in the induced topology. Then f can be extended to a continuous map on X with all values still in K.*

Theorem 2 will be the heart of our proof of the next theorem.

THEOREM 3. *Let U be a cozero-set in Y and let H be a separable Riesz subspace of $C_b(U)$. Then there exists an extensor T: $H \to C(Y)$ with $\|T\| \leqslant 1$.*

PROOF. The proof is divided into two steps.

Step 1. There exists a positive simultaneous extension operator:

Every element of H can uniquely be extended to an element of $C(\overline{U})$. Thus, we may assume that H is a Riesz subspace of $C(\overline{U})$. Denote $\partial U = \overline{U} \setminus U$ and define i: $\partial U \to (\overline{H}_{|\partial U})^*$ by $i(x)(f) = f(x)$ for all $x \in \partial U$ and all $f \in \overline{H}$. ($\overline{H}$ is the closure of H in the uniform norm.) Diagramatically we have

$$\begin{array}{ccc} \partial U & \xrightarrow{\ i\ } & (\overline{H}_{|\partial U})^* \\ \cap & \nearrow_{\alpha} & \\ Y \setminus U & & \end{array}$$

Consider $K = \{h \in (\overline{H}_{|\partial U})^{*+} \mid \|h\| \leqslant 1\}$ with the w^*-topology. Remark that K is convex and metrizable by the standard arguments (see Theorem 3.16 in [13]), compact by Alaoglu's theorem and thus metrically complete by 24C.3 in [16]. i is continuous and $i(\partial U) \subset K$. Thus, Theorem 2 yields the existence of a continuous map α: $Y \setminus U \to K$ extending i. For $f \in H$, we define

$$\bar{f}(y) = \begin{cases} f(y) & \text{if } y \in U, \\ \alpha(y)(f_{|\partial U}) & \text{if } y \in Y \setminus U. \end{cases}$$

Then $\bar{f}$ is a continuous function on Y and $\bar{f}_{|\bar{U}} = f$. Furthermore, $\bar{T}$: $H \to C(Y)$ defined by $f \to \bar{f}(f \in H)$ is linear and, because $\alpha(Y \setminus U) \subset K$, positive as well. Obviously, $\|\bar{T}\| \leqslant 1$. We may assume that $\bar{T}$: $\bar{H} \to C(Y)$.

Step 2. There exists an extensor:

Remark that $\bar{H} \times \bar{H}$ is separable and metrizable under the product topology. Define $\Delta = \{(f, g) \,|\, f, g \in \bar{H}$ and $f \wedge g = 0\}$. Δ is separable as well. Choose a countable dense subset $\bar{\Delta} = \{(f_n, g_n) \,|\, n \in \mathbf{N}\}$ of Δ. Define $A = \{f_n \,|\, n \in \mathbf{N}\} \cup \{g_n \,|\, n \in \mathbf{N}\}$. Remark that for all f and g from $\bar{H}$ with $f \wedge g = 0$ and all $\varepsilon > 0$, there exist $\bar{f}$ and $\bar{g}$ in A with $\|f - \bar{f}\|_{\bar{U}} \leqslant \varepsilon$, $\|g - \bar{g}\|_U \leqslant \varepsilon$ and $\bar{f} \wedge \bar{g} = 0$. Define $V = \{y \in Y \,|$ there exists $n, m \in \mathbf{N}$ with $(\bar{T}f_n)(y) \wedge (\bar{T}g_m)(y) \neq 0\}$ where $\bar{T}$ is as constructed in step 1. V is an open F_σ and $V \cap U = \varnothing$. Therefore, by 14N4 in [7], there exists a continuous function $0 \leqslant F \leqslant 1_Y$ with $F(V) = \{0\}$ and $F(U) = \{1\}$. We define T: $\bar{H} \to C(Y)$ by $T(f) = F\bar{T}(f)$ $(f \in \bar{H})$. We will show that T is a Riesz homomorphism. Therefore, suppose $f \wedge g = 0$ and $\varepsilon > 0$. Choose $\bar{f}$ and $\bar{g}$ from A with $\|f - \bar{f}\|_{\bar{U}} \leqslant \varepsilon$ and $\|g - \bar{g}\|_{\bar{U}} \leqslant \varepsilon$ and $\bar{f} \wedge \bar{g} = 0$. Surely $T(\bar{f}) \wedge T(\bar{g}) = 0$ and

$$\begin{aligned} \|T(f) \wedge T(g)\|_Y &= \left\|\left(T(f - \bar{f}) + T(\bar{f})\right) \wedge \left(T(g - \bar{g}) + T(\bar{g})\right)\right\|_Y \\ &\leqslant \left\|T(f - \bar{f}) \wedge T(g - \bar{g})\right\|_Y + \left\|T(g - \bar{g}) \wedge T(\bar{f})\right\|_Y \\ &\quad + \left\|T(f - \bar{f}) \wedge T(\bar{g})\right\|_Y \leqslant 3\varepsilon. \end{aligned}$$

Thus, $T(f) \wedge T(g) = 0$.

