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PAPER

Valuation of long-term care options embedded in life annuities

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Abstract

In most industrialised countries, one of the major societal challenges is the demographic change coming along with the ageing of the population. The increasing life expectancy observed over the last decades underlines the importance to find ways to appropriately cover the financial needs of the elderly. A particular issue arises in the area of health, where sufficient care must be provided to a growing number of dependent elderly in need of long-term care (LTC) services. In many markets, the offering of life insurance products incorporating care options and LTC insurance products is generally scarce. In our research, we therefore examine a life annuity product with an embedded care option potentially providing additional financial support to dependent persons. To evaluate the care option, we determine the minimum price that the annuity provider requires and the policyholder's willingness to pay for the care option. For the latter, we employ individual utility functions taking account of the policyholder's condition. We base our numerical study on recently developed transition probability data from Switzerland. Our findings give new and realistic insights into the nature and the utility of life annuity products proposing an embedded care option for tackling the financing of LTC needs.

Keywords: Annuity; Life insurance; Long-term care; Pricing; Utility

1. Introduction

Nowadays, ageing is set to become one of the main concerns in most developed economies. Medical advances and improvements in the overall quality of life have made it possible to extend the lifespan of humans. While this is a positive step forward for societies, it also brings its share of challenges. Indeed, ageing affects various aspects of the daily life including the ability to carry out autonomously day-to-day activities such as walking and bathing. Best known as long-term care (LTC) dependence, this impairment is now to become a stage of life rather than a fatality. From an individual's perspective, becoming care-dependent has both health and financial consequences (Guillén & Comas-Herrera, 2012; Scheil-Adlung & Bonan, 2013). For the latter, it is especially alarming that the households' care expenses have recently disproportionately increased (Zhou-Richter *et al.*, 2010). In Germany, e.g. the mean monthly co-payment of a dependent elderly for being nursed in an institution has grown by 4% in 2018, which is more than twice the inflation rate in that period (Szent-Ivanyi, 2018; BMAS, 2018; Vanella *et al.*, 2020). From the government's perspective, current schemes will soon face a lack in available care infrastructure (Katz, 2011), in professional caregivers (Nichols *et al.*, 2010; Colombo *et al.*, 2011) and in public financing (Brown & Finkelstein, 2009). For instance, in 2017, LTC public spendings amounted on average already to 1.1% of the GDP in OECD countries (OECD, 2020). The situation is exacerbated through the low pensions paid to retirees often belonging to the part of the population with the highest dependency risk (Christensen *et al.*, 2008; Engberg *et al.*, 2009). Therefore, assuring proper funding is cornerstone (Costa-Font *et al.*, 2015; Costa-Font *et al.*, 2017).

Insurance appears as an appropriate solution to stagger and mutualise the financing of care. Out-of-pocket spendings can be taken over by private insurance when available and purchased before by those who can afford it (Chen, 2001; Pestieau & Sato, 2008). However, the private market for LTC insurance encounters low take-up rates (Brown & Finkelstein, 2009; Braun *et al.*, 2019) due to the presence of comprehensive public LTC schemes (Colombo & Tapay, 2004) and excessive insurance premiums (Carrns, 2015; Konrad, 2018). This explains why stand-alone products have failed to succeed.

An appealing alternative is to combine life and LTC annuities for providing the appropriate financial protection in both situations, autonomy and care dependency, and meanwhile also proposing a lower overall premium due to diversification benefits (Wu *et al.*, 2018; Wu *et al.*, 2019). A possible concrete product approach consists of combining a life annuity paying as long as the policyholder is alive with a care option increasing the annuity amount in case of LTC needs (Brazell *et al.*, 2008; Pitacco, 2016). More precisely, the customer is able to decide if he just wants a regular life annuity, i.e. he does not contract the care option, or if he wants to obtain annuity payments, whose amounts depend on the degree of the individual's care dependency, i.e. he does contract the care option. Strictly speaking, the latter type of product is known as a life care annuity (Brown & Warshawsky, 2013), an enhanced annuity (Hoermann & Ruß, 2008; Brown & Scahill, 2010; Ramsay & Oguledo, 2020) or an embedded care annuity (Vidal-Melia *et al.*, 2016). Recently, significant literature has grown around this topic (Gatzert & Klotzki, 2016; Eling & Ghavibazoo, 2019; Lambregts & Schut, 2019). The fundamental idea behind the life care annuity is based on the combination of longevity and disability risks in a single product (Alegre *et al.*, 2002; Rickayzen, 2007; Levantesi & Menzietti, 2007; Levantesi & Menzietti, 2012), which often translates into shifting a part of the pension benefits into LTC benefits (Spillman *et al.*, 2003; Wu *et al.*, 2016; Vidal-Melia *et al.*, 2017). From an insurer's perspective, appropriate underwriting standards (Gatzert *et al.*, 2012b) and solvency concerns (Shao *et al.*, 2015; Juillard & Juillard, 2019) are key in ensuring the sustainability of such an insurance. Considering the important interdependence between the insurer and the policyholder in the success of a corresponding contract, it seems surprising that hardly any literature accounts for both viewpoints in their framework.

The objective of this article is to examine a life annuity with care option in a framework combining both the individual's and the insurer's perspectives, where the policyholder decides at inception whether or not to contract the care option. Specifically, we are interested in the value of the care option. For this purpose, we develop a multi-state LTC model (Guibert & Planchet, 2018; Denuit *et al.*, 2019) and derive two pricing approaches for, respectively, expressing both stakeholders' perspectives. In the first approach, we mirror the insurer's viewpoint by determining the actuarial value (Biessy, 2017; Hsieh *et al.*, 2018; Lim *et al.*, 2019). In the second approach, we compute the utility-based value for reflecting the policyholder's perspective (Berketi, 1999; Gatzert *et al.*, 2012a), where we employ condition-based utility functions. We make use of the expected utility theory to evaluate annuities as in, e.g. Yaari (1965), Yagi & Nishigaki (1993), Mitchell (2002), Davidoff *et al.* (2005) or Peijnenburg *et al.* (2016). In particular, we apply power utility functions as they are pointed out as the most frequently used utility functions to capture the preferences of individuals (Levy, 1994; Campbell & Viceira, 2002; Sharpe, 2017). The main condition for a contract to become possible is that the maximal fee the policyholder is ready to pay is equal or exceeds the minimal fee the insurer is ready to receive. In this way, we can answer the question when the care option is contracted. Indeed, we derive explicit expressions for the actuarial as well as the utility-based pricing methods for the care option reflecting the insurer's and policyholder's views, respectively.

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For providing a realistic computation of the theoretical aspects, we combine the mortality data extracted from the human mortality database (HMD, 2020) with the up-to-date LTC transition probability estimates stemming from Fuino & Wagner (2018). The latter emerge from unique fine-grained data covering the whole elderly population in need of care in Switzerland over a 20-year period. The possibility to use such reliable data makes the findings of this article particularly valuable given the scarcity of data on LTC.

Our main results are as follows: based on both realistic parameter values and the accurate data, we find that trade situations can occur. When such a situation appears, we detect that the solution is not unique and is belonging to a set of realistic parameter value combinations, i.e. we find that the policyholder is willing to pay a higher fee than the minimal fee the insurer requires in several instances. Having detailed the solution by gender sheds light on the concerns that need to be taken into account in case of favouring gender pooling and unisex premiums since significant gender differences exist. In particular, we observe that the differently calculated fees for the care option both increase if the policyholder is female. From a series of sensitivity and comparison analyses, we learn about various effects of, inter alia, the multiplier defining the payment of the increased care annuity, the policyholder's level of risk aversion, the impact of the care dependency on his utility or the characteristics of the individual's marginal utility. More specifically, we find that the policyholder's willingness to pay for the care supplement grows and thus, a trade relating thereto is more likely to occur if the risk aversion level declines. This is also the case when the impact of the care dependency on the utility of the individual becomes greater, provided that his marginal utility is larger when being in need of care than when being autonomous. If the latter condition is reversed, a greater utility impact of the care dependency reduces the maximal fee the policyholder is ready to pay.

The remainder of this article is organised as follows: section 2 introduces the model framework including the derivation of the pricing formulas and our theoretical statements. In section 3, we describe the numerical implementation, provide statistics on the data used and present the selected values for the parameters. In section 4, we reveal the numerical results, discuss the most influential parameters as well as the practical implementations and limitations of our research. Section 5 concludes the article.

2. Model Framework

In this section, we elaborate on the life annuity with care option and introduce a three-state model to describe the policyholder's health situation. Subsequently, we discuss two pricing concepts, namely the fair actuarial approach and the utility-based approach, in order to evaluate the life annuity and the care option, respectively. In the former approach, we take the annuity provider's point of view, where the resulting actuarial price can be considered as the lowest premium required by the provider to propose the care option. With this price, the annuity provider makes a zero-expected profit. In the latter approach, we take the policyholder's point of view, where the resulting utility indifference price can be considered as the highest price the policyholder is willing to pay for the care option embedded in the life annuity. Eventually, we elaborate on theoretical statements found when comparing both assessment approaches.

2.1 Life annuity with care option

Life annuities with care option have been traded on the German market (see, e.g. ALTE LEIPZIGER 2013; Münchener Verein 2020; Barmenia n.d.). They combine a regular life annuity paying as long as the policyholder is alive with the possibility of a supplementary annuity in case of LTC dependence. More precisely, prior to the first annuity payment, the individual decides whether or not to include the care option. If included, the individual receives a care annuity on top of the regular life annuity, i.e. a life care annuity, whose disbursed amount at each payment date

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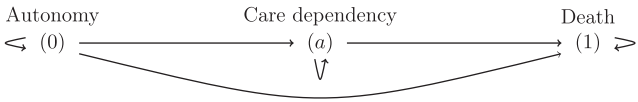

Figure 1. Three-state model.

depends on whether the individual is dependent or not. If the option is not included, the individual simply obtains the regular life annuity. The design of this product hence gives the policyholder a great flexibility and appropriately covers the individual's preferences and personal financial needs. Particularly, it covers two goals often considered crucial by people thinking ahead: first, it delivers more financial protection at older ages and can thus help to fight poverty amongst the elderly, especially when they are in need of care. Second, it provides the possibility for an enhanced support to policyholders who become dependent and need significant additional amounts of money to meet their increased expenses. In this context, care dependence is usually determined by medical doctors through the assessment of the individual's impairments based on activities of daily living and instrumental activities of daily living scales (see, e.g. Fuino *et al.*, 2020).

In the present article, with the implementation of the care option, the policyholder obtains annuity payments as with the regular life annuity before becoming care-dependent, and these payments are raised once the customer is in need of care[1].

For modelling the life annuity with care option, we need to distinguish the relevant health situations that the policyholder can experience. For simplicity, we make the assumptions that the policyholder is autonomous at contract inception and that there is only one level of care dependency. Hence, three possible health paths exist. First, the policyholder can remain autonomous until death while never requiring any LTC. Second, the policyholder can become dependent and stay in this health condition until death. Third, after becoming dependent, the policyholder can recover autonomy and again face any of the previous paths. However, as the recovery of lost capabilities by a dependent person is very infrequent, we disregard this third case (see, e.g. Levantesi & Menzietti, 2012; Fong *et al.*, 2017; Reinhard, 2018)[2].

In Figure 1, we illustrate our three-state model summarising the different transitions an autonomous policyholder can go through in our setting. The imperative "Autonomy" state, abbreviated by (0), refers to the alive independent policyholder, the "Care dependency" state, abbreviated by (a), covers the policyholder in need of LTC of any kind and the imperative "Death" state, abbreviated by (1), accounts for the dead policyholder.

