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EXTENSION OF SET FUNCTIONS TO MEASURES AND APPLICATIONS TO INVERSE LIMIT MEASURES

BY
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Introduction. In measure theory and probability it is often useful to be able to extend a set function g to a measure μ. One situation in which such an extension arises is that of obtaining limit measures for inverse (or projective) systems of measure spaces ([1], [5]).

Since such extensions do not exist in general, conditions must be placed on g in order to guarantee the existence of a measure which is an extension of g.

A condition frequently assumed for this purpose is that g can be approximated from below on every countable family of sets in its domain by a family of sets $\mathscr{C}$ with properties very similar to those of compact sets. Specifically, $\mathscr{C}$ is assumed to be such that for $C_1, C_2, \ldots \in \mathscr{C}$, $\bigcap_{k=1}^{\infty} C_k = \varnothing$ iff for some n, $\bigcap_{k=1}^{n} C_k = \varnothing$ (Marczewski [3]).

In order to obtain inverse limit measures one then places on the inverse system restrictions which will guarantee that the generating set function satisfies the above conditions.

In this paper we show that approximation of g by families of sets satisfying a weaker condition (descending property) than Marczewski's is sufficient to guarantee extension to a measure.

We then apply the result to inverse systems of measures and show that inverse limit measures exist for systems which satisfy conditions other than those required by previous workers.

1. Extension of Set Functions to Measures.

1.1. Definitions. A family of subsets $\mathscr{D}$ of a space X has the *descending property* iff for every sequence of non-empty sets $D_1, D_2, \ldots, \in \mathscr{D}$ with $D_1 \supset D_2 \supset D_3 \cdots$ we have $\bigcap_{k=1}^{\infty} D_k \neq \varnothing$.

(Clearly, families of sets with Marczewski's property have the descending property. However, the converse does not hold (see example 1.3).)

Let g be a real valued non-negative set function on a family of sets $\mathscr{H}$. Then $\mathscr{C} \subset \mathscr{H}$ is an *inner family* for g on $\mathscr{H}$ iff for every $H \in \mathscr{H}$,

$$g(H) = \sup\{g(C) : C \in \mathscr{C}, C \subset H\}.$$

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A family of sets $\mathscr{A}$ is a *semi-ring* iff for every $A, B \in \mathscr{A}$, $A \cap B \in \mathscr{A}$ and $A \sim B = \bigcup_{k=1}^{n} A_k$ for some $A_1, \ldots, A_n \in \mathscr{A}$.

1.2. THEOREM. Let g be an additive, real valued non-negative set function on a semi-ring $\mathscr{H}$ of subsets of a space X, and let $\mathscr{D} \subset \mathscr{H}$ have the descending property and be an inner family for g on $\mathscr{H}$. Then there exists a measure μ on the σ-field $S(\mathscr{H})$ generated by $\mathscr{H}$ such that $\mu(H) = g(H)$ for every $H \in \mathscr{H}$.

Proof. Let $H \subset \bigcup_{n=1}^{\infty} H_n$ where $H, H, \ldots, \in \mathscr{H}$. Then it follows from Theorem 1.2 Sec. I of [2] that it suffices to show that $g(H) \leq \sum_{n=1}^{\infty} g(H_i)$, i.e., that g is countably subadditive.

Let $t < g(H)$ and choose $D \in \mathscr{D}$, $D \subset H$ with $g(D) > t$. Then let $D \sim H_1 = \bigcup_{j_1=1}^{m} H^{j_1}$ where $\{H^{j_1} : j_1 = 1, 2, \ldots, m\}$ is a disjoint family in $\mathscr{H}$. Now choose $D^1, D^2, \ldots, D_m \in \mathscr{D}$ such that $D^j \subset H^j$ and

$$\sum_{j=1}^{m} g(D^j) + g(H_1) > t.$$

Similarly, by recursion on n choose disjoint families

$$\mathscr{H}^{j_1, \ldots, j_n} = \{H^{j_1, \ldots, j_n, j_{n+1}} : j_{n+1} = 1, 2, \ldots, m_{j_1, \ldots, j_n}\} \subset \mathscr{H}$$

