J. Austral. Math. Soc. (*Series A*) **42** (1987), 183–195

CLT GROUPS AND WREATH PRODUCTS

ROLF BRANDL

(Received 11 June 1985)

Communicated by H. Lausch

Abstract

In this paper the question is considered of when the wreath product of a nilpotent group with a CLT group G is a CLT group. It is shown that if the field with p^r elements is a splitting field of a Hall p'-subgroup of G, then $P \operatorname{wr} G$ is a CLT group for all p-groups P with $|P/P'| \geqslant p^r$. Moreover, the class of all groups G having the property that $N \operatorname{wr} G$ is a CLT group for every nilpotent group N is shown to be quite large. For example, every group of odd order can be embedded as a subgroup of a group belonging to this class.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 C 05, 20 D 60, 20 E 22, 20 F 16.

A *CLT group* is a finite group G of order n, say, having the property that for every divisor d of n there exists a subgroup of G having index d in G (by [5, p. 8] it suffices to know that this condition is satisfied for all prime powers d dividing n). It is well known (cf. [5, p. 663]) that every CLT group is soluble and that every soluble π-group G can be embedded as a subgroup of a CLT group which is a π-group. Indeed, the direct product of G with a cyclic group having the same order as G has this property [5, p. 663]. The alternating group G of degree four is not a CLT group, but $G \times \mathbf{Z}_2$ is. The latter group is isomorphic to the wreath product $\mathbf{Z}_2 \operatorname{wr} \mathbf{Z}_3$.

In this paper we investigate the question: for which CLT groups G and nilpotent groups N is the wreath product $N \operatorname{wr} G$ a CLT group? For example, it was shown in [7] that if G is a CLT group and p is a prime not dividing the order of G, then $N \operatorname{wr} G$ is a CLT group when N is an elementary abelian p-group of rank equal to $|G| - 2$.

https://doi.org/10.1017/S1446788700028196 Published online by Cambridge University Press

1. The results

If N is a cyclic group of prime order p, then the base group of $N \operatorname{wr} G$ is G-isomorphic to the group ring $\mathbb{F}_p G$, and so representation theory plays a central rôle in our investigations. The main problem comes from the fact that prime fields very rarely are splitting fields for our groups G. We now introduce the relevant parameter to cope with this situation.

DEFINITION. Let G be a finite soluble group and let p be any prime. Then $r(G,p)$ denotes the least positive integer r such that the field $\mathbb{F}_{p^r}$ with p^r elements is a splitting field for a Hall p'-subgroup of G.

It seems to be a nontrivial problem to determine $r(G, p)$ without knowing all absolutely irreducible representations of G (cf. [6, p. 31]). However, there is a criterion yielding an upper bound for $r(G, p)$ which can easily be computed: if H is a Hall p'-subgroup of G, and if $\exp(H) \equiv 1 \bmod p^r$, then $r(G, p)$ divides r [6, p. 31].

The following result provides some information about when the wreath product of p-group with some CLT group again is a CLT group.

THEOREM A. *Let G be a* CLT *group and let P be a p-group for some prime p. If $|P/P'| \geqslant p^{r(G,p)}$, then $P \operatorname{wr} G$ is a* CLT *group.*

A special case of Theorem A with $p \nmid |G|$, $r(G, p) = 1$, and $|P| = p$ has appeared in [4].

Theorem A implies that for a given CLT group G and a prime p the wreath product $P \operatorname{wr} G$ is a CLT group for all but finitely many abelian p-groups. Moreover, if $r(G, p) \leqslant 2$, then $P \operatorname{wr} G$ is a CLT group for all p-groups P of order $\geqslant p^{r(G,p)}$. However, for each $r \geqslant 3$ there exist examples of CLT groups G and primes p such that $r(G, p) = r$ and $P \operatorname{wr} G$ is not a CLT group for infinitely many p-groups P. For example, if $G = \langle a, b \mid a^{43} = b^7 = 1, a^b = a^4 \rangle$ and $p = 23$, then $r(G, p) = 3$, and for every p-group P with $P/P' \cong \mathbb{Z}_p \times \mathbb{Z}_p$, the group $P \operatorname{wr} G$ is not a CLT group. More detailed information concerning this and other examples may be found in §2.

In [2] T. Gagen proved that any soluble group H can be embedded as a subgroup of a direct indecomposable CLT group G. However, in these examples one has $\pi(G) \supsetneq \pi(H)$. This result can be improved by using Theorem A.

COROLLARY. *Every soluble π-group H can be embedded as a subgroup of a direct indecomposable* CLT *group which itself is a π-group.*

PROOF. We may assume that $H \neq 1$ is a CLT-group. Let p be a prime dividing the order of H and let P be an elementary abelian p-group of rank $r(H, p)$. By Theorem A, the wreath product $W = P \operatorname{wr} H$ is a CLT group.

Assume that $W = R \times S$. By [9, p. 354] we may assume without loss of generality that $R \leqslant Z(W)$ and $S \geqslant H$. Now [9, pp. 348ff] implies that $R \leqslant W'$, since p divides the order of H. Hence $R \leqslant W' \cap Z(W) \leqslant \Phi(W)$, and so $R = 1$ and $S = W$, as required.