2.2 Actuarial pricing of the care option

In the following, we derive formulas for the premiums of the life care annuity (with care option) and the regular life annuity (without care option) based on the framework of actuarial pricing. The resulting actuarially fair initial net premiums can be understood as premiums, at which the insurer makes a zero-expected profit. The difference between both premiums allows us to determine the minimum premium required by the insurer for the option.

Generally, we denote by $r \in \mathbb{R}$ the discount rate, by ζ_x the remaining lifetime of an individual aged $x \geq 0$ and by τ_x the first time, at which such an individual will be in need of care. Here, both ζ_x and τ_x are random variables.

[1]Theoretically, we can alternatively model the annuity payments, if the care option is included, as follows: the product disburses reduced annuity payments, compared to the regular life annuity, as long as the policyholder stays autonomous and increases these payments if the insured person gets dependent.

[2]For instance, in the Swiss empirical data processed in Fuino & Wagner (2018) who provide the transition probability estimates for our later numerical study, less than 0.05% deals with recovery transitions.

For reasons of simplicity, we assume that the policyholder under consideration pays a single premium at inception, i.e. at time 0, at which he is $x_0 \geq 0$ years old and autonomous[3]. Moreover, it is supposed that the first annuity payment starts immediately at the beginning of the insurance contract, i.e. at time 0. Consistently, the care option embedded in this immediate life annuity can only be chosen at inception. As long as the policyholder remains autonomous, he receives a regular annuity payment of $c > 0$. If the care option is included and in case of care dependence, the previous annuity level is multiplied by the constant $\alpha > 1$, so that the annuity payments after τ_x are increased to αc to financially support the arising care expenses. If the care option is not included, the policyholder continues to receive the regular payment of c even if he gets dependent[4].

To derive the initial actuarial price of the benefits guaranteed by the life care annuity, we decompose the associated valuation into two parts: first, we assess the payments amounting to c which the policyholder receives if he is autonomous. Afterwards, the payments in the amount of αc, that are disbursed to the policyholder in case he becomes dependent, are similarly evaluated. To find the actuarial price at time 0 of the benefits obtained by the policyholder while independent, we split the timeline of interest, namely $[0, \omega - x_0]$, where $\omega \geq x_0$ is the maximum age reachable by a human being, into $m \in \mathbb{N}_0$ time intervals of equal length. In practice (see also our numerical framework in section 3), the timeline would comprise $\omega - x_0$ years. For $j \in \{0, \ldots, m-1\}$, the starting point of the $(j+1)$th time interval is denoted by t_j and its end point by t_{j+1} with $0 = t_0 < t_1 < \ldots < t_j < t_{j+1} < \ldots < t_{m-1} < t_m = \omega - x_0$. As we suppose that all payments are made regularly in advance, i.e. at the beginning of the respective time intervals, the policyholder receives the annuity payment c at every point in time t_j as long as he is in the autonomy state. Consequently, for the $(j+1)$th annuity payment at time t_j, the individual benefit is given by

$$c\mathbb{1}_{\{\zeta_{x_0} > t_j, \tau_{x_0} > t_j\}} \tag{1}$$

where $\mathbb{1}_B$ is 1 if the event B occurs and 0 otherwise. On account of the actuarial equivalence principle ensuring fairly priced insurance policies, the corresponding actuarial price at time 0 (net present value) amounts to

$$P_0^{C_1} := \mathrm{E}\left[\sum_{j=0}^{m-1} \mathrm{e}^{-rt_j} c\mathbb{1}_{\{\zeta_{x_0} > t_j, \tau_{x_0} > t_j\}}\right] = c \underbrace{\sum_{j=0}^{m-1} \mathrm{e}^{-rt_j} \Pr\left(\zeta_{x_0} > t_j, \tau_{x_0} > t_j\right)}_{=:\Psi_0(r)} \tag{2}$$

where $\Psi_0(r)$ corresponds to the cumulative probability-weighted number of annuity payments in the autonomy state discounted to time 0. For the analysis of the second part of the life care annuity, i.e. the increased payments αc in the case of dependency, we follow a similar procedure for deriving the actuarial price at time 0: For $f \in \{1, \ldots, m-1\}$, let us assume that the policyholder becomes dependent at some time in the interval $\left(t_{f-1}, t_f\right]$. Then, contingent on this event and provided that the policyholder is alive, the $\left(i - f + 1\right)$th annuity payment at time t_i, $i \in \left\{f, \ldots, m-1\right\}$, is given by

$$\alpha c\mathbb{1}_{\{\zeta_{x_0} > t_i, t_{f-1} < \tau_{x_0} \leq t_f\}} \tag{3}$$

[3]For reasons of clarity, we assume, whenever no distinction between men and women is necessary, that the customer is male. Otherwise, we use $y_0 \geq 0$ as the initial age of the considered female policyholder.

[4]We neglect any administrative and other costs related to the insurance contract.

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Applying the actuarial equivalence principle and taking account of the domains of f, i.e. $\{1, \ldots, m-1\}$, and i, i.e. $\{f, \ldots, m-1\}$, result in the following value at time 0:

$$P_0^{C_2} := \mathrm{E}\left[\sum_{f=1}^{m-1}\sum_{i=f}^{m-1} \mathrm{e}^{-rt_i}\alpha c \mathbb{1}_{\{\zeta_{x_0}>t_i, t_{f-1}<\tau_{x_0}\leq t_f\}}\right] = \alpha c \underbrace{\sum_{f=1}^{m-1}\sum_{i=f}^{m-1} \mathrm{e}^{-rt_i}\Pr\left(\zeta_{x_0} > t_i, t_{f-1} < \tau_{x_0} \leq t_f\right)}_{=:\Psi_a(r)} \tag{4}$$

where $\Psi_a(r)$ corresponds to the cumulative probability-weighted number of annuity payments in the care dependence state discounted to time 0. The total actuarial price at time 0 of the benefits of the life care annuity is the sum of $P_0^{C_1}$ given in (2) and $P_0^{C_2}$ given in (4). We hence get for the single net premium P_0^C to be paid at inception of the life care annuity:

$$P_0^C = P_0^{C_1} + P_0^{C_2} = c\left(\Psi_0(r) + \alpha\Psi_a(r)\right) \tag{5}$$

Having derived P_0^C, we now also determine the single net premium P_0^L to be paid at time 0 to buy the regular life annuity (without care option). In our setup, the customer holding the regular life annuity receives annuity payments c until he dies. This payment structure is indeed a special case of the life care annuity: by setting $\alpha = 1$ in the above derivation, we obtain the single net premium for the regular life annuity as follows:

$$P_0^L = c\left(\Psi_0(r) + \Psi_a(r)\right) \tag{6}$$

The actuarial pricing technique eventually leads to the fee F_0 for the care option by defining

$$F_0 := P_0^C - P_0^L = c\,(\alpha - 1)\,\Psi_a(r) \tag{7}$$

This difference corresponds to the value at time 0 of the additional payments received in the case of care dependence from the care option.

Note that the pricing method presented above can be extended to allow for more flexible time-related product features such as the elimination period and the maximum benefit period. In our approach, we chronologically go along with the relevant timeline and can also incorporate the durations of stay in the different states. In contrast, other insurance pricing techniques, like the application of the generalised Thiele's differential equation, can often not be used directly if such features are included (see, e.g. Shao *et al.*, 2015). In order to avoid unnecessary complexity, we however omit the examination of the elimination period and the maximum benefit period.

2.3 Utility-based pricing of the care option

While the fee determined in section 2.2 can be understood as the minimum premium an insurer would ask to provide the care option, our second approach assesses the option value from the perspective of the policyholder. For this, we aim at determining the customer's willingness to pay for the care supplement.

In the following, we assume that the policyholder is rational and bases his decisions on his (discounted) expected lifetime utility. Further, we suppose that his subjective perceptions concerning the transition probabilities coincide with the respective information available to the insurer. Generally, for deciding to buy a life care annuity, the policyholder can compare with the situation, in which he buys a regular life annuity. If the two products lead to the same cost for the policyholder, he chooses the one which results in a higher lifetime utility level. However, a regular life annuity with lifelong payments c costs less than a life care annuity regularly paying out c in the autonomy state and αc, with $\alpha > 1$, in the care dependency state. If the life care annuity leads to a higher lifetime utility, this does not necessarily imply its superiority over the regular life annuity.

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Thus, to make both products comparable, we introduce the quantity θ which denotes the number of regular life annuities making the policyholder indifferent, in terms of his expected utility, between buying the life care annuity and these θ regular life annuities, i.e.

$$\mathrm{EU}_0^L(\theta c) \stackrel{!}{=} \mathrm{EU}_0^C(c) \tag{8}$$

where $\mathrm{EU}_0^L(\theta c)$ is the expected lifetime utility achieved from the θ regular life annuities (annuity payment is θc accordingly) and $\mathrm{EU}_0^C(c)$ is the expected lifetime utility achieved from the life care annuity (annuity payment is c or αc depending on health status). The utility indifference price $\widehat{P}_0^C$ the policyholder would be willing to pay for the life care annuity is hence given by

$$\widehat{P}_0^C = \theta P_0^L \tag{9}$$

where P_0^L is the actuarially fair single premium for the regular annuity determined in (6) in the previous section. Overall, after determining the utility-indifferent quantity θ by means of (8), we apply the actuarially fair valuation approach to the θ regular life annuities each with a periodic payment of c. As this approach entails a linear pricing rule, we obtain (9) right away. As a consequence, the utility-based fee $\widehat{F}_0$ for the care option is computed through

$$\widehat{F}_0 = \widehat{P}_0^C - P_0^L \tag{10}$$

As noted before, we interpret $\widehat{F}_0$ as the maximum a policyholder is willing to pay for the care option. A fee higher than $\widehat{F}_0$ would prevent the policyholder from buying the life care annuity. The care option will only be chosen by the policyholder if $F_0 \leq \widehat{F}_0$, i.e. the annuity provider requires a fee F_0 smaller than or equal to the willingness to pay $\widehat{F}_0$ of the policyholder. In this sense, if $\widehat{F}_0 - F_0$ is positive, this difference can be used by the provider as the upper bound for a potential safety loading.

Following, e.g. Yaari (1965) and using the formula for P_0^C given in (5), the individual's expected utility $\mathrm{EU}_0^C(c)$ for the life care annuity can be written as follows:

$$\begin{aligned} \mathrm{EU}_0^C(c) &= u_0(c) \sum_{j=0}^{m-1} \mathrm{e}^{-\eta t_j} \mathrm{Pr}\left(\zeta_{x_0} > t_j, \tau_{x_0} > t_j\right) + u_a(\alpha c) \sum_{f=1}^{m-1} \sum_{i=f}^{m-1} \mathrm{e}^{-\eta t_i} \mathrm{Pr}\left(\zeta_{x_0} > t_i, t_{f-1} < \tau_{x_0} \leq t_f\right) \\ &= u_0(c)\Psi_0(\eta) + u_a(\alpha c)\Psi_a(\eta) \end{aligned} \tag{11}$$

where $\eta \in \mathbb{R}$ is the subjective discount rate of the policyholder, which captures the individual's time preferences, i.e. the utilities are exponentially weighted by η over time. One way to realistically estimate the subjective discount rate can be, e.g. to assess suitable survey data (see, e.g. Booij & van Praag, 2009). The mappings $u_0(z) \in \mathbb{R}$ and $u_a(z) \in \mathbb{R}$ are the pivotal utility functions describing the customer's utility of an incoming cash flow $z > 0$ depending on his health status. Thereby, u_0 describes the utility of an autonomous individual, whereas u_a describes the utility of a dependent individual[5]. For the regular life annuity, we can derive the expected utility $\mathrm{EU}_0^L(c)$ by setting $\alpha = 1$:

$$\mathrm{EU}_0^L(c) = u_0(c)\Psi_0(\eta) + u_a(c)\Psi_a(\eta) \tag{12}$$

[5]Such condition-based utility functions have been frequently discussed in the literature, such as in Evans & Viscusi (1991). By employing this two-tier utility modelling approach, we strive to emphasise that the perception of a person regarding a certain (financial) outcome naturally depends on his health status, i.e. in our context, whether he is dependent or not. Consequently, we are able to create a more realistic situation, in contrast to the case, in which just one universal utility function is used. Note that it is often supposed that utility functions are (strictly) concave, which implies what is typically observed, namely that the considered individual is risk averse.