such that

$$D^{j_1, \ldots, j_n} \sim H_{n+1} = \bigcup_{j_{n+1}=1}^{m_{j_1, \ldots, j_n}} H^{j_1, \ldots, j_n, j_{n+1}}$$

and

$$\mathscr{D}^{j_1, \ldots, j_n} = \{D^{j_1, j_2, \ldots, j_{n+1}} : j_{n+1} = 1, 2, \ldots, m_{j_1, \ldots, j_n}\} \subset \mathscr{D}$$

such that $D^{j_1, \ldots, j_{n+1}} \subset H^{j_1, \ldots, j_{n+1}}$ and

$$\sum_{(j_1, \ldots, j_{n+1}) \in T} g(D^{j_1, \ldots, j_{n+1}}) + \sum_{k=1}^{n} g(H_k) > t$$

where T is the set of all n-tuples used as indices.

If for every n, $\sum_{i=1}^{n} g(H_i) < t$, then there exists a sequence $k_1, k_2, \ldots$ such that $D^{k_1, \ldots, k_n} \neq \varnothing$ for every n (because the families at each stage are finite). Letting $D^m = D^{k_1, \ldots, k_m}$ we see that $D_1, D_2, \ldots$ is a descending sequence of non-empty sets in $\mathscr{D}$, so that $\bigcap_{n=1}^{\infty} D_n \neq \varnothing$ and thus $D \not\subset \bigcup_{n=1}^{\infty} H_n$, contradicting the choice of D.

Thus $\sum_{n=1}^{\infty} g(H_n) > t$, hence $\sum_{n=1}^{\infty} g(H_n) \geq g(H)$.

The following example shows that if a family with the descending property is extended so as to be closed under finite unions or intersections the resulting family may not have the descending property.

1.3. EXAMPLE. Let $Z = \mathscr{R} \times (0, 1)$ and for $t \in \mathscr{R}$, $0 < \varepsilon < 1$ let

$$\begin{aligned} D(t, \varepsilon) = {} & \{(x, y) : |x - t| < \varepsilon\} \\ & \cup \{(x, y) : x > t, y > \varepsilon(x - t - 1) + 1\} \\ & \cup \{(x, y) : x < t, y > \varepsilon(t - x - 1) + 1\} \end{aligned}$$

($D(t, \varepsilon)$ has a "cocktail glass" shape.)

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Then

$$\mathscr{D} = \{D(t, \varepsilon): t \in R, 0 < \varepsilon < 1\}$$

has the descending property since

$$D(t_1, \varepsilon_1) \subset D(t_2, \varepsilon_2) \quad \text{iff} \quad t_1 = t_2, \varepsilon_1 < \varepsilon_2.$$

(note that

$$\sup\{x-t: \text{for some } y \in (0, 1), (x, y) \in D(t, \varepsilon)\} = 1)$$

so that for a descending sequence $D_1, D_2, \ldots, \in \mathscr{D}$, for some t the "line" $x=t$ is contained in every D_n.

However, for $t_1 \neq t_2$ and $|t_1-t_2|<1$,

$$F_n = D\left(t_1, \frac{1}{n+1}\right) \cap D\left(t_2, \frac{1}{n+1}\right) \neq \varnothing$$

and $F_1, F_2, \ldots$ is a descending sequence with $\bigcap_{n=1}^{\infty} F_n = \varnothing$.

Furthermore, if we let

$$G_n = D\left(-\frac{1}{n+1}, \frac{1}{n+1}\right) \cup D\left(\frac{1}{n+1}, \frac{1}{n+1}\right).$$

Then $G_{n+1} \subset G_n$ but $\bigcap_{n=1}^{\infty} G_n = \varnothing$.

1.4. Remark. Let for each $i \in I (X_i, \mathscr{M}_i, \mu_i)$ be a measure space, and if for each $i \in I$, $\mathscr{D}_i$ is a family of subsets of X_i, let

$$\text{Rect}(\mathscr{D}) = \left\{D: D = \prod_{i \in I} D_i, D_i = X_i\right.$$

for all but a finite number of $i \in I$, $D_j \in \mathscr{D}_i$ otherwise$\}$.