In another direction, we consider the class $\mathscr{R}$ of all groups G such that $N \operatorname{wr} G$ is a CLT group for every nilpotent group N. From Theorem A it follows immediately that $\mathscr{R}$ contains all elementary abelian 2-groups (this can be seen more directly, as any extension of a nilpotent group by an elementary abelian 2-group is supersoluble and so is a CLT group [1]). However, somewhat surprisingly, the class $\mathscr{R}$ is much larger than one might expect at first glance.

THEOREM B. (a) *$\mathscr{R}$ contains every 2-group, $\mathbb{Z}_3$, and every noncyclic 3-group.*
(b) *If $G \in \mathscr{R}$ is a p-group, then $p \leqslant 3$.*
(c) *Every group of odd order can be embedded as a subgroup of an $\mathscr{R}$-group.*

In particular, $\mathscr{R}$ contains groups of arbitrary Fitting length, and for every prime p there exists some $\mathscr{R}$-group of order divisible by p. The class $\mathscr{R}$ is not closed with respect to taking subgroups or quotients, and it is an open question as to whether $\mathscr{R}$ is closed under direct products. For more details the reader is referred to §3.

All groups considered in this paper are finite. If X is a subgroup of Y, and if Z is a kX-module for some field k, then Z^Y denotes the induced module. A similar convention will be used for characters.

2. Particular base groups

In this section we consider any given CLT group G and study the question: For which p-groups P is the wreath product $P \operatorname{wr} G$ a CLT group? It turns out that the results depend heavily upon the distribution of the degrees of the irreducible modules of (a Hall p'-subgroup of) G.

The proof of the following preparatory result on group rings is based on some ideas from [4].

LEMMA 1. *Let $G \neq 1$ be a soluble group and let K be a splitting field for G whose characteristic does not divide the order of G. Then for any λ with $0 \leqslant \lambda \leqslant |G| - 1$, the group ring $M = KG$ contains a submodule S with $\dim_K(S) = \lambda$ such that the trivial KG-module is not a composition factor of S.*

PROOF. We use induction on the derived length d of G. If G is abelian, then every irreducible KG-module has dimension one and everything is clear, so let $d > 1$. Let N be the last nontrivial term of the derived series of G, so that N is an abelian normal subgroup of G. Let U be the sum of all irreducible submodules of M having N in their kernel, so that U is isomorphic with the regular KG/N-module. By induction, U contains submodules S_1 of dimension ρ with $C_{S_1}(G) = 1$ for every ρ such that $0 \leqslant \rho \leqslant |G/N| - 1$. If $M = U \oplus V$, then by a result of N. Itô (see [5, p. 570]) every composition factor V_0 of V satisfies $\dim_K(V_0) \leqslant |G/N|$.

Now let λ with $0 \leqslant \lambda \leqslant |G| - 1$ be given. By the above, there exists a submodule S_2 of V which satisfies $0 \leqslant \lambda - \dim_K(S_2) < |G/N|$. Also, there exists $S_1 \leqslant U$ with $\dim_K(S_1) = \lambda - \dim_K(S_2)$ and $C_{S_1}(G) = 1$. Since obviously $C_{S_2}(G) = 1$, the module $S = S_1 \oplus S_2$ has the required properties.

The argument that produced the submodule S_2 having dimension "close to" λ in the proof of Lemma 1 will be used several times in the sequel and will be referred to as the "*sandwich technique*".

Before we state the next result, we introduce some notation that we propose to use throughout the remainder of this paper without further notice. For a group G and a group X we shall denote by $B(X)$ the base group of $X \operatorname{wr} G$, endowed with the obvious action of G. It will always be clear from the context which group G is meant. Moreover, if Y is a normal subgroup of X, then $B(X/Y)$ will be considered to be a quotient of $B(X)$, both viewed as G-groups in a natural way. We say that $B(X)$ *contains G-invariant subgroups of every possible order* if for each divisor d of $|B(X)|$ there exists some G-invariant subgroup of $B(X)$ which has order d.

LEMMA 2. *Let G be any soluble group and let p be a prime not dividing the order of G. Let A be an elementary abelian p-group of rank $r(G, p)$. Then $B(A)$ contains G-invariant subgroups of every possible order.*

PROOF. Let $r = r(G, p)$ and let K be the field with p^r elements. We need to show that $B(A)$ contains G-invariant subgroups of order p^m for any m such that $0 \leqslant m \leqslant r \cdot |G|$. Let $m = \lambda \cdot r + \rho$ for some nonnegative integers λ and ρ satisfying $0 \leqslant \rho < r$. Without loss of generality we may assume that $m < r \cdot |G|$, and so $\lambda < |G|$.

Now A is isomorphic to the additive group of K, and so $B(A)$ is isomorphic to the additive group of the group ring KG, viewed as additive groups acted upon by G. By Lemma 1 there exists a KG-submodule S_1 of KG with $\dim_K(S_1) = \lambda$ such that the trivial KG-module T is not a summand of S_1. In particular, S_1 is a G-invariant subgroup of $B(A)$ of rank $\lambda \cdot r$. Moreover, T contains G-invariant subgroups of every possible order, so let S_2 be one of rank ρ. Hence the subgroup $S = \langle S_1, S_2 \rangle = S_1 \oplus S_2$ has order $p^{\lambda t+\rho} = p^m$ as desired.