Before moving to an explicit formulation for the quantity θ (cf. (8)), we discuss general properties of it.

Proposition 1. *If there exists a solution θ fulfilling (8), then, as long as u_0 and u_a are strictly monotone (in the same direction), it holds that $\theta \in (1, \alpha)$.*

Proof. See Appendix A. $\square$

To explicitly illustrate what kind of formulas for θ and eventually $\widehat{F}_0$ can be obtained, we need to specify the utility functions u_0 and u_a. For that, we use u_0 as the reference utility function and alter its structure to get u_a. In particular, we set $u_a(z) = \kappa_a u_0(z)$, where $\kappa_a \geq 0$ reflects the impact of being in need of care on the customer's utility. To realistically valuate subjective impact parameters like κ_a, empirical research methods, such as interviews, can be deployed. Like Evans & Viscusi (1991), we naturally stipulate the individual's preference for good health, i.e.

$$u_0(z) \geq u_a(z) \tag{13}$$

If we assume that $u_0(z)$ can only be either non-negative or non-positive, we make the following observation: As, for the case $u_0(z) \geq 0$, the condition in (13) is fulfilled if $\kappa_a \leq 1$ and if we require, as for our illustration, that the utility functions are increasing, we get the following relation between the marginal utilities:

$$\frac{\partial u_0(z)}{\partial z} \geq \frac{\partial u_a(z)}{\partial z} \tag{14}$$

which means that the policyholder proportionally profits more from an extra income unit as long as he is autonomous compared to when he is dependent. On the other hand, for the case $u_0(z) \leq 0$, the condition in (13) is fulfilled if $\kappa_a \geq 1$ and thus, this relation is reversed:

$$\frac{\partial u_0(z)}{\partial z} \leq \frac{\partial u_a(z)}{\partial z} \tag{15}$$

The literature does not exclude a priori one of the relations in (14) and (15), but embraces their coexistence (see, e.g. Evans & Viscusi, 1991). That is why, we also consider both possibilities subsequently. It is important to note that it is substantial whether (14) or (15) holds, as we will see later on. In the following, we consider the often used power utility function. Denoting with $u_0 = u_0^{\mathrm{PU}}$ and $u_a = u_a^{\mathrm{PU}}$ the utility functions, we assume

$$u_0^{\mathrm{PU}}(z) = \frac{z^{1-\gamma}}{1-\gamma} \quad \text{and} \quad u_a^{\mathrm{PU}}(z) = \left(\underline{\kappa_a}\mathbb{1}_{\{\gamma \in [0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma > 1\}}\right) \frac{z^{1-\gamma}}{1-\gamma} \tag{16}$$

where $\gamma \geq 0$ adhering to $\gamma \neq 1$ is the measure of the individual's risk aversion, $\underline{\kappa_a} \in [0, 1]$ implying (13) and (14) and $\overline{\kappa_a} \geq 1$ implying (13) and (15)[6]. As u_a^{PU} increases in $\underline{\kappa_a}$ and decreases in $\overline{\kappa_a}$, respectively, the policyholder's care dependency affects his utility to a great extent if $\underline{\kappa_a}$ is small and $\overline{\kappa_a}$ is large, respectively. As a result of the definitions in (16), we can find $\theta = \theta^{\mathrm{PU}}$ that fulfils (8) as

$$\theta^{\mathrm{PU}} = \left(\frac{\Psi_0(\eta) + \left(\underline{\kappa_a}\mathbb{1}_{\{\gamma \in [0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma > 1\}}\right) \alpha^{1-\gamma} \Psi_a(\eta)}{\Psi_0(\eta) + \left(\underline{\kappa_a}\mathbb{1}_{\{\gamma \in [0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma > 1\}}\right) \Psi_a(\eta)} \right)^{\frac{1}{1-\gamma}} \tag{17}$$

[6]The mapping $u_0^{\mathrm{PU}}(z)$ is actually the common standard power utility function. The mapping $u_a^{\mathrm{PU}}(z)$, that constitutes a modified power utility function, maintains the properties of the standard power utility function, namely having a constant relative risk aversion of γ and a decreasing absolute risk aversion of $\frac{\gamma}{z}$.

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By combining (6), (9), (10) and (17), we obtain the policyholder's willingness to pay for the care option under power utility. The corresponding fee $\widehat{F}_0 = \widehat{F}_0^{\mathrm{PU}}$ is hence given by

$$\begin{aligned}\widehat{F}_0^{\mathrm{PU}} &= \left(\theta^{\mathrm{PU}} - 1\right) P_0^L \\ &= \left(\left(\frac{\Psi_0(\eta) + \left(\underline{\kappa_a}\mathbb{1}_{\{\gamma\in[0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma>1\}}\right)\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta) + \left(\underline{\kappa_a}\mathbb{1}_{\{\gamma\in[0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma>1\}}\right)\Psi_a(\eta)}\right)^{\frac{1}{1-\gamma}} - 1\right) c\left(\Psi_0(r) + \Psi_a(r)\right) \qquad (18)\end{aligned}$$

The policyholder's willingness to pay depends on several parameters, particularly on the risk aversion level γ and the impact parameters $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively. Naturally, we are interested in the relation between F_0 given in (7) and $\widehat{F}_0^{\mathrm{PU}}$ given in (18) and in answering the question when the policyholder contracts the care option with the annuity provider. For this, we first examine the condition, under which both fees are identical. In Proposition 2, we explicitly determine this condition for $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively.

Proposition 2. *It holds that $F_0 = \widehat{F}_0^{\mathrm{PU}}$ if $\underline{\kappa_a} = \underline{\kappa_a}^*$ and $\overline{\kappa_a} = \overline{\kappa_a}^*$, respectively, where*

$$\underline{\kappa_a}^*\mathbb{1}_{\{\gamma\in[0,1)\}} + \overline{\kappa_a}^*\mathbb{1}_{\{\gamma>1\}} = \frac{\Psi_0(\eta)\left(1 - \left(\frac{\Psi_0(r)+\alpha\Psi_a(r)}{\Psi_0(r)+\Psi_a(r)}\right)^{1-\gamma}\right)}{\Psi_a(\eta)\left(\left(\frac{\Psi_0(r)+\alpha\Psi_a(r)}{\Psi_0(r)+\Psi_a(r)}\right)^{1-\gamma} - \alpha^{1-\gamma}\right)} \qquad (19)$$

Proof. See Appendix B. □

In other words, for a policyholder with a given risk aversion level γ, the formula stated in (19) indicates the critical values for the impact parameters. As it can be shown that $\widehat{F}_0^{\mathrm{PU}}$ strictly increases in $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively, we conclude that the option is not contracted because of $F_0 > \widehat{F}_0^{\mathrm{PU}}$ if $\underline{\kappa_a} < \underline{\kappa_a}^*$ and $\overline{\kappa_a} < \overline{\kappa_a}^*$, respectively. This means that, if the marginal utilities of the policyholder behave as in (14), i.e. $\underline{\kappa_a}$ applies, the trade is not achieved if the policyholder's care dependency affects his utility to a great extent. On the contrary, if the marginal utilities of the policyholder behave as in (15), i.e. $\overline{\kappa_a}$ applies, the trade is not achieved if the effect is rather small.

Another parameter of interest, which influences the magnitude of the difference $\widehat{F}_0^{\mathrm{PU}} - F_0$, is the multiplier α that increases the annuity payment in case of care dependency. As both $\widehat{F}_0^{\mathrm{PU}}$ and F_0 increase in α, the difference $\widehat{F}_0^{\mathrm{PU}} - F_0$ is not necessarily monotone in α. In fact, it is possible to find an explicit formula for α, so that $\widehat{F}_0^{\mathrm{PU}} - F_0$ is maximised (the maximal difference is non-negative). We prove the following Proposition 3.

Proposition 3. *If $\gamma \neq 0$, $\underline{\kappa_a} \neq 0$ and*

$$\Psi_0(\eta) > \frac{\underline{\kappa_a}\mathbb{1}_{\{\gamma\in[0,1)\}}\Psi_a(\eta)\left(\left(\Psi_0(r) + \Psi_a(r)\right)^{1-\gamma} - \Psi_a(r)^{1-\gamma}\right)}{\Psi_a(r)^{1-\gamma}} \qquad (20)$$

it holds that $\widehat{F}_0^{\mathrm{PU}} - F_0$ is maximised with respect to α if $\alpha = \alpha^*$, where

$$\begin{aligned}\alpha^* = \Bigg(\Bigg(\Bigg(&\frac{\Psi_a(r)}{\left(\underline{\kappa_a}\mathbb{1}_{\{\gamma\in[0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma>1\}}\right)\Psi_a(\eta)(\Psi_0(r) + \Psi_a(r))}\Bigg)^{\frac{1-\gamma}{\gamma}} \\ &\cdot\left(\Psi_0(\eta) + \left(\underline{\kappa_a}\mathbb{1}_{\{\gamma\in[0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma>1\}}\right)\Psi_a(\eta)\right)^{\frac{1}{\gamma}} - \left(\underline{\kappa_a}\mathbb{1}_{\{\gamma\in[0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma>1\}}\right)\Psi_a(\eta)\Bigg)\Big/\Psi_0(\eta)\Bigg)^{\frac{1}{\gamma-1}}\end{aligned} \qquad (21)$$

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Table 1. Specification of model parameters in baseline case: symbols, descriptions and values.

Symbol	Description	Value
x_0	Initial age of male policyholder	66
y_0	Initial age of female policyholder	66
ω	Maximum age of policyholder	100
r	Discount rate	0.02
c	Payment of regular life annuity and of life care annuity in state (0)	12,000
α	Multiplier for payment of life care annuity in state (a)	1.4
η	Subjective discount rate	0.02
γ	Measure of policyholder's risk aversion	2
$\overline{\kappa_a}$	Measure of impact of state (a) on policyholder's utility	1.5

Proof. See Appendix C. □

The condition given in (20) is actually only technical since, in practice, it is very likely always fulfilled due to the fact that $\Psi_0(\eta)$ usually attains relatively high values[7]. Proposition 3 indicates that with α^* given in (21) the fee range $\left[F_0, \widehat{F}_0^{\mathrm{PU}}\right]$ is maximised. In other words, with certain choices of the multiplier α, the insurer has a wide scope to determine a fee acceptable to the policyholder and incentivise the policyholder to contract the care option.

Beyond the impact parameters $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively, and the multiplier α, we study the influence of other interesting parameters like the risk aversion level γ on $\widehat{F}_0^{\mathrm{PU}} - F_0$ through our numerical results in section 4.1. Indeed, mostly no explicit results can be found for these other parameters.

3. Numerical Framework

In order to numerically examine the proposed model and the theoretical findings introduced in section 2, we present the appropriate framework, the available data and the approximations used in this section.

For the numerical illustration, we apply the transition probability data from Fuino & Wagner (2018) and the HMD. Further, suppose for the utility-based pricing approach that the individual's preferences are expressed via the power utility functions defined in (16). In the following, we describe the calibration of our model by specifying the different parameters and needed transition probabilities.