Then if g is a non-negative additive set function on Rect($\mathscr{M}$) for which $g(\pi_i^{-1}[M]) = \mu_i(M)$ (π_i representing projection onto the i-th coordinate) whenever $i \in I$, $M \in \mathscr{M}_i$, the extension of g to a measure is a (indirect) product measure (cf. [4]).

Clearly, if for each $i \in I$, $\mathscr{C}_i$ is an inner family for μ_i on $\mathscr{M}_i$ and $\mathscr{C}_i$ has the descending property then Rect($\mathscr{C}$) has the descending property and is an inner family for Rect($\mathscr{M}$) whenever the measures μ_i are σ-finite. Hence by Theorem 1.2 g could be extended to a measure, i.e., an indirect product measure would exist.

2. Inverse Limit Measures.

2.1. Definitions. An *inverse system of measure spaces* will be denoted by (X, f, μ, I) where:

(i) I is an index set directed by $<$.

(ii) For each $i \in I$ X_i is a space.

(iii) For $i<j \in I$, $f_{ij}: X_j \to X_i$ and if $i<j<k \in k \in I$, $f_{ij} \cdot t_{jk} = f_{ik}$.

(iv) For each $i \in I$ μ_i is a measure on a σ-field $\mathscr{M}_i$ of subsets of X_i.

(v) For $i<j \in I$ and $M \in \mathscr{M}_i$, $f_{ij}^{-1}[M] \in \mathscr{M}_j$ and $\mu_j(f_{ij}^{-1}[M]) = \mu_i(M)$.

If (X, f, I) satisfies (i), (ii) and (iii) above we will say (X, f, I) is an *inverse system of spaces*. The *inverse limit set* of such a system is

$$L = \left\{x \in \prod_{i \in I} X_i : f_{ij}(x_j) = x_i \quad \text{for all } i, j \in I, i < j\right\}.$$

For (X, f, μ, I) an inverse system of measure spaces, ν is an *inverse limit measure* iff ν is a measure on a σ-field $\mathscr{M}_\nu$ of subsets of L and for all $i \in I$, $M \in \mathscr{M}_i$, $\pi_i^{-1}[M] \cap L \in \mathscr{M}_\nu$ and $\nu(\pi^{-1}[M] \cap L) = \mu_i(M)$ (where again π_i represents projection onto the ith coordinate).

To obtain our result on existence of inverse limit measures we first establish a set theoretic property on a special class of inverse systems.

A family of sets $\mathscr{C}$ will be called *compact* iff for every subfamily $\{C_\lambda : \lambda \in \Lambda\} \subset \mathscr{C}$, $\bigcap_{\lambda \in \Lambda} C_\lambda \in \mathscr{C}$ and we have $\bigcap_{\lambda \in \Lambda} C_\lambda = \varnothing$ iff for some $\lambda_1, \ldots, \lambda_n \in A$, $\bigcap_{j=1}^n C_{\lambda j} = \varnothing$.

2.2. LEMMA. *Let (X, f, I) be an inverse system of spaces such that f_{ij} is onto whenever $i<j$, $i, j \in I$ and $I = \{i_0, i_1, i_2, \ldots\}$ where $i_n < i_{n+1}$ for every non-negative integer n. Also for every $i \in I$, let $\mathscr{C}_i$ be a compact family of sets such that for $i<j$, $C \in \mathscr{C}_j$ we have $f_{ij}[C] \in \mathscr{C}_i$.*

Then

$$\overline{\mathscr{C}} = \{\pi_i^{-1}[C] \cap L : i \in I, C \in \mathscr{C}_i\}$$

has the descending property.

Proof. Let $C_1, C_2, \ldots$ be a descending sequence of non-empty elements of $\overline{\mathscr{C}}$. Let, for each n, $C_n = \pi_{i_{m(n)}}^{-1}[K_n] \cap L$ where $K_n \in C_{i_{m(n)}}$ and $m(n)$ is the smallest integer for which this can be done.