We are now going to drop the assumption that the group A in Lemma 2 is elementary. The notation that we now introduce will be used in the next two results. For a group X we denote by $\mathcal{O}(X)$ the set of all orders of G-invariant subgroups of $B(X)$. Clearly $B(X)$ contains subgroups of every possible order if $\mathcal{O}(X)$ consists of all divisors of the order of $B(X)$. It is easy to see that $\mathcal{O}(X \times Y)$ contains the set $\mathcal{O}(X) \cdot \mathcal{O}(Y)$ of all products of an element in $\mathcal{O}(X)$ with one of $\mathcal{O}(Y)$.

LEMMA 3. *Let G be a finite group, let p be a prime not dividing the order of G and let E be an elementary abelian group of order p^r. Then $\mathcal{O}(\mathbf{Z}_{p^r}) \supseteq \mathcal{O}(E)$.*

PROOF. Let $x \in \mathcal{O}(E)$ and let U be a G-invariant subgroup of $B(E)$ of order x. As p does not divide the order of G, we have $B(\mathbf{Z}_p) \cong \oplus_{i=1}^{t} C_i$ with irreducible $\mathbb{F}_pG$-modules C_i. Now $B(E) \cong B(\mathbf{Z}_p) \times \cdots \times B(\mathbf{Z}_p)$ as an $\mathbb{F}_pG$-module and so we infer that $U \cong \oplus_{i=1}^{t}(\oplus_{j=1}^{t_i} C_i)$ for some t_i with $0 \leqslant t_i \leqslant r$.

We now produce a G-invariant subgroup V of $M = B(\mathbf{Z}_{p^r})$ of order x. Let $M/\Phi(M) = \oplus_{i=1}^{t} M_i/\Phi(M)$ where $M_i/\Phi(M) \cong C_i$ as an $\mathbb{F}_pG$-module. It is well known that $M = \oplus_{i=1}^{t} M_i$, that each M_i is homocyclic of exponent p^r and that M_i has exactly r composition factors that are all $\mathbb{F}_pG$-isomorphic with C_i. In fact, for $1 \leqslant j \leqslant r$ one has $\Omega_j(M_i)/\Omega_{j-1}(M_i) \cong C_i$. From this it follows easily that $V = \oplus_{i=1}^{t} \Omega_{t_i}(M_i)$ is G-invariant and $|V| = x$.

From the above observation the following is immediate.

LEMMA 4. *Let G be a finite group and let p be a prime not dividing the order of G. Let A be an abelian group of order p^r and let E be elementary abelian of order p^r. Then $\mathcal{O}(A) \supseteq \mathcal{O}(E)$.*

We need another preparatory remark.

LEMMA 5. *Let G be a finite group and let p be a prime not dividing the order of G. Let A and A^* be p-groups with $|A| \geqslant |A^*|$. If $B(A)$ contains subgroups of every possible order then $B(A \times A^*)$ contains subgroups of every possible order.*

PROOF. Let d be a divisor of $|B(A \times A^*)|$. If $d \leqslant |B(A)|$ then we can select a G-invariant subgroup of order d in $B(A)$. So let $d > |B(A)| \geqslant |B(A^*)|$. As A and A^* are p-groups, we can select a G-invariant subgroup S of $B(A)$ having order $d|B(A^*)|^{-1}$. So $S \times B(A^*)$ is G-invariant of order d.

We are now going to prove that for any given CLT group G and a p-group P the wreath product $P \operatorname{wr} G$ very often is a CLT group.

First, we introduce some additional notation. For a prime p and a CLT group G, let H denote some Hall p'-subgroup of G. Further, let $\mathscr{L}(G, p)$ denote the class of all p-groups $P \neq 1$ having the property that $B(P)$ contains H-invariant subgroups of every possible order. Clearly, the class $\mathscr{L}(G, p)$ does not depend on the choice of H.

THEOREM 1. *Let G be a* CLT *group and let P be a p-group for some prime p.*

(a) *If $|P/P'| \geqslant p^{r(G,p)}$, then $P \operatorname{wr} G$ is a* CLT *group.*

(b) *If $P = P_1 \geqslant P_2 \geqslant \cdots \geqslant P_{t+1} = 1$ is a normal series of P with $P_i/P_{i+1} \in \mathscr{L}(G, p)$ for $1 \leqslant i \leqslant t$, then $P \in \mathscr{L}(G, p)$. In particular, $P \operatorname{wr} G$ is a* CLT *group.*

PROOF. (a) It is sufficient to prove that $W = P \operatorname{wr} G$ contains subgroups of index d for every prime power d dividing the order of W. As G is a CLT group, we only need to consider the case when d is a power of p. In this case, it is sufficient to prove the stronger assertion that $B(P)$ contains H-invariant subgroups of every possible order, where H denotes some Hall p'-subgroup of G. As $B(P)$, considered as an H-group, is isomorphic to a direct sum of $[G:H]$ copies of the base group of $P \operatorname{wr} H$, it is sufficient to prove the theorem for p'-groups G.