First of all, we assume the discretisation of the decisive timeline $[0, \omega - x_0]$ to be on an annual basis. Thus, all numbers related to time (t_j) and age ($x_0 + t_j$) are given in (integer) years, so that, at the times $0 = t_0 < 1 = t_1 < \ldots < j = t_j < t_{j+1} = j + 1 < \ldots < t_{m-1} = \omega - x_0 - 1 < t_m = \omega - x_0$, the corresponding ages are given by $x_0 < x_0 + 1 < \ldots < x_0 + j < x_0 + j + 1 < \ldots < \omega - 1 < \omega$.

As our baseline case parameter setup, we fix the parameter values as specified in Table 1. Given the available data for ages between 66 and 100 years, we set the initial ages of the male and female policyholders to $x_0 = y_0 = 66$ and the maximum age to $\omega = 100$. Indeed, too few observations to calibrate dependence probabilities are available beyond the age of 100. If the policyholder surpasses ω, we simplistically assume that the annuity stops the payments from that age on. Because

[7] Given the probability data and our baseline case parameter setup (cf. Table 1) that are introduced in the next section for our numerical framework, the condition in (20) is fulfilled as it becomes 14.219 > 0 (male) and 15.372 > 0 (female). For the case $\gamma \in [0, 1)$, if, e.g. $\gamma = 0.5$ and $\underline{\kappa_a} = 0.5$ (and the remaining parameters still take their baseline values given in Table 1), the condition is also fulfilled as it is then equivalent to 14.219 > 1.557 (male) and 15.372 > 1.959 (female).

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of the discretisation assumption and the range of ages, it holds that the maximal remaining lifetime is $m = 100 - 66 = 34$. Further, in the baseline case, we fix the value of the discount rate r at 2%, where we are guided by the long-term nature of the annuity and the numbers provided by Statista (2020)[8]. For the subjective discount rate η, we use the simplifying assumption equating the value of η with the one of r. Note that we further consider the cases where $\eta < r$ or $\eta > r$ by varying the values of r and η in the subsequent sensitivity analyses (see section 4.1). For the remaining parameters, we choose the following values: The regular basic annuity payment is set to $c = 12,000$ and the multiplier α takes the value 1.4. This implies that the life care annuity for dependent policyholders pays $\alpha c = 16,800$. For the policyholder's risk aversion level, we stipulate $\gamma = 2$ (see, e.g. Havranek *et al.*, 2015). Finally, by choosing $\overline{\kappa_a} = 1.5$ in the baseline case, we suppose that the individual's utility level declines by half in case of care dependency.

The relevant probabilities affecting the two fees F_0 defined in (7) and $\widehat{F}_0^{\mathrm{PU}}$ defined in (18) are covered by the quantities $\Psi_0(r)$, $\Psi_0(\eta)$, $\Psi_a(r)$ and $\Psi_a(\eta)$. In our numerical application, we need to quantify

$$\Pr\left(\zeta_{x_0} > t_j, \tau_{x_0} > t_j\right) \tag{22}$$

$$\text{and} \quad \Pr\left(\zeta_{x_0} > t_i, t_{f-1} < \tau_{x_0} \le t_f\right) \tag{23}$$

where, using the previous specifications, $j \in \{0, \ldots, m-1 = 33\}$ implying $t_j \in \{0, \ldots, 33\}$, $f \in \{1, \ldots, m-1 = 33\}$ implying $t_{f-1} \in \{0, \ldots, 32\}$ and $t_f \in \{1, \ldots, 33\}$ and $i \in \left\{f, \ldots, m-1 = 33\right\}$ implying $t_i \in \left\{f, \ldots, 33\right\}$. In order to approximate the above probabilities with the help of the available data, we need to rewrite them. For the probability to stay in the autonomy state (0) between the ages x_0 and $x_0 + t_j$ given in (22), we deploy the complementary probability:

$$\begin{aligned}\Pr\left(\zeta_{x_0} > t_j, \tau_{x_0} > t_j\right) &= 1 - \Pr\left(\zeta_{x_0} > t_j, \tau_{x_0} \le t_j\right) - \Pr\left(\zeta_{x_0} \le t_j\right) \\ &= 1 - \left[\Pr\left(\zeta_{x_0} > t_j, t_0 < \tau_{x_0} \le t_1\right) + \Pr\left(\zeta_{x_0} > t_j, t_1 < \tau_{x_0} \le t_2\right)\right. \\ &\quad \left. + \ldots + \Pr\left(\zeta_{x_0} > t_j, t_{j-1} < \tau_{x_0} \le t_j\right)\right] - \Pr\left(\zeta_{x_0} \le t_j\right)\end{aligned} \tag{24}$$

where the probabilities appearing in the square brackets are described in (23). For probabilities as given in (23), we use the following breakdown:

$$\begin{aligned}\Pr\left(\zeta_{x_0} > t_i, t_{f-1} < \tau_{x_0} \le t_f\right) &= \Pr\left(\zeta_{x_0} > t_{f-1}, \tau_{x_0} > t_{f-1}\right) \\ &\quad \cdot \Pr\left(\zeta_{x_0} > t_f, \tau_{x_0} \le t_f \middle| \zeta_{x_0} > t_{f-1}, \tau_{x_0} > t_{f-1}\right) \\ &\quad \cdot \Pr\left(\zeta_{x_0} > t_i \middle| \zeta_{x_0} > t_f, t_{f-1} < \tau_{x_0} \le t_f\right)\end{aligned} \tag{25}$$

The three factors on the right-hand side of equation (25) can be interpreted as follows: the first one describes the probability to stay in state (0) between the ages x_0 and $x_0 + t_{f-1}$ (cf. (22)); the second one is the probability to become care-dependent between the ages $x_0 + t_{f-1}$ and $x_0 + t_f$, i.e. to start in state (0), to move at some time from the autonomy state (0) to the dependency state (a) in the corresponding time interval and then to stay in state (a) for the remaining time; the last one is the probability to stay in state (a) between the ages $x_0 + t_f$ and $x_0 + t_i$. After the rewriting, it is possible to specify the approximation procedure for the relevant probabilities given in (22) and (23). As mentioned, the data used for this purpose stems from two data sources. For the probabilities $\Pr\left(\zeta_{x_0} > t_f, \tau_{x_0} \le t_f \middle| \zeta_{x_0} > t_{f-1}, \tau_{x_0} > t_{f-1}\right)$ and $\Pr\left(\zeta_{x_0} > t_i \middle| \zeta_{x_0} > t_f, t_{f-1} < \tau_{x_0} \le t_f\right)$ appearing in (25), we use an extract of the results from Fuino & Wagner (2018). In their article, the authors compute semi-Markov transition probabilities for care-dependent elderly based on a longitudinal data set covering the whole Swiss care-dependent population being older than

[8]Statista (2020) provides the average risk-free rates of investment for 29 selected countries in Europe as of 2020. The estimated mean of theses rates approximately amounts to 2.32%. Ignoring the three countries with the highest rates, i.e. Turkey, Greece and Russia, still leads to a mean of 1.7%.

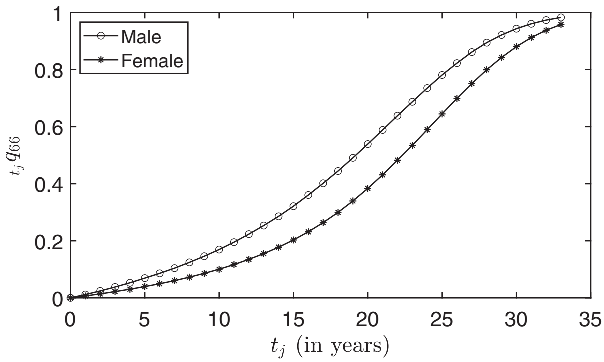

Figure 2. Illustration of available data for $_{t_j}q_{66}$ approximating relevant probabilities.

65 years over a 20-year period. More precisely, the authors provide appropriate data for our three-state model, with which both probabilities can be approximated by age and by gender[9]. Further, we add HMD data to approximate the probability $\Pr\left(\zeta_{x_0} \le t_j\right)$ occurring in (24). More precisely, the following approximations are applied throughout our numerical study:

- The probability $\Pr\left(\zeta_{x_0} > t_f, \tau_{x_0} \le t_f \middle| \zeta_{x_0} > t_{f-1}, \tau_{x_0} > t_{f-1}\right)$ is approximated by the average prevalence rate for the age $x_0 + t_{f-1}$, which is denoted by $\overline{\pi}_{x_0+t_{f-1}}$ and provided in Table 3 in Fuino & Wagner (2018). Due to the rate's annual structure, it matches our setting seamlessly.
- The probability $\Pr\left(\zeta_{x_0} > t_i \middle| \zeta_{x_0} > t_f, t_{f-1} < \tau_{x_0} \le t_f\right)$ is approximated by the appropriate semi-Markov transition probability that is denoted by $p^{aa}_{x_0+t_f}(t_i - t_f)$ and derived similarly as the semi-Markov transition probabilities in Fuino & Wagner (2018) if only one state of care dependency exists. The authors assume the Weibull distribution for the duration law when establishing the transition probabilities. As we consider a three-state model and disregard recovery from care dependency (see Figure 1), our semi-Markov transition probability is in fact a simple sojourn probability.
- The probability $\Pr\left(\zeta_{x_0} \le t_j\right)$ is approximated by the transition probability to die between the ages x_0 and $x_0 + t_j$ which is typically denoted by $_{t_j}q_{x_0}$. It can be directly derived from the Swiss mortality tables provided in the HMD.

Finally, the probabilities given in (22) and (23) are approximated by using the relations in (24) and (25), their interdependence and the three approximations listed above. More technical details can be found in Appendix D.

In Figures 2, 3, 4, 5 and 6, we illustrate the data underlying our study and the resulting approximated probabilities. Specifically, we plot the transition probabilities $_{t_j}q_{66}$, $\overline{\pi}_{66+t_{f-1}}$ and $p^{aa}_{66+t_f}(t_i - t_f)$ in Figures 2, 3 and 4 for both male and female. The courses of the depicted curves follow natural expectations: In Figure 2, the death probability $_{t_j}q_{66}$ increases with t_j and approaches one for larger values of t_j as it becomes more likely to die at older ages. Regarding the average prevalence rate $\overline{\pi}_{66+t_{f-1}}$ describing the probability to become care-dependent between the ages $66 + t_{f-1}$ and $66 + t_f$ displayed in Figure 3, we also detect an increasing curve shape, i.e. the likelihood to enter the care dependency state grows the older the policyholder gets. At more advanced ages, this probability increases quite rapidly for the female and even exceeds 30% for women approaching the age of 100 years. In Figure 4, the semi-Markov transition probability $p^{aa}_{66+t_f}(t_i - t_f)$ describing the probability to stay care-dependent at age $66 + t_f$ for a duration of $t_i - t_f$ years decreases not only if this duration time draws out, but also if the policyholder is

[9]We distinguish policyholders only with respect to age and to gender in our numerical study and ignore cohort effects resulting from different birth years.

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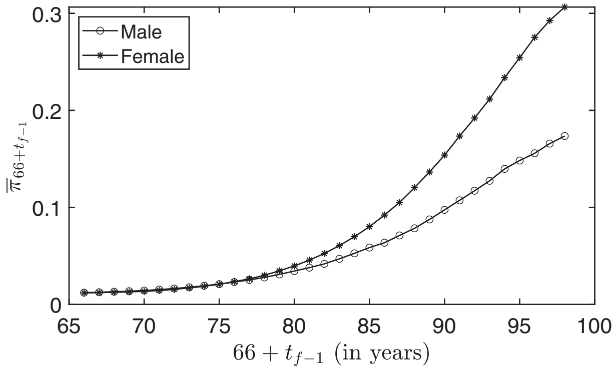

Figure 3. Illustration of available data for $\overline{\pi}_{66+t_{f-1}}$ approximating relevant probabilities.