We wish to remark that the proof of step 1 is similar to the proof of Theorem 5.2 in Arens' paper [1]. The same technique was used in [2] for a slightly different setting. The following corollary is immediate.

COROLLARY 4. *For a compact space Z the following are equivalent.*

(1) *Z is an F-space.*

(2) *For every cozero-set U in Z and every separable Riesz subspace H of $C_b(U)$ there exists an extensor $H \to C(Z)$.*

Getting a bit closer in nature to Proposition 1, we state the following.

PROPOSITION 5. *Each of the conditions* (1) *and* (2) *in Corollary* 4 *is equivalent to*

(3) *For every cozero-set U in Z, every compact metrizable space X and every continuous map τ: $U \to X$, there exist an open subset W of Z containing $\overline{U}$ and a continuous map $\bar{\tau}$: $W \to X$ which extends τ.*

PROOF. (3) $\Rightarrow$ (1): Apply Urysohn's lemma.

(2) $\Rightarrow$ (3): By the fact that Z is an F-space, we can extend τ: $U \to X$ to τ: $\overline{U} \to X$ (indeed, $\overline{U} = \beta U$). We may assume that $\tau(\overline{U}) = X$. Therefore, we can embed $C(X)$ into $C(\overline{U})$ and by (2) there exists an extensor T: $C(X) \to C(Z)$. Denoting $W = \{z \in Z \mid$ there exists $f \in C(X)$ such that $T(f)(z) \neq 0\}$ it is easy to see that there exist a continuous map $\bar{\tau}$: $W \to X$ and a continuous function ρ: $W \to \mathbf{R}^+$ such that $T(f)(z) = \rho(z)(f \circ \bar{\tau})(z)$ if $z \in W$ and $(Tf)(z) = 0$ otherwise, for all $f \in C(X)$. (In fact a more general representation theorem for Riesz homomorphisms will be proved in Section 2, Theorem 9.) Remark that $V = \{z \in Z \mid T(1_X)(z) > \frac{1}{2}\}$ is a cozero-set, that $U \subset V$ and that $\bar{\tau} | V$ extends τ.

Even closer to Proposition 1 is the following.

PROPOSITION 6. *For a compact space Z the following are equivalent.*

(1) *Z is a totally disconnected F-space.*

(2) *For every cozero-set U in Z and every separable Riesz subspace H of $C_b(U)$ which contains 1_U, there exists an extensor $H \to C(Z)$ which sends 1_U to 1_Z.*

(3) *For every cozero-set U in Z and every compact metrizable space X and every continuous map τ: $U \to X$, there exists an extension $\bar{\tau}$: $Z \to X$.*

PROOF. (1) $\Rightarrow$ (2): It is clear from Proposition 5 that (given U and H) an extensor $H \to C(Z)$ does exist. Let T: $H \to C(Z)$ be an extensor with $\|T\| \leqslant 1$. Let $A = \{z \in Z \mid T(1_U)(z) > \frac{1}{2}\}$ and $B = \{z \in Z \mid T(1_U)(z) < \frac{1}{2}\}$. Then $\overline{A} \cap \overline{B} = \varnothing$ because Z is an F-space and because Z is totally disconnected there exists an open-and-closed set W with $\overline{A} \subset W$ and $\overline{B} \subset Z \setminus W$. Let $z_0 \in U$. Define

$$T_1(f) = T(f)1_W + f(z_0)1_{Z\setminus W} \quad (f \in H).$$

It follows that $T_1(1_U)_{|W} = T(1_U)_{|W}$ and $e = T_1(1_U) \geqslant \frac{1}{2}1_Z$. Furthermore, T_1 is a Riesz homomorphism. Finally, define $\overline{T}(f) = e^{-1}T_1(f)$ $(f \in H)$. $\overline{T}$ is an extensor and $\overline{T}(1_U) = 1_Z$.