We proceed by induction on the order of G. Let $A \neq 1$ be an abelian normal subgroup of G. Let $|P| = p^\alpha$ and $|P'| = p^\rho$. We need to show that for any ι with $0 \leqslant \iota \leqslant \alpha \cdot |G|$ the group $B(P)$ contains a G-invariant subgroup of order p^ι.

If $\rho \cdot |G| \leqslant \iota \leqslant \alpha \cdot |G|$, then $B(P/P')$ contains a G-invariant subgroup of order $p^{\iota - \rho \cdot |G|}$ by Lemma 2 and Lemma 4, and its natural preimage in $B(P)$ has order p^ι, as required.

So let $0 \leqslant \iota < \rho \cdot |G|$ and let $\iota = \lambda \cdot |G| + \delta$, where $0 \leqslant \delta < |G|$. Then we have $0 \leqslant \lambda < \rho$. Let $R \trianglelefteq P$, where $R \leqslant P'$ and $|R| = p^\lambda$. It clearly suffices to show that $B(P/R)$ contains a G-invariant subgroup of order p^δ, and so without loss of generality we may assume that $R = 1$ and that $\iota = \delta$.

Let $C = C_{B(P)}(A)$. Then there is a direct decomposition $C = D \times D^{g_2} \times \cdots \times D^{g_t} = D^G$, where D consists of those elements of $B(P)$ that are constant on A and trivial on $G \setminus A$, and where $\{g_1 = 1, g_2, \ldots, g_t\}$ constitutes a system of coset representatives of A in G. Moreover, $D \cong P$. Also, the action of G on C is similar to the action of G/A on the base group of D wr(G/A). As $r(G/A, p) \leqslant r(G, p)$ and $|D/D'| = |P/P'| \geqslant p^{r(G,p)}$, we infer by induction that C possesses G-invariant subgroups of every order p^δ for $0 \leqslant \delta = \alpha \cdot [G:A]$.

Now let $Z \trianglelefteq P$, where $Z \leqslant P'$ and $|Z| = p$. Let $B(Z) = M \oplus M^*$ as a G-module, where $M = C_{B(Z)}(A)$. So A acts nontrivially on every G-composition factor of M^*, and hence $C \cap M^* = 1$. By [5, p. 570] the dimension of every G-composition factor of M^* is $\leqslant [G:A] \cdot r(G, p)$. So the sandwich technique yields a G-invariant subgroup V of M^* with $|V| = p^\nu$, where $0 \leqslant \delta - \nu < [G:A] \cdot r(G, p) \leqslant [G:A] \cdot \alpha$, since $r(G, p) \leqslant \alpha$. By the above, there exist a G-invariant subgroup U of C with $|U| = p^{\delta-\nu}$, and so $\langle U, V \rangle = U \times V$ is G-invariant of order p^δ as required.

(b) This follows by induction on t. The technique is similar to that used in the proof of part (a).

COROLLARY. *Let G be a* CLT *group and let p be a prime.*

(a) *If Q is the direct product of $r(G, p)$ copies of P, then Q* wr *G is a* CLT *group.*

(b) *If $r(G, p) \leqslant 2$, then P* wr *G is a* CLT *group for all p-groups P with $|P| \geqslant p^{r(G,p)}$.*

(c) *If P is any p-group, then P* wr *$\mathbb{Z}_3$, is a* CLT *group.*

PROOF. (a) This follows immediately from Theorem 1.

(b) If $r = 1$ or $r = 2$, then for any p-group P the conditions $|P/P'| \geqslant p^r$ and $|P| \geqslant p^r$ are equivalent.

(c) If $p \neq 3$, then we have $B(\mathbb{Z}_p) = M_1 \oplus M_2$, where $\dim_{\mathbb{F}_p}(M_i) = i$, and so $\mathbb{Z}_p \in \mathscr{L}(G, p)$ for all primes p.

In particular, if $r(G, p) \leqslant 2$, then P wr G is a CLT group for almost all p-groups P. The case $r(G, p) \geqslant 3$ is more interesting. Before dealing with this, we need a preparatory result concerning a very special group.

LEMMA 6. *Let $G = \langle a, b \mid a^{43} = b^7 = 1, a^b = a^4 \rangle$ be the nonabelian group of order $43 \cdot 7$ and let $p = 23$. Then*:

(a) $r(G, p) = 3$.

(b) $B(\mathbb{Z}_p) = \oplus_{i=1}^{17} V_i$, *where all the V_i are irreducible, and where* $\dim V_1 = 1$, $\dim V_2 = \dim V_3 = 3$, *and* $\dim V_i = 21$ *for* $4 \leqslant i \leqslant 17$.