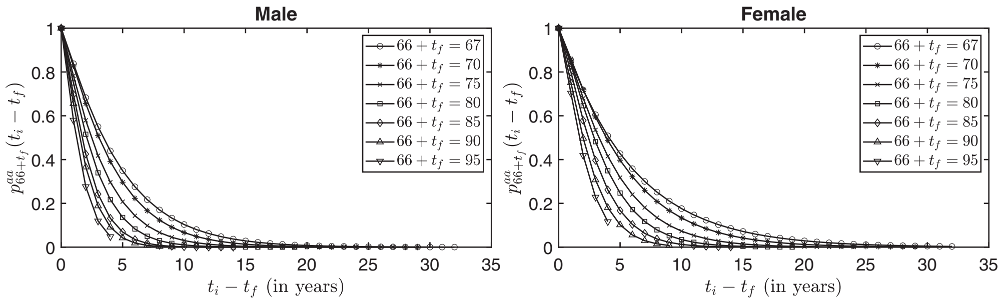

Figure 4. Illustration of selected available data for $p^{aa}_{66+t_f}(t_i - t_f)$ approximating relevant probabilities.

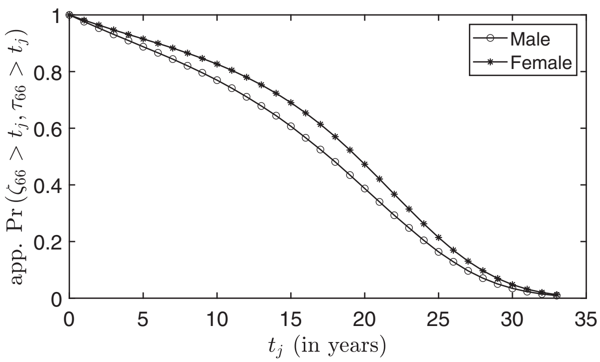

Figure 5. Illustration of approximated (app.) probabilities $\Pr(\zeta_{66} > t_j, \tau_{66} > t_j)$.

initially of greater age. Both of these observations can also be traced back to the fact that the death probability increases with age.

In Figures 5 and 6, we illustrate the numerical approximations of the two relevant probabilities given in (22) and (23). In Figure 5, we see that the probability to stay autonomous between the ages 66 and $66 + t_j$ decreases and approaches zero for growing values of t_j as it becomes more likely either to become dependent or to die at higher ages. If t_f is fixed, the decreasing nature of the curves in Figure 6 coincides with the behaviour of $p^{aa}_{66+t_f}(t_i - t_f)$ observed in Figure 4. If, however,

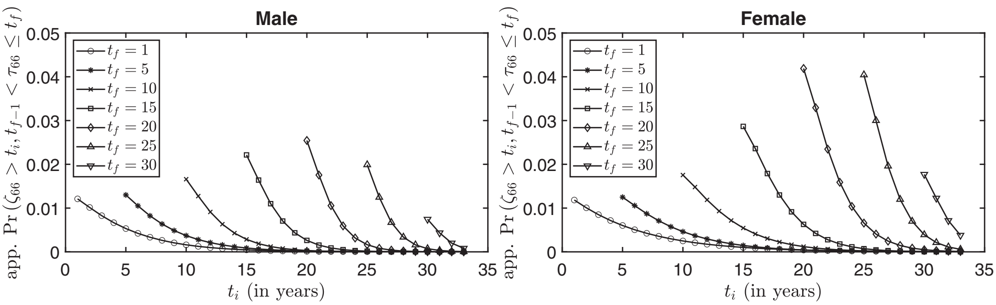

Figure 6. Illustration of selected approximated (app.) probabilities $\Pr\left(\zeta_{66} > t_i, t_{f-1} < \tau_{66} \leq t_f\right)$.

the starting time of the care dependency is shifted further into the future, i.e. if t_f goes up, we detect that the probability to stay dependent from this starting time on until at least t_i first of all grows, but eventually decreases. This is mainly due to the reverse behaviours of $\Pr\left(\zeta_{66} > t_{f-1}, \tau_{66} > t_{f-1}\right)$ and $\Pr\left(\zeta_{66} > t_f, \tau_{66} \leq t_f \middle| \zeta_{66} > t_{f-1}, \tau_{66} > t_{f-1}\right)$ affecting heavily $\Pr\left(\zeta_{66} > t_i, t_{f-1} < \tau_{66} \leq t_f\right)$ (cf. (25)). In particular, there is the increasing course of $\overline{\pi}_{66+t_{f-1}}$ illustrated in Figure 3, which dominates in the beginning and there is the decreasing curve shape of the probability to stay autonomous illustrated in Figure 5 that dominates towards the end.

Concerning the differences between male and female revealed in Figures 2, 3, 4, 5 and 6, we observe that, on average, female policyholders live longer in the autonomy state, are at higher risk of becoming dependent and also stay longer in need of care.

4. Results and Discussion

In this section, we compute the two fees for the care option, F_0 and $\widehat{F}_0^{\text{PU}}$, and analyse the relation between them. In particular, we carry out a series of sensitivity analyses to find out under what parameter combinations the care option becomes tradable, i.e. when it holds that $F_0 \leq \widehat{F}_0^{\text{PU}}$. Further, we examine how the results depend on the gender of the policyholder. At the end, we reflect upon practical implications and limitations of our research.

4.1 Numerical results and sensitivity analyses

In the following, we apply the specifications from section 3 that calibrate our model. These include the discretisation assumption, the baseline parameter values from Table 1 (if not otherwise stated, particularly for the sensitivity analyses) and the approximated relevant probabilities shown in Figures 5 and 6.

Baseline case. Using the pricing formulas in (7) and (18) and fixing all parameters to their baseline values leads to the following numbers for the fees F_0 and $\widehat{F}_0^{\text{PU}}$:

$$\begin{aligned} F_0 &= 5,822.549 \quad \text{and} \quad \widehat{F}_0^{\text{PU}} = 6,203.615 \quad \text{for male} \\ \text{and} \quad F_0 &= 9,782.734 \quad \text{and} \quad \widehat{F}_0^{\text{PU}} = 10,394.587 \quad \text{for female.} \end{aligned} \tag{26}$$

As it holds that $F_0 < \widehat{F}_0^{\text{PU}}$, we conclude that, in the baseline case, the care option is offered by the insurer and contracted by the policyholder. For both male and female, we find that the policyholder is willing to pay about 6% more than the minimum fee required by the insurer. Further, when comparing the fees by gender, we observe that both the actuarial fee and the utility-based fee are significantly higher for women than for men (by a factor larger than 1.6). This is due to the on average higher chances for a woman to become dependent and to the lower female mortality in

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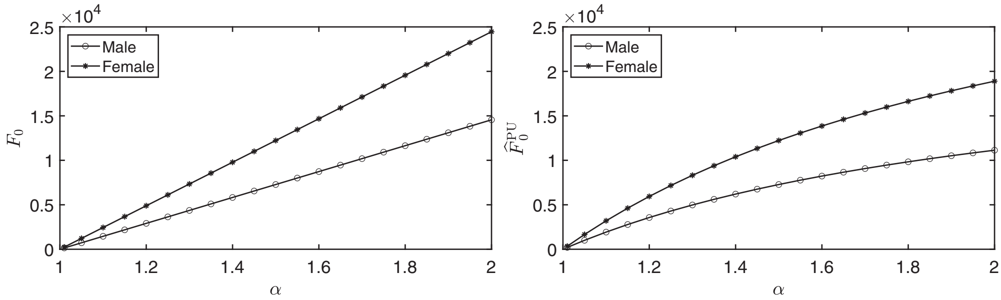

Figure 7. Illustration of F_0 and $\widehat{F}_0^{\text{PU}}$ as functions of α in baseline case.

the care dependency state entailing longer stays in dependence. Note that, when calculating $\widehat{F}_0^{\text{PU}}$, the quantity θ^{PU} given in (17) amounts to

$$\begin{aligned} &\theta^{\text{PU}} = 1.033 \quad \text{for male} \\ \text{and} \quad &\theta^{\text{PU}} = 1.050 \quad \text{for female} \end{aligned} \tag{27}$$

which shows that the individual is indifferent between holding the life care annuity and the regular life annuity with a payment constantly increased by 3.3% and by 5% for male and female, respectively. In absolute terms, this means that the male (female) policyholder is indifferent between annually receiving $\theta^{\text{PU}}c = 12,396\,(12,600)$ for the rest of his (her) life, and annually obtaining $c = 12,000$ if he (she) is autonomous and $\alpha c = 16,800$ if he (she) is dependent. Moreover, we note that the numbers in (27) verify Proposition 1 that, with $\alpha = 1.4$, limits the range of θ^{PU} to the interval $(1, 1.4)$.

Sensitivity analyses regarding $\boldsymbol{\alpha}$. In this paragraph, we take a closer look at α, the multiplier for the payment of the life care annuity in the care dependency state. We begin with the examination of the sensitivities of the insurer's actuarial and the policyholder's utility-based fees, F_0 and $\widehat{F}_0^{\text{PU}}$, towards this parameter by means of Figure 7. As already noticed in section 2.3, both fees increase in α. Raising the payment of the care annuity by increasing α clearly makes the care option more valuable for the insurer and the policyholder. However, because of the risk aversion of the individual, we observe that $\widehat{F}_0^{\text{PU}}$ only concavely grows in α and thus, the policyholder's willingness to pay flattens more and more compared to the linear increase of F_0.

We are also interested in analysing the relation between F_0 and $\widehat{F}_0^{\text{PU}}$. Specifically, we aim at illustrating the theoretical results found in section 2.3, which consider the question when a trade for the care option takes place between the policyholder and the annuity provider. The related numerical findings involving the parameter α are presented in Figure 8 and Table 2. In Figure 8, we plot the difference $\widehat{F}_0^{\text{PU}} - F_0$ depending on α. We detect that the difference first increases and then decreases in α. This shows that the policyholder's willingness to pay grows faster compared to the insurer's demand for premium for small values of α. If α exceeds a threshold value, the situation turns around and F_0 begins to prevail. This behaviour of $\widehat{F}_0^{\text{PU}} - F_0$ entails that there exists, as laid out in Proposition 3, a positive maximum for the difference if the multiplier α is the input variable. Furthermore, it implies that a value for α can be found, so that $F_0 = \widehat{F}_0^{\text{PU}}$. Up to this value, the policyholder would contract the care option and could even reach a point, at which, in terms of the expected utility, it can no longer get any better. As soon as α surpasses this value, the care option becomes too expensive. Note that the value of α in Figure 8, for which $\widehat{F}_0^{\text{PU}} - F_0 = 0$, is given by 1.5 which is the (baseline) value of $\overline{\kappa_a}$ here. The reason why $F_0 = \widehat{F}_0^{\text{PU}}$ holds if α attains the (given) value of $\overline{\kappa_a}$ and why the sex of the policyholder does not play a role in this is that, for the baseline assumptions $\eta = r$ and $\gamma = 2$, it follows from Proposition 2 (cf. (19)) that $\overline{\kappa_a}^* = \alpha$.

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Table 2. α^* and resulting (res.) $\widehat{F}_0^{\mathrm{PU}} - F_0$ with regard to varying γ and $\overline{\kappa_a}$.