(2) $\Rightarrow$ (3): Proceeds as the lines of (2) $\Rightarrow$ (3) of Proposition 5, but is easier.

(3) $\Rightarrow$ (1): That Z is an F-space follows from Proposition 5. To prove that Z is totally disconnected, let A and B be disjoint closed sets in Z. Choose cozero-sets U_1, U_2 with $\overline{U}_1 \cap \overline{U}_2 = \varnothing$, $A \subset U_1$ and $B \subset U_2$. It is not hard to see (use

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Corollary 1.61 and Proposition 1.64 of [15]) that continuous surjections $\overline{U}_1 \to [0,1]$ and $\overline{U}_2 \to [2,3]$ do exist. The map $\overline{U}_1 \cup \overline{U}_2 \to [0,1] \cup [2,3]$ which is thus given can be extended to a continuous map τ: $Z \to [0,1] \cup [2,3]$. It follows that $\tau^{-1}[0,1]$ and $\tau^{-1}[2,3]$ are open-and-closed sets containing A and B respectively.

Concluding Section I we point to the relation between Proposition 6 and Proposition 1. We may or may not be able to extend id: $\overline{U} \to \overline{U}$ to a continuous map $Y \to \overline{U}$ for every compact totally disconnected F-space Y and every cozero-set U in Y. However, every continuous map $\overline{U} \to X$ for a compact metrizable space (under the same conditions for Y and U) does extend to a continuous map $Y \to X$. The method which we have been using here is restricted to the separable case. It seems not unlikely, however, that extensors from bigger subspaces of $C_b(U)$ exist if $C(Y)$ has the α-interpolation property for $\alpha > \omega_0$. At the same time, i.e. under the same conditions, it may not be impossible to relax the countability condition on U.

2. Extension of $C(Y)$-valued Riesz homomorphisms

In this section we will prove Theorem C which was mentioned in the introduction. We introduce some notation. For two Banach lattices E and F we say that (E, F) has property (uI) if for every ideal $I \subset E$ and every continuous Riesz homomorphism $I \to F$ there exists a Riesz homomorphic extension $E \to F$. Though only defined for Banach lattices here, we remark that by introducing the right topologies on the Riesz spaces E and F one can define a property (uI) very generally. Though we intend to return to this subject later, we do not need such generality in this paper. We do need the following theorem from [4], which we state for the reader's convenience.

THEOREM 7. *Suppose Y is a compact Hausdorff space. Then the following are equivalent.*

(1) *Y is an F-space.*

(2) *For every disjoint, countable and order bounded subset $A \subset C(Y)^+$ there exists an element $g \in C(Y)^+$ such that $(g - a) \wedge a = 0$ for all $a \in A$.*

Let c denote the space of all convergent sequences. We have the following lemma.

LEMMA 8. *If Y is a compact Hausdorff space and $(c, C(Y))$ has property (uI) then Y is an F-space.*

PROOF. It suffices to prove by Theorem 7 that for every disjoint order bounded set $\{f_n \mid n \in \mathbf{N}\} \subset C(Y)^+$ there exists $g \in C(Y)^+$ such that $(g - f_n) \wedge f_n = 0$ for all $n \in \mathbf{N}$. Therefore, let $\{f_n \mid n \in \mathbf{N}\}$ be an order bounded disjoint subset of $C(Y)^+$. Define in the obvious way a Riesz homomorphism ϕ: $c_0 \to C(Y)$ such that $\phi(1_{\{n\}}) = f_n$ for all $n \in \mathbf{N}$. The order boundedness of $\{f_n \mid n \in \mathbf{N}\}$ and the fact that $(c, C(Y))$ has property (uI) imply that there is a Riesz homomorphism Φ: $c \to C(Y)$ such that $\Phi_{|c_0} = \phi$. It follows that $(\Phi(1_{\mathbf{N}}) - f_n) \wedge f_n = 0$ for all $n \in \mathbf{N}$.

Actually the converse of Lemma 8 is also true as we will show later. Further relevant remarks concerning c in the problem of extending Riesz homomorphisms can be found in [3]. The following theorem in one form or another may be well-known. For the sake of convenience and later reference we will prove it in the following rather general form. Note that $\bigcap Z[I]$ for an ideal $I \subset C(X)$ refers to the set $\{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$.