PROOF. (a) Let L be the field with p^3 elements. As $p^3 \equiv 1 \bmod 7$, all characters of G having $A = \langle a \rangle$ in their kernel can be realized over L. Any faithful and irreducible character of G over a splitting field of characteristic p is of the form $(\varepsilon_A)^G$, where ε_A is a linear character of A (see [5, p. 561]). So from $p^{21} \equiv 1$

mod 43 it follows that the field K with p^{21} elements is a splitting field for G (see [6, p. 31]). From the structure of G we infer that $\varepsilon_A^G(g) = 0$ for $g \in G \setminus A$, and for $g \in A$ we have $\varepsilon_A^G(g) = \zeta + \zeta^4 + \zeta^{16} + \cdots + \zeta^{4^6}$, where $\zeta \in K$ denotes some primitive 43^{rd} roots of unity.

We now prove that $\varepsilon_A^G(g) \in L$ for any $g \in G$. Let $\varphi \in \mathrm{Gal}(K \mid \mathbb{F}_p)$ be defined by $\varphi(x) = x^{p^3}$. Then

$$\begin{aligned}\varphi\left(\varepsilon_A^G(g)\right) &= \zeta^{41} + \zeta^{35} + \zeta^{11} + \zeta + \zeta^4 + \zeta^{16} + \zeta^{21} \\ &= \varepsilon_A^G(g) \quad \text{for all } g \in A,\end{aligned}$$

and so all values of ε_A^G belong to $\mathrm{Fix}(\varphi) = L$; moreover, L is the smallest splitting field for G by [6, p. 31].

(b) The degrees of the irreducible $\mathbb{F}_pG$-modules which have A in their kernels are easily seen to be 1, 3 and 3, so let V be a faithful and irreducible $\mathbb{F}_pG$-module. Let $\overline{V} = V \otimes_{\mathbb{F}_p} L = \oplus_{i=1}^t \overline{V}_i$, where the $\overline{V}_i$ are absolutely irreducible LG-modules. Then we have $\dim_L(\overline{V}_i) = 7$, and $\mathrm{Gal}(L \mid \mathbb{F}_p)$ acts on the $\overline{V}_i$. As $\mathrm{GL}(7, p)$ does not contain any element of order 43, we have $t = 3$ and $\dim_{\mathbb{F}_p}(V) = 21$.

The next result shows that Theorem 1 and part (b) of its corollary cannot be improved in general. In some special cases, however, there are some slightly stronger results.

THEOREM 2. (a) *Let G be a group of prime order, let p be a prime different from the order of G and let $P \neq 1$ be a p-group. Then*

$$P \in \mathscr{L}(G, p) \Leftrightarrow |P/P'| \geqslant p^{r(G,p)-1}.$$

In particular, if $r(G, p) = 3$, then $P \in \mathscr{L}(G, p)$ if and only if $|P| \geqslant p^2$.

(b) *Let $G = \langle a, b \mid a^{43} = b^7 = 1, a^b = a^4 \rangle$ and let $p = 23$. Then for any p-group P with $P/P' \cong \mathbf{Z}_p \times \mathbf{Z}_p$, the wreath product P wr G is not a* CLT *group. In particular, if $|P| = p^{r(G,p)}$, then P wr G need not be a* CLT *group.*

(c) *For any $r \geqslant 3$ there exist a* CLT *group G and a prime p with $r(G, p) = r$ such that P wr G is not a* CLT *group for infinitely many p-groups P.*

PROOF. (a) Let $r = r(G, p)$ and let $W = P$ wr G. First, assume that $|P/P'| < p^{r-1}$. We claim that W does not contain any subgroup of index p^{r-1}. Indeed, otherwise there would exist a G-invariant subgroup U of $B = B(P)$ with $[B : U] = p^{r+1}$. Let $U = U_0 \lhd U_1 \lhd \cdots \lhd U_t = B$ be a G-composition series between U and B. As all irreducible $\mathbb{F}_pG$-modules have dimension 1 or r, we infer that the U_{i+1}/U_i are all centralized by G. Let $N = \mathrm{Core}_W(U)$. By the Jordan-Hölder Theorem, all G-composition factors between N and B are trivial, and so, by a standard argument on coprime automorphisms, we have $[B, G] \leqslant N$. From [9,

p. 348] we infer that $[B, G] = \{(a_1, \dots, a_{|G|}) \in B \mid a_1 \cdots a_{|G|} \in P'\}$, and so $B' \leqslant [B, G]$. Hence we have $B' \leqslant N \leqslant U$, and so B/B' contains a G-invariant subgroup of index p^{r-1}. But $B/B' \cong B(P/P')$ contains only $r-2$ trivial composition factors, and all others have order p^r. This is a contradiction.

The converse follows from the argument used in the proof of Theorem 1(a). The last assertion is trivial.

(b) We claim that $B(P)$ does not contain any G-invariant subgroup U of index p^{15}. As above, let $N = \mathrm{Core}_W(U)$. By Lemma 6, all composition factors of B/N have dimension 1 or 3, and so $A = \langle a \rangle$ centralises B/N. By the argument used in [9, pp. 348ff] we get $B(P') \leqslant [B(P), A] \leqslant N \leqslant U$, and so $B(P/P')$ would contain a G-invariant subgroup having index p^{15}, which contradicts the structure of $B(\mathbb{Z}_p \times \mathbb{Z}_p) \cong B(\mathbb{Z}_p) \times B(\mathbb{Z}_p)$ as given in Lemma 6.