$\gamma > 1$	$\overline{\kappa_a}$	α^* (male)	α^* (female)	res. $\widehat{F}_0^{\mathrm{PU}} - F_0$ (male)	res. $\widehat{F}_0^{\mathrm{PU}} - F_0$ (female)
1.2	1.9	1.7354	1.7489	3,386.293	5,410.790
	1.5	1.4100	1.4138	1,168.373	1,870.927
	1.1	1.0829	1.0831	53.667	86.284
2	1.9	1.3859	1.3894	1,865.556	2,962.703
	1.5	**1.2271**	**1.2283**	**665.755**	**1,062.967**
	1.1	1.0489	1.0490	31.830	51.151
2.8	1.9	1.2610	1.2626	1,286.834	2,039.182
	1.5	1.1569	1.1575	465.413	742.270
	1.1	1.0347	1.0347	22.624	36.350

Note: If not varying, all parameters, except α, take their baseline values given in Table 1. Bold numbers give results for baseline parametrisation.

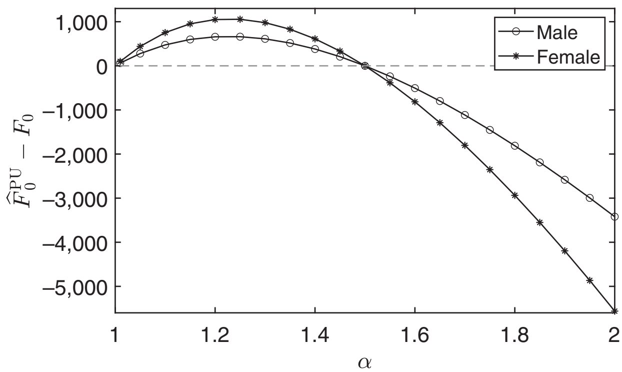

Figure 8. Illustration of $\widehat{F}_0^{\mathrm{PU}} - F_0$ as function of α in baseline case.

In Table 2, we take a closer look at Proposition 3 within our numerical study. In particular, the values for α^* from (21) are shown for different values of $\gamma > 1$ and $\overline{\kappa_a}$. Moreover, we also report the resulting maximised difference $\widehat{F}_0^{\mathrm{PU}} - F_0$. We detect that α^* decreases in the risk aversion level γ and increases in the impact parameter $\overline{\kappa_a}$, which, as we will see later on (cf. Table 3), indicates that the behaviour of α^* goes hand in hand with the one of $\widehat{F}_0^{\mathrm{PU}}$. The maximised difference naturally follows this pattern as the actuarial fee F_0 does not depend on γ nor on $\overline{\kappa_a}$. We can summarise that, if $\gamma > 1$ and hence the relation between the marginal utilities is given by (15), the policyholder can optimise his or her profit if his or her risk aversion level is comparatively small and the impact of the care dependency on his or her utility is large. The optimised multiplier α^* is then rather big to ensure that the increased regular annuity payment suffices. The displayed positive values for $\widehat{F}_0^{\mathrm{PU}} - F_0$ outline that the care option is contracted for all presented parameter combinations if $\alpha = \alpha^*$. In the case of $\gamma \in [0, 1)$, it is not possible to come upon $\alpha^* > 1$ if we consider solely different value combinations of the parameters γ and $\overline{\kappa_a}$, such as $\gamma \in \{0.2, 0.5, 0.8\}$ and $\overline{\kappa_a} \in \{0.1, 0.5, 0.9\}$ (cf. Table 3). However, if we set, e.g. η sufficiently small, e.g. $\eta = 0$, we obtain $\alpha^* = 1.0402$ (male) and $\alpha^* = 1.1342$ (female) for $\gamma = 0.5$ and $\overline{\kappa_a} = 0.9$.

In addition to α, in the real world, the insurer's actuarial fee and the policyholder's utility-based fee will also depend on the actual additional expenses occurring, e.g. for nursing homes. The utility function under consideration might be unable to tackle this possibly non-linear effect appropriately. We disregard this realistic aspect in our analysis.

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Table 3. $\widehat{F}_0^{\mathrm{PU}}$ with regard to varying γ and $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively.

$\gamma \in [0, 1)$	$\underline{\kappa_a}$	$\widehat{F}_0^{\mathrm{PU}}$ (male)	$\widehat{F}_0^{\mathrm{PU}}$ (female)	$\gamma > 1$	$\overline{\kappa_a}$	$\widehat{F}_0^{\mathrm{PU}}$ (male)	$\widehat{F}_0^{\mathrm{PU}}$ (female)
0.2	0.1	605.020	1,056.037	1.2	1.9	8,638.866	14,233.780
	0.5	2,929.748	5,027.025		1.5	6,990.368	11,651.978
	0.9	5,112.357	8,634.494		1.1	5,257.413	8,872.050
0.5	0.1	574.443	1,002.912	2	1.9	7,686.258	12,743.783
	0.5	2,786.467	4,786.561		1.5	**6,203.615**	**10,394.587**
	0.9	4,870.165	8,240.822		1.1	4,653.237	7,884.205
0.8	0.1	545.849	953.201	2.8	1.9	6,858.522	11,427.961
	0.5	2,651.878	4,560.021		1.5	5,524.489	9,294.532
	0.9	4,641.710	7,867.669		1.1	4,135.280	7,028.516

Note: If not varying, all parameters take their baseline values given in Table 1. Bold numbers give results for baseline parametrisation. In all considered cases, it holds $F_0 = 5,822.549$ (male) and $F_0 = 9,782.734$ (female).

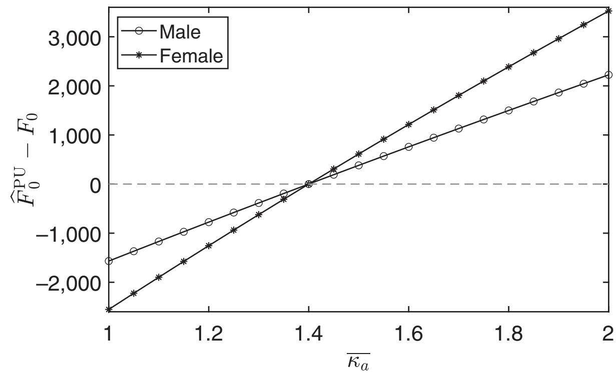

Figure 9. Illustration of $\widehat{F}_0^{\mathrm{PU}} - F_0$ as function of $\overline{\kappa_a}$ in baseline case.

Sensitivity analyses regarding $\underline{\kappa_a}$ and $\overline{\kappa_a}$. We numerically study the subjective measures $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively, by examining the sensitivity of the fee $\widehat{F}_0^{\mathrm{PU}}$ in Table 3. In this table, we let $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively, and the risk aversion level γ vary simultaneously and report the resulting (maximum) values of the care option fee. If $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively, increase, we detect that the fee increases as well. This however does not imply a consistent statement for the relation between $\widehat{F}_0^{\mathrm{PU}}$ and these impact parameters: For $\gamma \in [0, 1)$, i.e. (13) and (14) hold, the increase of $\widehat{F}_0^{\mathrm{PU}}$ in $\underline{\kappa_a}$ implies that the policyholder is willing to only spend a smaller amount of money for the care option if the impact of the care dependency becomes greater. Recall that the impact of losing autonomy is important if $\underline{\kappa_a}$ is small. This observation is a consequence of (14) that tells us that the policyholder's marginal utility is greater in the autonomy state than in the care dependency state. For $\gamma > 1$, i.e. (13) and (15) hold, the opposite is true as the increase of $\widehat{F}_0^{\mathrm{PU}}$ in $\overline{\kappa_a}$ implies that the policyholder is actually willing to spend a larger amount of money for the care option (if the impact of dependency is larger). Here, recall that the impact of care dependency is important if $\overline{\kappa_a}$ is large. This observation is a consequence of (15) entailing that the policyholder's marginal utility is greater in the dependency state than in the autonomy state. On the whole, we see again how critical the relationship between the marginal utilities is.

For the examination of the relation between F_0 and $\widehat{F}_0^{\mathrm{PU}}$ (see the theoretical results in section 2.3), we present the numerical findings, with respect to $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively, in Figure 9 and Table 4. In Figure 9, the difference $\widehat{F}_0^{\mathrm{PU}} - F_0$ as a function of the parameter $\overline{\kappa_a}$ is displayed. The increase of $\widehat{F}_0^{\mathrm{PU}} - F_0$ in $\overline{\kappa_a}$ is shown (as already visible from Table 3). Furthermore, Proposition 2

Table 4. $\underline{\kappa_a}^*$ and $\overline{\kappa_a}^*$, respectively, with regard to varying γ and η.

$\gamma \in [0, 1)$	η	$\underline{\kappa_a}^*$ (male)	$\underline{\kappa_a}^*$ (female)	$\gamma > 1$	η	$\overline{\kappa_a}^*$ (male)	$\overline{\kappa_a}^*$ (female)
0.2	0	0.9139	0.8755	1.2	0	1.0839	1.0381
	0.02	–	–		0.02	1.2282	1.2277
	0.04	–	–		0.04	1.3868	1.4476
0.5	0	0.9627	0.9221	2	0	1.2354	1.1837
	0.02	–	–		0.02	**1.4**	**1.4**
	0.04	–	–		0.04	1.5808	1.6507
0.8	0	–	0.9705	2.8	0	1.4013	1.3439
	0.02	–	–		0.02	1.5880	1.5895
	0.04	–	–		0.04	1.7930	1.8741

Note: If not varying, all parameters, except $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively, take their baseline values given in Table 1. Bold numbers give results for baseline parametrisation.

is illustrated here as $\overline{\kappa_a}^*$ given in (19) and entailing $F_0 = \widehat{F}_0^{\mathrm{PU}}$ can be directly read from this graph. Since we assume for the baseline case that $\gamma > 1$, which implies that the marginal utilities behave as in (15), the policyholder rather decides for buying the care option if the impact of the care dependency gets higher, i.e. $\overline{\kappa_a}$ goes up and possibly exceeds $\overline{\kappa_a}^*$.

To elaborate on Proposition 2, we display, in Table 4, the attained values of $\underline{\kappa_a}^*$ and $\overline{\kappa_a}^*$, respectively, if the risk aversion level γ and the subjective discount rate η vary. We find that $\underline{\kappa_a}^*$ and $\overline{\kappa_a}^*$ increase in γ, i.e. the more risk averse the policyholder is, the smaller, in case of $\gamma \in [0, 1)$, respectively, the greater, in case of $\gamma > 1$, the utility impact of the care dependency must be in order for the trade of the care option to occur. Considering $\gamma > 1$, we also observe that $\overline{\kappa_a}^*$ increases in η. This seems plausible since we will see later on (cf. Figure 11) that the policyholder's willingness to pay reduces if the subjective discount rate increases. Therefore, to raise the interest for the care option, the impact parameter $\overline{\kappa_a}$ needs to grow. This actually also applies to the case where $\gamma \in [0, 1)$. However, similar to Table 2, if η is too large, it is not possible any more to find $\underline{\kappa_a}^* \in [0, 1]$ and thus, in this case and if (14) holds, respectively, the care option is only interesting if η is relatively small. Clearly, the interest for the care option is also higher if the policyholder is less risk averse.

Sensitivity analyses regarding γ. For the assessment of the sensitivity of the policyholder's willingness to pay for the care option, i.e. $\widehat{F}_0^{\mathrm{PU}}$, towards the individual risk aversion level γ, we again consult Table 3. Overall, for a given $\underline{\kappa_a}$ and a given $\overline{\kappa_a}$, we record that $\widehat{F}_0^{\mathrm{PU}}$ decreases in $\gamma \in [0, 1)$ and in $\gamma > 1$, respectively, which means that, if the policyholder becomes more risk averse, the willingness to pay for the option gets lower. This can be explained by the following line of reasoning: Compared to the regular life annuity, the life care annuity provides higher payments in the dependency state, but, at the same time, leads to a less smooth payment pattern. A more risk averse individual typically prefers smoother consumption patterns, which has the consequence that his or her corresponding willingness to pay diminishes.