THEOREM 9. *Suppose X is realcompact and Y is any completely regular space. Assume $I \subset C(X)$ is an ideal and ϕ: $I \to C(Y)$ is continuous with respect to the compact-open topologies on $C(X)$ and $C(Y)$. Then there exist an open subset U of Y and continuous mappings τ: $U \to X \setminus \cap Z[I]$ and ω: $U \to (0, \infty)$ such that for all $f \in I$,*

$$\phi(f)(y) = \begin{cases} 0 & \text{if } y \notin U, \\ \omega(y)(f \circ \tau)(y) & \text{if } y \in U. \end{cases}$$

Furthermore, if $\overline{U}$ is compact then $\overline{\tau(U)}$ is compact as well. In particular if Y is compact, then $\overline{\tau(U)}$ is compact.

For its proof we need the following easy lemma of which we leave the proof to the reader.

LEMMA 10. *Suppose X is realcompact and $I \subset C(X)$ is an ideal. For a nonzero Riesz homomorphism ϕ the following are equivalent.*

(1) *ϕ is a continuous Riesz homomorphism with respect to the compact-open topology on $C(X)$.*

(2) *There exist a unique $x \in X$ and $\lambda \in \mathbf{R}$ such that $\phi = \lambda\delta_x$, where $\delta_x(f) = f(x)$ for all $f \in C(X)$.*

PROOF OF THEOREM 9. Assume $I \subset C(X)$ is an ideal and ϕ: $I \to C(Y)$ is continuous with respect to the compact-open topologies on $C(X)$ and $C(Y)$. Define $U = \{y \in Y \mid \text{ there exists } f \in I \text{ such that } \phi(f)(y) \neq 0\}$. U is an open subset of Y and for every $y \in U$ there exist unique $x = \tau(y) \in X \setminus \bigcap Z[I]$ and

$\omega(y) \in \mathbf{R}^+$ such that $\delta_y \circ \phi = \omega(y)\delta_x$ because of Lemma 10. We prove that τ is continuous. Therefore, suppose $U' \subset X$ is open and $y_0 \in \tau^{-1}(U')$. We can find $f \in I^+$ such that $f(\tau(y_0)) = 1$ and $f_{|U'^c} = 0$ by using Urysohn's lemma and the fact that I is an ideal. Define $g = \phi(f)$. Then $g(y_0) = \phi(f)(y_0) = \omega(y_0)f(\tau(y_0)) = \omega(y_0) \neq 0$ and $g_{|\tau^{-1}(U')^c} = 0$. It follows that $\tau^{-1}(U')$ is open, because the set $\{y \in Y \mid g(y) > 0\}$ is open in Y, contains y_0 and is contained in $\tau^{-1}(U')$. The continuity of ω follows easily from the continuity of τ. Suppose furthermore that $\overline{U}$ is compact. Define for each $f \in C(Y)$, $p(f) = \sup_{y \in U}|f(y)|$. By the continuity of ϕ that we assumed, there exists a seminorm q on $C(X)$ such that for all $f \in I$, $p \circ \phi(f) \leqslant q(f)$. More precisely, there exist a compact set B and $C \in \mathbf{R}$ such that $p \circ \phi(f) \leqslant C\|f\|_B$ for all $f \in I$. We claim that $\tau(U) \subset B$. If not, choose $x \in A^c \cap \tau(U)$. Choose $f \in I^+$ such that $f(x) = f(\tau(y)) = 1$ and $f(A) = \{0\}$. It follows that $1 \leqslant 0$.

Our first result about extension of Riesz homomorphisms has now met all necessary preparations.

THEOREM 11. *Suppose X is compact and metrizable and Y is a compact F-space. Then $(C(X), C(Y))$ has property (uI).*

PROOF. Suppose $I \subset C(X)$ is an ideal and ϕ: $I \to C(Y)$ is a norm continuous Riesz homomorphism. According to Theorem 9 there exist an open subset U of Y and continuous mappings ω: $U \to (0, \infty)$ and τ: $U \to X \setminus \bigcap Z[I]$ such that for all $f \in I$

$$\phi(f)(y) = \begin{cases} 0 & \text{if } y \notin U, \\ \omega(y)f(\tau(y)) & \text{if } y \in U. \end{cases}$$