(c) Let q be a prime with $q \equiv 1 \bmod r$. Let c be such that $c \bmod q$ has order r and let p be a prime with $p \equiv c \bmod q$. Then $r(\mathbb{Z}_q, p) = r$. By part (a), we have $P \notin \mathscr{L}(\mathbb{Z}_q, p)$ for every p-group P with $|P/P'| \leqslant p^{r-2}$; moreover, for each $r \geqslant 4$ there are infinitely many such P. For $r = 3$, infinitely many examples may be found in part b).

3. The universal class

In this section we consider the class $\mathscr{R}$ of all groups G having the property that, for every nilpotent group N, the wreath product $N \operatorname{wr} G$ is a CLT group. Clearly, every group in $\mathscr{R}$ is a CLT group, and if $G \in \mathscr{R}$, then $W = P \operatorname{wr} G$ is a CLT group for every p-group P. The converse of this is true, because for every prime p one has $B(N) = B(O_p(N)) \times B(O_{p'}(N))$, and because, by assumption, $W/B(O_{p'}(N)) \cong O_p(N) \operatorname{wr} G$ is a CLT group and so has subgroups of all possible indices of the form p^a. Moreover, the class $\mathscr{R}$ is nontrivial, as, for example, the wreath product of any nilpotent group with an elementary abelian 2-group is supersoluble, and by [1] it is a CLT group. Also, $\mathbb{Z}_3 \in \mathscr{R}$ by the corollary to Theorem 1.

We first deal with nilpotent $\mathscr{R}$-groups.

THEOREM 3. (a) *If $G \in \mathscr{R}$ is a q-group for some prime q, then $q \leqslant 3$.*

(b) *Every 2-group belongs to $\mathscr{R}$.*

(c) *Every noncyclic 3-group belongs to $\mathscr{R}$.*

(d) *$\mathbb{Z}_9 \notin \mathscr{R}$; in particular, $\mathscr{R}$ is not closed with respect to taking subgroups or quotients.*

PROOF. (a) Assume that $q \geqslant 5$. Then there exists some positive integer a such that $a \bmod q$ has order $\varphi(q) \geqslant 4$. By Dirichlet's Theorem there exists a prime $p \equiv a \bmod q$. Let $k = \mathbb{F}_p$. We now consider the degrees of the irreducible kG-modules V. If $G' \leqslant \ker(V)$, then $\dim_k(V) = 1$ for exactly one such V, and all others are of dimension $\varphi(q) \geqslant 4$. If $G' \nleqslant \ker(V)$, then $\dim_k(V) \geqslant q$. Hence $B(\mathbb{Z}_p)$ contains exactly one G-submodule of dimension one, and all other irreducible submodules have dimension at least four. This implies that $\mathbb{Z}_p \operatorname{wr} G$ does not contain any subgroup of order $p^2 \cdot |G|$ or $p^3 \cdot |G|$, and so $G \notin \mathscr{R}$.

(b) We need to show that $\mathbb{Z}_q \operatorname{wr} G$ is a CLT group for any prime q. Without loss of generality, we may assume that $q \neq 2$. Let Z be a normal subgroup of G with $|Z| = 2$. By induction on the order of G, the regular $\mathbb{F}_q(G/Z)$-module M contains submodules of every possible dimension. Now $B(\mathbb{Z}_q)$ is completely reducible, and so $B(\mathbb{Z}_q) = M \oplus M^*$, where the $\mathbb{F}_qG$-module M^* has dimension $|G|/2$. The assertion follows because M has submodules of every possible dimension, and the dimensions of M and M^* are equal.

(c) We first claim that the degrees of the irreducible representations of a noncyclic group G of order 3^α over any prime field $k = \mathbb{F}_p$ are bounded by $2 \cdot 3^{\alpha-2}$. We may assume that $p \neq 3$. The assertion is true for $\alpha = 2$, since the field with p^2 elements is a splitting field for G in this case, and so the degrees in question are at most two. So let $\alpha \geqslant 3$. Let U be a noncyclic subgroup of G with $|U| = 3^{\alpha-1}$, and let $kU = \oplus_{i=1}^t U_i$, where the U_i are irreducible kU-modules. By induction, we have $\dim_k(U_i) \leqslant 2 \cdot 3^{\alpha-3}$.

We now consider induced modules (see [5, pp. 552ff]). We have

$$kG = 1^G = (1^U)^G = (kU)^G = \bigoplus_{i=1}^{t} (U_i^G).$$

Every irreducible kG-module V is a constituent of U_i^G for some i, and so we get

$$\dim_k(V) \leqslant \dim_k(U_i^G) = 3 \cdot \dim_k(U_i) \leqslant 2 \cdot 3^{\alpha-2},$$

as required.

We now prove that $G \in \mathscr{R}$. As in the proof of part b), we may assume that $p \neq 3$, and we get $B(\mathbb{Z}_p) = M \oplus M^*$, where $\dim_k(M) = 3^{\alpha-1}$, and where M contains submodules of every possible dimension. By the above, all composition factors of M^* have dimension less than or equal to the dimension of M, and so the sandwich argument proves the result.