As before, we are interested in examining the relation between the fees F_0 and $\widehat{F}_0^{\mathrm{PU}}$. For that, we plot the difference $\widehat{F}_0^{\mathrm{PU}} - F_0$ as a function of the parameter $\gamma > 1$ in Figure 10. As already visible from Table 3, the fee $\widehat{F}_0^{\mathrm{PU}}$ decreases in γ and, as F_0 does not depend on this subjective parameter, this observation obviously holds for the difference. Similarly, as in the analyses of the parameters α and $\underline{\kappa_a}$ and $\overline{\kappa_a}$, respectively, we can spot a critical value for γ in the graph, for which both the required premium and the willingness to pay are identical. Consequently, the care option is rather contracted if the policyholder is less risk averse. Note that the critical value for γ in Figure 10 is approximately equal to 2.4342 in the male case and approximately equal to 2.4315 in the female case.

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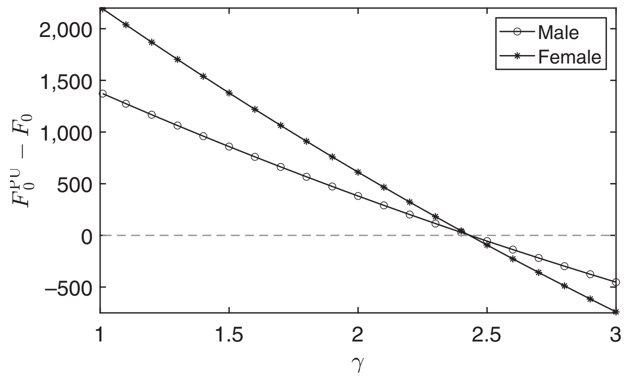

Figure 10. Illustration of $\widehat{F}_0^{\mathrm{PU}} - F_0$ as function of γ in baseline case.

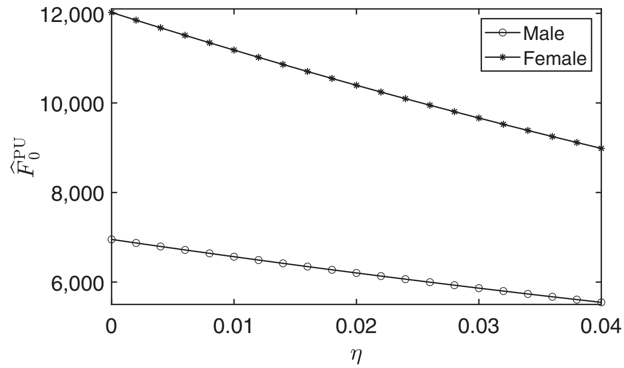

Figure 11. Illustration of $\widehat{F}_0^{\mathrm{PU}}$ as function of η in baseline case.

Sensitivity analyses regarding η. The next parameter that we inspect is the subjective discount rate η. By means of Figure 11, we examine the sensitivity of the fee $\widehat{F}_0^{\mathrm{PU}}$ towards η. The decrease of $\widehat{F}_0^{\mathrm{PU}}$ in η can be traced back to the fact that a higher subjective discount rate can be interpreted in such a way that the policyholder is less patient about the future, i.e. the policyholder reduces the weighting of future transactions in his or her total utility. Accordingly, as care dependency is more likely to occur at more advanced ages, the willingness to pay for the care option becomes lower if the individual's subjective discount rate is larger. Hence, as shown in Figure 11, $\widehat{F}_0^{\mathrm{PU}}$ decreases in η. Note that, as the minimum fee F_0 required by the insurer does not depend on the subjective discount rate, this automatically implies that the policyholder rather decides for buying the care option if he or she is more patient about the future.

Sensitivity analyses regarding r and c. For reasons of completeness, we also discuss the influences of the discount rate r and the regular annuity payment c the policyholder receives while remaining autonomous. It is not surprising that both fees, F_0 and $\widehat{F}_0^{\mathrm{PU}}$, exponentially decrease in r and linearly increase in c. This entails that the difference $\widehat{F}_0^{\mathrm{PU}} - F_0$ is not necessarily monotone in r and in c. However, in our framework, we find that $\widehat{F}_0^{\mathrm{PU}}$ declines slower than F_0, and thus, the care option is rather contracted, if r grows within a range corresponding to the persistently low interest rate environment, i.e. a small range around zero. Let us compare, e.g. the respective values of F_0 and of $\widehat{F}_0^{\mathrm{PU}}$ when going from $r = 0\%$ to the baseline case value $r = 2\%$. For $r = 0\%$, we obtain $F_0 = 7,984.307$ and $\widehat{F}_0^{\mathrm{PU}} = 7,585.537$ for male, and $F_0 = 14,152.803$ and $\widehat{F}_0^{\mathrm{PU}} = 12,986.603$

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for female. Consequently, for $r = 0\%$, the care option is not contracted. When proceeding to the baseline case with $r = 2\%$ which leads to the fees given in (26), we observe a decrease of about 27% (male) and 31% (female) in F_0, and of 18% (male) and 20% (female) in $\widehat{F}_0^{PU}$. We see that the impact of r is here more substantial on F_0 than on $\widehat{F}_0^{PU}$. The critical value for r, so that $F_0 = \widehat{F}_0^{PU}$, is approximately equal to 0.89% in the male case and 1.18% in the female case. Regarding c, we realise that the condition $F_0 = \widehat{F}_0^{PU}$ does not depend on this parameter, which implies that, if c grows, $\widehat{F}_0^{PU} - F_0$ is always positive (negative) and linearly increases (decreases) or remains zero. Consequently, the amount c of the basic payment can at most make the policyholder's decision for buying the care option either easier or harder, but can never change the decision itself.

Gender analysis. At the end of this section, let us come back to the impact of gender on the outcomes. From (26), Figures 7 and 11 and Table 3, we observe huge differences between the amounts applying to men and to women for both F_0 and $\widehat{F}_0^{PU}$. In all cases, the female policyholder, compared to her male companion, not only has to pay (significantly) more for the care option, but is also willing to spend (significantly) more money for it. This is due to the fact that the probabilities to become dependent and to stay long in need of care are higher for women than for men (see also Figures 2, 3, 4, 5 and 6 in section 3). The more extreme behaviour of the results for the female policyholder in Figures 8, 9 and 10 and Tables 2 and 4 considering the relation of F_0 and $\widehat{F}_0^{PU}$ can also be traced back to this fact. In particular, we detect much wider ranges for the difference of the two fees if α, $\overline{\kappa_a}$ or γ varies (cf. Figures 8, 9 and 10) as well as for $\overline{\kappa_a}^*$ equating the fees if η varies (cf. Table 4) in case the individual is female. Furthermore, the optimal multiplier α^* and the resulting maximised difference $\widehat{F}_0^{PU} - F_0$ always attain larger values for different combinations of the risk aversion level and the impact parameter for the female policyholder (cf. Table 2). Based on these results, a (mandatory) unisex tariff implemented in practice for the life annuity including a care option can generate underinsurance of the male population. Apparently, females will find the unisex tariff attractive, while males will be less interested in the resulting product. The unisex tariff can thus cause the presence of adverse selection, i.e. males might have less incentives to invest in this life care annuity.

4.2 Practical implications and limitations

Our work brings significant knowledge to the insurance sector. First, it gives a clear insight on the premium for receiving a care supplement within a life annuity by providing both theoretical and numerical results. Our theoretical framework reflects important aspects in the decision-making process when underwriting a life annuity with care option. Considering both the insurer's and the individual's perspectives when it comes to the offer of financial coverage for (a certain part of) the LTC costs to the policyholder can lead to enlarged market opportunities in a still underdeveloped market. As far as the numerical application is concerned, the possibility to use fine-grained probability data enriches our research by providing numerical results often not accessible to the insurance world.

For a life care annuity contract to exist, i.e. the care option to be contracted, both parties involved have to find a situation that suits them. This is modelled in our framework by finding situations, in which the maximal price the policyholder is ready to pay exceeds the minimum premium the insurer expects. In the previous section, we have illustrated many situations, in which a life care annuity contract can exist, which is rather good news in terms of extending the set of financing solutions for LTC. However, those situations also appear to be highly sensitive to the parameter values. Beyond the parameter values allowing for a trade, many realistic parameter value combinations do not result in market opportunities. This finding raises awareness of the importance for insurers to know their customers' profiles, such as their individual preferences. Furthermore, we have highlighted the existing differences in the fees when genders are considered separately. Indeed, the fees applying to women appear to be significantly higher than the ones applying to men. Concrete figures of this finding allow to get a better idea of the associated issue induced by risk pooling, particularly in the case of unisex tariffs as required, e.g. in the European Union.

Although we are convinced that our research enlarges the current body of literature, we are aware of some limitations it might face. One of them is surely the absence of any consideration of other external factors such as regulation concerns, the presence of competition and the existence of social insurance. Regarding the regulation, setting solvency requirements would entail an increase in the fee requested by the insurer and would therefore lead to a reduction in the set of solutions bringing on a trade. Conversely, the presence of tax-deductible premiums could significantly shape the policyholder's utility in a positive manner and thus increase the chances for a trade. Further, the insurance market is in practice more complex because an insurer has to evolve in a market where competitors are also active. Such difficulties are compounded by the fact that the demand comes from policyholders having preferences sometimes hard to anticipate, which leads to the problems of adverse selection and moral hazard (see, e.g. Zhou-Richter & Gründl, 2011). Finally, the existence of social insurance is an important matter since it yields crowding-out effects that make it hard to propose other insurance policies (see, e.g. Brown & Finkelstein, 2009). Integrating all incoming (insurance and social security benefits) and outgoing (payment for at-home and institutional care) cash flows from care dependency that a policyholder faces would provide a more comprehensive picture.

5. Conclusion

In the present article, we examine an LTC option embedded in a life annuity which, if contracted for a fee, entails an increase of the annuity payments in case of care dependency. Particularly, we are interested in its pricing not only from the perspective of the annuity provider, but also from the policyholder's point of view. Hence, we develop a suitable model framework, where we derive, on the one hand, an appropriate actuarial pricing approach that leads to the minimum fee the insurer requires, and, on the other hand, an appropriate utility-based pricing approach that brings on the fee as the policyholder's willingness to pay for the care option. For the latter, we apply a condition-based utility function which takes account of the impact of the care dependency on the utility of the individual. Under certain assumptions, we are able to show some theoretical statements about the relation between the two different fees, which also allows us to respond to the question when the care option is contracted by the policyholder. Furthermore, by using the Swiss LTC transition probability data from Fuino & Wagner (2018) for a realistic calibration, we numerically illustrate the model and the implied theoretical findings. Amongst others, we detect the following within our numerical study: If the policyholder is less risk averse, his willingness to pay increases and thus, the trade becomes more likely to take place. In the case where the individual proportionally profits more from the annuity when being in need of care, this is also true if the impact of the care dependency on his utility grows. In the resulting situation, the optimal supplementary annuity amount in case of dependency increases as well. On the contrary, if the individual proportionally profits more when being autonomous, a greater utility impact of the care dependency reduces the policyholder's willingness to pay, and only under fairly specific circumstances, the option is bought. If the policyholder is female, both fees are higher due to the greater survival and LTC probabilities for women. Finally, at the end of the article, we reflect upon practical implications and limitations of our research. The limitations suggest possible extensions for our framework, namely the inclusion of other external factors, which can be incorporated within further related research studies. Future applications can encompass the specific environment of each country, taking into account social security, taxes and the out-of-pocket payments required from dependent individuals on the one hand, and the regulatory and solvency requirements for insurers on the other hand. Additionally, heterogeneity can be suspected amongst policyholders, who may also differ with respect to family situation and richness. Moreover, it can be of interest to extend our analyses in the future by using different available probability data sets from selected countries other than Switzerland in order to come up with a cross-country comparison. Furthermore, the analysis of other product variants can lead to a useful ranking giving recommendations for policyholders and insurers. With that concern, practical matters regarding adverse selection, cost of capital as well as unknown future mortality could also be the subject of future research. As for our applied model framework and the assumption to use only one state of care dependency, a reasonable enhancement can be to increase the level of granularity concerning the degree of care dependency and to appropriately adjust the condition-based utility function in a proper way.