Actually, as $U = \{y \in Y \mid \text{there exists an } f \in I \text{ such that } \phi(f)(y) \neq 0\}$ and I is separable, it follows from the continuity of ϕ that U is an open F_σ. Also, $\omega(U) \subset (0, \|\phi\|]$. Therefore, for every $f \in C(X)$, the function $\bar{f}$: $y \to \omega(y)f(\tau(y))$ is a bounded continuous function on U. Also, the map $f \to \bar{f}$ is a Riesz homomorphism of $C(X)$ into $C_b(U)$. The image H of $C(X)$ under this map is therefore a Riesz subspace of $C_b(U)$ and it is separable by 16.4 of [16]. Choose, by Theorem 3, an extensor T: $H \to C(Y)$ with $\|T\| \leqslant 1$. It follows that Φ: $f \to T(\bar{f})$ is a Riesz homomorphism such that $\Phi_{|I} = \phi$ and $\|\Phi\| = \|\phi\|$.

We have now proved, using Lemma 8, the following characterization of F-spaces.

THEOREM 12. *For a compact Hausdorff space Y the following are equivalent.*
(1) Y is an F-space.
(2) $(C(X), C(Y))$ has property (uI) for all compact metrizable spaces X.

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(3) $(c, C(Y))$ *has property* (uI). *Moreover, Riesz homomorphic extensions that exist by means of this theorem may be chosen norm preserving.*

If, instead of $C(X)$ with X compact and metrizable, we wish to consider separable Banach lattices in general (as we promised in the introduction) the proofs do not change significantly. However, there is one important difference with Theorem 12. Possibly the norm of Riesz homomorphic extensions differs from the given Riesz homomorphisms. Looking at Theorem A or its positive analogue, we see that we have come so far in asking more and more structure to be preserved, that finally some friction, in this case between the lattice structure and the normed (partially ordered) structure, starts to appear. By the way, exactly the same phenomenon occurs if one considers majorizing Riesz subspaces instead of ideals (see Theorem B in the introduction and [5]).

THEOREM 13. *For a compact Hausdorff space* Y *the following are equivalent.*

(1) Y *is an* F*-space.*

(2) $(E, C(Y))$ *has property* (uI) *for all separable Banach lattices* E.

PROOF. Suppose E is a separable Banach lattice, Y is a compact F-space and $I \subset E$ is an ideal. The norm on E is denoted by $\| \; \|$. Suppose ϕ: $I \to C(Y)$ is a continuous Riesz homomorphism. Let $g \in E^+$. By uniform completeness of E there exists a compact Hausdorff space X such that the ideal generated by g, (g), is Riesz isomorphic to $C(X)$. Let T: $C(X) \to (g)$ be a Riesz isomorphism. Define $J = T^{-1}(I \cap (g)) \subset C(X)$ and ψ: $J \to C(Y)$ by $\psi(f) = \phi(T(f))$. Then J is an ideal in $C(X)$ and ψ is a Riesz homomorphism. We are going to prove that ψ can be extended to a Riesz homomorphism $C(X) \to C(Y)$. Because this does not follow from Theorem 11, we will have to repeat some former arguments. By Theorem 10.3 in [8] there exists $C \in \mathbf{R}^+$ such that for all $f \in C(X)$ we have $\|T(f)\| \leqslant C\|f\|_\infty$. It follows that ψ is continuous as a map from J with $\| \; \|_\infty$ to $C(Y)$ with its supremum norm. So there exist an open F_σ-set $U \subset Y$ and continuous mappings ω: $U \to \mathbf{R}^+$ and τ: $Y \to X$ such that for all $f \in J$,

$$\psi(f)(y) = \begin{cases} \omega(y) f(\tau(y)) & \text{if } y \in U, \\ 0 & \text{if } y \notin U. \end{cases}$$

For any $f \in C(X)$, $\bar{f}$: $y \to \omega(y) f(\tau(y))(y \in U)$ is an element of $C_b(U)$ and the map Ψ: $f \to \bar{f}$ $(f \in C(X))$ is a Riesz homomorphism. We denote for all $f \in C(X)$, $\|f\|_E = \|T(f)\|$ and $\|\Psi(f)\|_\infty = \sup\{|\Psi(f)(u)| \; |u \in U\}$. It readily follows that Ψ: $(C(X), \| \; \|_E) \to (C_b(U), \| \; \|_\infty)$ is continuous. Using Theorem 3, one can now find a Riesz homomorphism S: $C(X) \to C(Y)$ (namely by composing Ψ with an extensor) which is continuous with respect to $\| \; \|_E$ and such that

$f \to S(T^{-1}(f))$ $(f \in (g))$ is a continuous (with respect to $\| \; \|$) Riesz homomorphic extension $(g) \to C(Y)$ of $\phi_{|I \cap (g)}$. The rest of the proof follows by choosing a special element $g \in E^+$, namely a quasi-interior point, and some easy density arguments. Remark furthermore that (2) ⇒ (1) follows from Theorem 12.