(d) For $G = \mathbb{Z}_9$ we have $B(\mathbb{Z}_2) = M_1 \oplus M_2 \oplus M_6$, where $\dim_{\mathbb{F}_2}(M_i) = i$, and where the M_i are irreducible. Hence $\mathbb{Z}_2 \operatorname{wr} G$ does not contain any subgroup of order $2^4 9$ or $2^5 9$, and so $G \notin \mathscr{R}$.

COROLLARY. *Let G be a* CLT *group and assume that $\pi(G) = \{2, 3\}$. If the Sylow* 3*-subgroup of G belongs to $\mathscr{R}$, and if $r(G, p) = 1$ for all $p \geqslant 5$, then $G \in \mathscr{R}$. In particular, the symmetric groups S_3 and S_4 are $\mathscr{R}$-groups.*

https://doi.org/10.1017/S1446788700028196 Published online by Cambridge University Press

PROOF. This follows immediately from Theorem 1 and Theorem 3.

If G is a rational group, i.e. if all irreducible complex characters are rational valued, then every prime field $\mathbb{F}_p$ is a splitting field for G (see [3]), so that, in particular, we have $r(G, p) = 1$ for all $p \nmid |G|$. Hence, if G is a CLT group, then $P \operatorname{wr} G$ is a CLT group for every p-group P. Even stronger, we would have $G \in \mathscr{R}$ if the following were true.

CONJECTURE. *Let G be soluble and let k be a splitting field for G. Then kG possesses submodules of every possible dimension.*

Before we deal with some types of nonnilpotent $\mathscr{R}$-groups, we need some more prelimininaries.

DEFINITION. Let $\mathscr{R}_0$ be the class of all finite groups G having the property that, for all primes p, the regular $\mathbb{F}_pG$-module has submodules of every possible dimension.

Obviously, every CLT group in $\mathscr{R}_0$ belongs to $\mathscr{R}$.
The following is well known.

LEMMA 7. *Let k be a field of characteristic $p > 0$ and let G be p-nilpotent. Then $kG = \oplus P_i$, where, for any i, all the composition factors of P_i are G-isomorphic.*

PROOF. This follows immediately from [8, p. 545], since the Cartan matrix of kG is a diagonal matrix.

The following result is also presumably known, but we have been unable to find any reference.

LEMMA 8. *Let p and q be primes and let r be a divisor of $q - 1$. Let $Q = AB$ be the Frobenius group of order qr and let N be an irreducible $\mathbb{F}_pQ$-module. Then* $\dim_{\mathbb{F}_p}(N) \leqslant q - 1$.

PROOF. If A centralises N, the result is clear, and so, henceforth, we assume that N is faithful, whence $p \neq q$.

We first claim that $C_N(B) \neq 1$. First, assume that p does not divide the order of Q. Let K be a splitting field for Q. We need to show that $N \otimes K$ contains the trivial KB-module as a summand. Let $\overline{N}$ be an irreducible KQ-submodule of $N \otimes K$. Then $\overline{N}$ is faithful, and so its character χ is induced from some linear

https://doi.org/10.1017/S1446788700028196 Published online by Cambridge University Press

character ε_A of A (see [5, p. 561]). Hence $\chi = 0$ on $Q \setminus A$. So we have $(\chi|_B, 1_B) = \chi(1)/|B| = \varepsilon_A(1) > 0$, and so $C_{\overline{N}}(B) \neq 1$. This implies that $C_N(B) \neq 1$.

Now let p be a divisor of the order of B and let $B = B_1 \times B_2$, where $B_1 = O_p(B)$. Let $Q_2 = AB_2$ and let $N = \oplus N_i$, where all the N_i are irreducible $\mathbb{F}_p Q_2$-modules. By the above, we have $C_{N_i}(B_2) \neq 1$ for all i, and so $C_2 := C_N(B_2) \neq 1$. As B_2 is normal in B, the group B_1 acts on C_2, and so we have $C_{C_2}(B_1) \neq 1$. Hence $1 \neq C_{C_2}(B_1) \leqslant C_N(B_1 B_2) = C_N(B)$, and the claim is proved.

To prove the lemma, let $c \in C_N(B)$ satisfy $o(c) = p$. We consider $N_0 = \langle c^{a^i} \mid 1 \leqslant i \leqslant q \rangle$, where a denotes some generator of A. Obviously, $N_0 = N_0^A$. For $b \in B$ we have $c^{a^i b} = c^{b a^j}$ for some j depending on i, and so $N_0 = N_0^B$. It follows that $N = N_0$. By construction of N_0, we have $\dim_{\mathbb{F}_p}(N_0) = q$. If N were of dimension q, then $c, c^a, \ldots, c^{a^{q-1}}$ would be linearly independent, and so $\langle c + c^a + \cdots + c^{a^{q-1}} \rangle$ would be a nontrivial Q-invariant subspace of N, which is a contradiction.

Finally, we collect some information about nonnilpotent $\mathscr{R}$-groups.