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A. Proof of Proposition 1

Proof. We can write (8) equivalently as

$$\underbrace{\left(u_0(\theta c) - u_0(c)\right)\, \Psi_0(\eta)}_{\text{(I)}} + \underbrace{\left(u_a(\theta c) - u_a(\alpha c)\right)\, \Psi_a(\eta)}_{\text{(II)}} = 0 \tag{A.1}$$

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Note that $\Psi_0(\eta) > 0$ and $\Psi_a(\eta) > 0$. Without loss of generality, we assume that both u_0 and u_a are strictly increasing. Then, if $\theta \leq 1$, the fulfilment of (8) is not possible as (I) ≤ 0 and (II) < 0 due to $\alpha > 1$. Furthermore, if $\theta \geq \alpha > 1$, the fulfilment of (8) is not possible either, as (I) > 0 and (II) ≥ 0. Thus, $\theta \in (1, \alpha)$. □

B. Proof of Proposition 2

Proof. For reasons of clarity, we define $\kappa_a^{\mathrm{PU}} := \underline{\kappa_a}\mathbb{1}_{\{\gamma\in[0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma>1\}}$. Given (7) and (18), we derive the following equivalences for $F_0 = \widehat{F}_0^{\mathrm{PU}}$:

$$
\begin{aligned}
& F_0 = \widehat{F}_0^{\mathrm{PU}} \\
\Leftrightarrow\ & c\,(\alpha-1)\,\Psi_a(r) = \left(\left(\frac{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}\right)^{\frac{1}{1-\gamma}} - 1\right) c\,(\Psi_0(r)+\Psi_a(r)) \\
\Leftrightarrow\ & \left(\frac{(\alpha-1)\,\Psi_a(r)}{\Psi_0(r)+\Psi_a(r)}+1\right)^{1-\gamma} = \frac{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)} \\
\Leftrightarrow\ & \kappa_a^{\mathrm{PU}}\left(\left(\frac{\Psi_0(r)+\alpha\Psi_a(r)}{\Psi_0(r)+\Psi_a(r)}\right)^{1-\gamma}\Psi_a(\eta) - \alpha^{1-\gamma}\Psi_a(\eta)\right) = \Psi_0(\eta)\left(1-\left(\frac{\Psi_0(r)+\alpha\Psi_a(r)}{\Psi_0(r)+\Psi_a(r)}\right)^{1-\gamma}\right) \\
\Leftrightarrow\ & \kappa_a^{\mathrm{PU}} = \frac{\Psi_0(\eta)\left(1-\left(\frac{\Psi_0(r)+\alpha\Psi_a(r)}{\Psi_0(r)+\Psi_a(r)}\right)^{1-\gamma}\right)}{\Psi_a(\eta)\left(\left(\frac{\Psi_0(r)+\alpha\Psi_a(r)}{\Psi_0(r)+\Psi_a(r)}\right)^{1-\gamma}-\alpha^{1-\gamma}\right)}
\end{aligned}
\tag{B.1}
$$

As the last line above coincides with (19), the claim is shown. □

C. Proof of Proposition 3

Proof. For reasons of clarity, we also use $\kappa_a^{\mathrm{PU}} = \underline{\kappa_a}\mathbb{1}_{\{\gamma\in[0,1)\}} + \overline{\kappa_a}\mathbb{1}_{\{\gamma>1\}}$ here. Given (7) and (18), it holds $\widehat{F}_0^{\mathrm{PU}} - F_0 = \theta^{\mathrm{PU}}P_0^L - P_0^C$, so that the necessary first-order condition reads as follows:

$$
\begin{aligned}
& \frac{\partial\left(\widehat{F}_0^{\mathrm{PU}}-F_0\right)}{\partial\alpha} = \frac{\partial\theta^{\mathrm{PU}}}{\partial\alpha}P_0^L - \frac{\partial P_0^C}{\partial\alpha} \overset{!}{=} 0 \\
\Leftrightarrow\ & \frac{1}{1-\gamma}\left(\frac{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}\right)^{\frac{\gamma}{1-\gamma}}\frac{\kappa_a^{\mathrm{PU}}(1-\gamma)\,\alpha^{-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}c\,(\Psi_0(r)+\Psi_a(r)) = c\Psi_a(r) \\
\Leftrightarrow\ & \left(\frac{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}\right)^{\frac{\gamma}{1-\gamma}}\alpha^{-\gamma} = \frac{\Psi_a(r)\left(\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)\right)}{\kappa_a^{\mathrm{PU}}\Psi_a(\eta)\,(\Psi_0(r)+\Psi_a(r))} \\
\Leftrightarrow\ & \frac{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}\alpha^{\gamma-1} = \left(\frac{\Psi_a(r)\left(\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)\right)}{\kappa_a^{\mathrm{PU}}\Psi_a(\eta)\,(\Psi_0(r)+\Psi_a(r))}\right)^{\frac{1-\gamma}{\gamma}}
\end{aligned}
$$

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$$\Leftrightarrow \Psi_0(\eta)\alpha^{\gamma-1} + \kappa_a^{\mathrm{PU}}\Psi_a(\eta) = \left(\frac{\Psi_a(r)}{\kappa_a^{\mathrm{PU}}\Psi_a(\eta)\left(\Psi_0(r)+\Psi_a(r)\right)}\right)^{\frac{1-\gamma}{\gamma}} \left(\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)\right)^{\frac{1}{\gamma}}$$

$$\Leftrightarrow \alpha^* = \left(\frac{\left(\frac{\Psi_a(r)}{\kappa_a^{\mathrm{PU}}\Psi_a(\eta)(\Psi_0(r)+\Psi_a(r))}\right)^{\frac{1-\gamma}{\gamma}} \left(\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)\right)^{\frac{1}{\gamma}} - \kappa_a^{\mathrm{PU}}\Psi_a(\eta)}{\Psi_0(\eta)}\right)^{\frac{1}{\gamma-1}} \tag{C.1}$$

where the last equivalence uses (20). In order to show that we indeed find the maximum of $\widehat{F}_0^{\mathrm{PU}} - F_0$, we check the appropriate sufficient second-order condition, for which we need

$$\begin{aligned}
\frac{\partial^2\left(\widehat{F}_0^{\mathrm{PU}}-F_0\right)}{\partial\alpha^2} &= \frac{\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)} c\left(\Psi_0(r)+\Psi_a(r)\right)\left(\frac{\gamma}{1-\gamma}\left(\frac{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}\right)^{\frac{\gamma}{1-\gamma}-1}\right.\\
&\quad \left.\cdot\frac{\kappa_a^{\mathrm{PU}}(1-\gamma)\alpha^{-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}\alpha^{-\gamma} - \left(\frac{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}\right)^{\frac{\gamma}{1-\gamma}}\gamma\alpha^{-\gamma-1}\right)\\
&= \frac{\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)} c\left(\Psi_0(r)+\Psi_a(r)\right)\gamma\left(\frac{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)}{\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\Psi_a(\eta)}\right)^{\frac{\gamma}{1-\gamma}}\alpha^{-\gamma-1}\\
&\quad \cdot\left(\left(\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)\right)^{-1}\kappa_a^{\mathrm{PU}}\alpha^{-\gamma+1}\Psi_a(\eta)-1\right)
\end{aligned} \tag{C.2}$$

Assuming that the second-order condition holds leads to

$$\begin{aligned}
&\frac{\partial^2\left(\widehat{F}_0^{\mathrm{PU}}-F_0\right)}{\partial\alpha^2} < 0\\
&\Leftrightarrow \left(\Psi_0(\eta)+\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)\right)^{-1}\kappa_a^{\mathrm{PU}}\alpha^{-\gamma+1}\Psi_a(\eta)-1<0\\
&\Leftrightarrow \kappa_a^{\mathrm{PU}}\alpha^{-\gamma+1}\Psi_a(\eta)-\kappa_a^{\mathrm{PU}}\alpha^{1-\gamma}\Psi_a(\eta)<\Psi_0(\eta)\\
&\Leftrightarrow 0<\Psi_0(\eta)
\end{aligned} \tag{C.3}$$

which is a true statement and thus, choosing $\alpha = \alpha^*$ indeed results in the maximisation of $\widehat{F}_0^{\mathrm{PU}} - F_0$. □

D. Approximation Details

The probabilities $\Pr\left(\zeta_{x_0} > t_j, \tau_{x_0} > t_j\right)$ and $\Pr\left(\zeta_{x_0} > t_i, t_{f-1} < \tau_{x_0} \le t_f\right)$ are approximated by using (24) and (25), their interdependence and the three approximations listed in section 3. In order for this to work, we stick to the following procedure:

Starting point: $\Pr\left(\zeta_{x_0} > 0, \tau_{x_0} > 0\right) = 1$

$\Downarrow$

results in approximations for

$\Pr\left(\zeta_{x_0} > 1, 0 < \tau_{x_0} \le 1\right), \Pr\left(\zeta_{x_0} > 2, 0 < \tau_{x_0} \le 1\right), \ldots, \Pr\left(\zeta_{x_0} > 33, 0 < \tau_{x_0} \le 1\right)$

$\Downarrow$

results in approximation for $\Pr\left(\zeta_{x_0} > 1, \tau_{x_0} > 1\right)$

$\Downarrow$

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$$
\begin{gathered}
\text{results in approximations for} \\
\Pr\left(\zeta_{x_0} > 2, 1 < \tau_{x_0} \leq 2\right), \Pr\left(\zeta_{x_0} > 3, 1 < \tau_{x_0} \leq 2\right), \ldots, \Pr\left(\zeta_{x_0} > 33, 1 < \tau_{x_0} \leq 2\right) \\
\Downarrow \\
\text{results in approximation for } \Pr\left(\zeta_{x_0} > 2, \tau_{x_0} > 2\right) \\
\Downarrow \\
\text{results in approximations for} \\
\Pr\left(\zeta_{x_0} > 3, 2 < \tau_{x_0} \leq 3\right), \Pr\left(\zeta_{x_0} > 4, 2 < \tau_{x_0} \leq 3\right), \ldots, \Pr\left(\zeta_{x_0} > 33, 2 < \tau_{x_0} \leq 3\right) \\
\Downarrow \\
\ldots \\
\Downarrow \\
\text{results in approximation for } \Pr\left(\zeta_{x_0} > 31, \tau_{x_0} > 31\right) \\
\Downarrow \\
\text{results in approximations for } \Pr\left(\zeta_{x_0} > 32, 31 < \tau_{x_0} \leq 32\right), \Pr\left(\zeta_{x_0} > 33, 31 < \tau_{x_0} \leq 32\right) \\
\Downarrow \\
\text{results in approximation for } \Pr\left(\zeta_{x_0} > 32, \tau_{x_0} > 32\right) \\
\Downarrow \\
\text{results in approximation for } \Pr\left(\zeta_{x_0} > 33, 32 < \tau_{x_0} \leq 33\right) \\
\Downarrow \\
\text{results in approximation for } \Pr\left(\zeta_{x_0} > 33, \tau_{x_0} > 33\right).
\end{gathered}
$$

In the end, all relevant probabilities for $t_j \in \{0, \ldots, 33\}$, $t_{f-1} \in \{0, \ldots, 32\}$, $t_f \in \{1, \ldots, 33\}$, and $t_i \in \left\{t_f, \ldots, 33\right\}$ are approximated.

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