Then if $k<l$ and $m(k)<m(l)$

$$f_{i_{m(k)} i_{m(l)}}[K_l] \subset K_k,$$

and if $k<l$ and $m(l)<m(k)$

$$f_{i_{m(l)} i_{m(k)}}^{-1}[K_l] \subset K_k.$$

So that if we let for every integer $p \geq 0$

$$N_p = \{n : m(n) \geq p\}$$

and

$$T_p = \bigcap_{n \in N_p} f_{i_p i_{m(n)}}[K_n]$$

we have

$$\varnothing \neq f_{i_p i_{p+1}}[T_{p+1}] \subset T_p$$

whenever N_{p+1} is infinite, since for any $n \in N_p$ there exists $n_1 > n$ with $m(n_1) > p$.

If there exists p with N_p finite, let

$$r = \text{l.u.b.}\{m : N_m \neq \varnothing\}.$$

Then

$$T = \bigcap_{n=0}^{\infty} f_{i_{m(n)} i_r}^{-1}[T_{m(n)}] \neq \varnothing$$

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(Note that only finitely many $m(n)$ exist.). Choose $x_r \in T$ and let $x_{m(n)} = f_{i_r i_{m(n)}}(x_r) \in T_{m(n)}$ for all n. Thus $f_{i_{m(n_1)} i_{m(n_2)}}(x_{m(n_2)}) = x_{m(n_1)}$ if $m(n_1) < m(n_2)$. Then

$$\varnothing \neq \bigcap_{N_{m(n)} \neq \varnothing} \pi_{i_{m(n)}}^{-1}[\{x_{m(n)}\}] \cap L \subset \bigcap_{n=1}^{\infty} C_n.$$

If no such p exists, define by recursion for every $i \in I$ and ordinal α, $T_{i,0} = T_i$ and

$$T_{i,\alpha} = \bigcap_{i<j} f_{ij}[T_{j,\alpha-1}]$$

unless α is a limit ordinal, in which case let $T_{i,\alpha} = \bigcap_{\beta<\alpha} T_{i,\beta}$. Then $T_{i,\alpha} \neq \varnothing$ for every i, α. Now let for every $x \in \prod_{i\in I} T_i$

$$0(x) = \inf\left\{\alpha : x \notin \prod_{i\in I} T_{i,\alpha}\right\},$$

provided there exists α with $x \notin \prod_{i\in I} T_{i,\alpha}$, and let

$$\gamma = \sup\left\{0(x) : x \in \prod_{i\in I} T_i\right\}.$$

Then $\prod_{i\in I} T_{i,\gamma+1} = \prod_{i\in I} T_{i,\gamma}$, and since $T_{i,\alpha+1} \subset T_{i,\alpha}$ for every i, α we have $T_{i,\gamma+1} = T_{i,\gamma}$ for every $i \in I$. But for $i<j$

$$f_{ij}[T_{j,\gamma+1}] = f_{ij}[T_{j,\gamma}] \subset T_{i,\gamma} = T_{i,\gamma+1}$$

and

$$f_{ij}[T_{j,\gamma+1}] = f_{ij}[T_{j,\gamma}] \supset T_{i,\gamma+1}$$

so that $f_{ij}[T_{j,\gamma+1}] = T_{i,\gamma+1}$.

Let $x_0 \in T_{i_0,\gamma+1}$, and choose by recursion on n $x_{n+1} \in T_{i_{n+1}}, \gamma+1$ such that $f_{i_n i_{n+1}}(x_{n+1}) = x_n$. Then

$$\varnothing \neq \{x\} = \bigcap_{n=0}^{\infty} \pi_i^{-1}[\{x_n\}] \subset L \quad \text{and} \quad \{x\} \subset \bigcap_{n=1}^{\infty} C_n \neq \varnothing.$$

Clearly, a similar argument would hold provided the families $\mathscr{C}_i$ contained the intersection of subfamilies of sufficiently high cardinality.

The following example shows that the family $\bar{\mathscr{C}}$ need not have the Marczewski property.