THEOREM 4. (a) *Let q be a prime and let $r \neq 1$ be a divisor of $q - 1$. If the Frobenius group $G = AB$ of order qr belongs to $\mathscr{R}$, then $r = q - 1$.*

(b) *Let q be a prime. If $\mathbb{Z}_{q-1} \in \mathscr{R}_0$, then the Frobenius group $G = AB$ of order $q(q-1)$ belongs to $\mathscr{R}$.*

(c) *Every group of odd order can be embedded as a subgroup of an $\mathscr{R}$-group.*

PROOF. (a) Let z be a generator of the multiplicative group of $\mathbb{Z}_q$ and let p be a prime with $p \equiv z \bmod q$. Let us choose p so that p does not divide the order of G. Let $k = \mathbb{F}_p$. Then $B(\mathbb{Z}_p) = M \oplus M^*$ is a kG-module, where all composition factors of M have A in their kernel, and where all composition factors of M^* are faithful. It follows that $\dim_k(M) = r$.

Let V be a composition factor of M^* and let d be its dimension. Then $\mathrm{GL}(d, p)$ contains an element of order q, and so $p^{d_0} \equiv 1 \bmod q$ for some $d_0 \leqslant d$. Hence we get $d \geqslant q - 1$, and the sandwich argument shows that $r = \dim_k(M) \geqslant q - 2$. As r divides $q - 1$, we arrive at $r = q - 1$, as claimed.

(b) Let p be a prime and let $k = \mathbb{F}_p$. If $p = q$, then $\mathbb{Z}_p \operatorname{wr} G$ is supersoluble and [1] implies that it is CLT. So let $p \neq q$. We have $B(\mathbb{Z}_p) = M \oplus M^*$, where $M \cong kB$, and where all composition factors of M^* are faithful. Indeed, this is obvious if p does not divide the order of G, and it follows readily from Lemma 7 if p divides $q - 1$. By assumption, M contains submodules of every possible dimension, and by Lemma 8 all composition factors of M have dimension $\leqslant q - 1$. The result follows.

https://doi.org/10.1017/S1446788700028196 Published online by Cambridge University Press

(c) Let B be a group of odd order. We wish to embed B as a subgroup of a $\mathscr{R}$-group. Without loss of generality, we may assume that B is a CLT group. Let A be an elementary abelian 2-group with $|A| \geqslant |B|$. We claim that $G = A \times B$ is a $\mathscr{R}$-group. Let p be a prime and let k be the field with p elements.

First, assume that p is odd. Let C be a Hall p'-subgroup of B. Then $H = A \times C$ is a Hall p'-subgroup of G. As A is an elementary abelian 2-group, we have $kA = \bigoplus_{i=1}^{|A|} V_i$, where all the V_i are one-dimensional. So $kH = kA \otimes kC = \bigoplus_{i=1}^{|A|}(V_i \otimes kC)$. Now if S is a kC-submodule of kC, then $V_i \otimes S$ is an H-submodule of kH which has the same dimension. As p does not divide the order of C, the trivial kC-submodule is a direct summand of kC, and so the assertion follows from the sandwich technique.

Now let $p = 2$ and let $kB = \bigoplus_{j=1}^{h} X_j$, where the X_j are irreducible kB-modules. Let X_1 be the trivial kB-module. We then have $kG = \bigoplus_{j=1}^{h}(kA \otimes X_j)$; moreover, all G-composition factors of $kA \otimes X_j$ are isomorphic to $1_A \otimes X_j$ and so have the same dimension $\dim_k(X_j) \leqslant |B|$. As $|A| \geqslant |B|$, the module $kA \otimes X_1$ contains submodules of any dimension d with $0 \leqslant d \leqslant |B|$, and so the sandwich technique proves the result.

References

[1] H. G. Bray, 'A note on CLT groups', *Pacific J. Math.* **27** (1968), 229–231.

[2] T. Gagen, 'A note on groups with the inverse Lagrange property' (*Group Theory*, Canberra, 1975. Lecture Notes in Mathematics, Vol. 573, Springer-Verlag, Berlin and New York, 1977, pp. 51–52).

[3] R. Gow, 'Groups whose characters are rational-valued', *J. Algebra* **40** (1976), 280–299.

[4] J. F. Humphreys and D. L. Johnson, 'On Lagrangian groups', *Trans. Amer. Math. Soc.* **180** (1973), 291–300.

[5] B. Huppert, *Endliche Gruppen* I (Springer-Verlag, Berlin and New York, 1967).

[6] B. Huppert and N. Blackburn, *Finite Groups* II (Springer-Verlag, Berlin and New York, 1982).

[7] D. Kindermann, 'Kranzprodukte und ein Satz von Lagrange', Zulassungsarbeit Würzburg (1979).

[8] G. O. Michler, 'Blocks and centers of group algebras' (Lecture Notes in Mathematics, Vol. 246, Springer-Verlag, Berlin and New York, 1972, pp. 430–552).

[9] P. M. Neumann, 'On the structure of standard wreath products of groups', *Math. Z.* **84** (1964), 343–373.

Mathematisches Institut
Am Hubland 12
D-8700 Würzburg
Federal Republic of Germany