3. Concluding remarks and a summary

It turns out that spaces of the form $C(Y)$ with the σ-interpolation property, or equivalently spaces of the form $C(Y)$ where Y is an F-space, behave extremely nicely as far as it concerns all kinds of $C(Y)$-valued mappings. To place the extension properties that we derived in this paper in their proper surroundings, we next give a summary of $C(Y)$ in this respect. Proofs, insofar not in this paper, can be found in [2], [5] and (for part of (1) ⇔ (6) of Theorem 15) [6].

THEOREM 14. *For a compact Hausdorff space the following are equivalent.*

(1) *Y is totally disconnected and an F-space.*

(2) *For every cozero-set $U \subset Y$ and every separable Riesz subspace H of $C_b(U)$ there exists an extensor $H \to C(Y)$ and if H contains 1_U then the extensor can be chosen to send 1_U to 1_Y.*

(3) *For every compact metrizable space X, every Riesz subspace H of $C(X)$ and every continuous Riesz homomorphism ϕ: $H \to C(Y)$ there exists a norm preserving Riesz homomorphic extension $C(X) \to C(Y)$.*

(4) *For every cozero-set U in Y, every compact metric space X and every continuous map τ: $U \to X$ there exists a continuous extension $Y \to X$.*

(5) *For every two metrizable compact spaces X and Z and every pair of continuous mappings f: $X \to Z$ and g: $Y \to Z$ there exists a continuous map h: $Y \to X$ with $f \circ h = g$.*

(6) *For every compact metrizable space X and every majorizing Riesz subspace H of $C(X)$ and every Riesz homomorphism $H \to C(Y)$ there exists a Riesz homomorphic extension $C(X) \to C(Y)$.*

(7) *For every pair of separable Banach lattices $B_1 \supset B_2$ such that B_2 is majorizing in B_1 and for every Riesz homomorphism $B_2 \to C(Y)$ there exists a Riesz homomorphic (not necessarily norm preserving) extension $C(X) \to C(Y)$.*

THEOREM 15. *For a compact Hausdorff space Y the following are equivalent.*

(1) *Y is an F-space.*

(2) *For every cozero-set U in Y and every separable Riesz subspace H of $C_b(U)$ there exists an extensor $H \to C(Y)$.*

(3) *For every cozero-set* $U \subset Y$ *and every compact metrizable space* X *and every continuous map* $U \to X$ *there exists a continuous extension to a neighbourhood* W *of* $\bar{U}$.

(4) *For every compact metrizable space* X *and every ideal* H *of* $C(X)$ *and every continuous Riesz homomorphism* $H \to C(Y)$ *there exists a Riesz homomorphic norm preserving extension* $C(X) \to C(Y)$.

(5) *For every compact metrizable space* X, *every Riesz subspace* H *of* $C(X)$ *and every positive operator* $H \to C(Y)$ *there exists a norm preserving positive extension* $C(X) \to C(Y)$. (*Dodds, Schep and Buskes*).

(6) *For every pair of separable Banach lattices* $B_1 \supset B_2$ *and every* (*positive*) *operator* $B_2 \to C(Y)$ *there exists a norm preserving* (*positive*) *extension* $B_1 \to C(Y)$. (*Cohen, Dodds, Schep and Buskes*).

(7) *For every pair of separable Banach lattices* $B_1 \supset B_2$ *such that* B_2 *is an ideal in* B_1 *and for every continuous Riesz homomorphism* $B_2 \to C(Y)$ *there exists a Riesz homomorphic extension* $B_1 \to C(Y)$.

It may be clear from these theorems that Riesz homomorphisms behave rather differently with respect to extension than just positive linear maps or linear maps. If one goes beyond the initial spaces that we considered here (separable ones), this becomes even more clear. For instance, assuming the continuum hypothesis, Lindenstrauss in [10] was able to make positive remarks for extending $C(Y)$-valued linear operators defined on subspaces of 1^∞, all the way up to 1^∞ (in a norm preserving way). However, to conclude this paper, we remark that [2] contains an example of an F-space Y such that $(1^\infty, C(Y))$ does not have property (uI).

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