2.3. Example. Let $I = \{0, 1\}$, $X_0 = \{0, 1\}$, $X_1 = \{1/n : n \text{ an integer, } n>0\} \cup \{0\}$ and $\mathscr{C}_1$, $\mathscr{C}_2$ be the families of sets which are compact in the topology induced by the usual topology on the real line. Also let

$$\begin{aligned} f_{01}(x) &= 0 \quad \text{if} \quad x = 0 \\ &= 1 \quad \text{if} \quad x \neq 0 \end{aligned}$$

Then the system satisfies the hypotheses of Lemma 2.2 but if $K_n = [0, 1/n] \cap X_1$ for $n \in N$, and $K_0 = \{1\}$ then for every n

$$\pi_0^{-1}[K_0] \cap \bigcap_{m=1}^{n} \pi_1^{-1}[K_m] \cap L \neq \varnothing$$

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but

$$\pi_0^{-1}[K_0] \cap \bigcap_{m=1}^{\infty} \pi_1^{-1}[K_m] \cap L = \varnothing.$$

We now turn to the construction of an inverse limit measure.

For (X, f, μ, I) an inverse system of measure spaces let

$$\mathscr{B} = \{\pi_i^{-1}[A] \cap L : i \in I, A \in \mathscr{M}_i\}$$

and for $\alpha \in \mathscr{B}$ let $g(\alpha) = \mu_i(A)$ where $A \in \mathscr{M}_i$ and $\alpha = \pi_i^{-1}[A] \cap L$.

We shall assume that (X, f, μ, I) satisfies *sequential maximality*, i.e., if $i_0 < i_1 < i_2 \cdots$ is a sequence in I and if $\{y_n : y_n \in i_n\}$ is such that $f_{i_m i_n}(y_n) = y_m$, $m < n$ then there exists $y \in L$ with $\pi_{i_n}(y) = y_n$ for every n.

It is easily seen (cf. [1], [5]) that the above definitions and conditions imply that $\mathscr{B}$ is a ring and that g is finitely additive on $\mathscr{B}$, hence an inverse limit measure exists iff g is countably subadditive on $\mathscr{B}$.

2.4. THEOREM. *Let (X, f, μ, I) be an inverse system of measure spaces which satisfies sequential maximality such that for each $i \in I$ there exists a compact family $\mathscr{C}_i \subset \mathscr{M}_i$ which is an inner family for μ_i on $\mathscr{M}_i$, and for $i < j \in I$, $f_{ij}[C] \in C_i$ whenever $C \in C_j$. Then (X, f, μ, I) has an inverse limit measure.*

Proof. In view of the above we have only to show that g is countably subadditive on $\mathscr{B}$.

Let $B_0, B_1, B_2 \cdots \in \mathscr{B}$ and $B_0 \subset \bigcup_{n=1}^{\infty} B_n$. For each m let $B_m = \pi_{i_m}^{-1}[A_m] \cap L$ where $A_m \subset \mathscr{M}_{i_m}$ and $i_0 < i_1 < i_2 \ldots$. Let $\{i_0, i_1, \ldots\} = I'$, and consider the subsystem (X, f, μ, I') with inverse limit set L'. Define g', $\mathscr{B}'$, π_i' for this system as g, $\mathscr{B}$, π_i are defined for (X, f, μ, I). Then if we define the mapping $s: L \to L'$ by $s(x) = y$ where $x_{i_n} = y_{i_n}$ for all integers $n \geq 0$ then

$$s[\pi_{i_n}^{-1}[A_n] \cap L] = \pi_{i_n}'^{-1}[A_n] \cap L'$$

and

$$s^{-1}[\pi_{i_n}'^{-1}[A_n] \cap L'] = \pi_{i_n}^{-1}[A_n] \cap L$$

(because of sequential maximality). We also have

$$g(\pi_{i_n}^{-1}[A_n] \cap L) = g'(\pi_i'^{-1}[A_n] \cap L').$$

But

$$\overline{\mathscr{C}} = \{\pi_{i_n}'^{-1}[C] \cap L' : C \in \mathscr{C}_i\}$$

is an inner family for g' on $\mathscr{B}'$, and has the descending property by Lemma 2.2. Hence by Theorem 1.2

$$g'(\pi_{i_0}'^{-1}[A_0] \cap L') \leq \sum_{n=1}^{\infty} g'([\pi_{i_n}'^{-1}[A_n] \cap L')$$

so that

$$g(B_0) \leq \sum^{\infty} g(B_n)$$

and an inverse limit measure exists.

Note that previously obtained existence theorems along similar lines require either special properties of inverse images of points ([1], [5]) or stronger conditions on the measures of the system ([2]).

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