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GROUP RINGS WITH UNITS OF BOUNDED EXPONENT OVER THE CENTER

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Let KG be the group ring of a group G over a field K, and U its group of units. Given a group H, we shall denote by $\xi(H)$ the center of H and by $T(H)$ the set of all its torsion elements.

The following question appears in [**5**, p. 231]: When is $U^n \subset \xi(U)$, for some n? It was considered by G. Cliff and S. K. Sehgal in [**1**], where G is assumed to be a solvable group. A complete answer at characteristic zero is given there. Also they obtain partial results at characteristic $p \neq 0$, with certain restrictions on the exponent n.

In this note, we shall answer the question at characteristic p assuming that G is either a solvable or an FC-group. In fact, we shall need specially the following property which is common to both these families of groups: if H is a finitely generated subgroup of G such that $H/\xi(H)$ is torsion, then both $T(H)$ and H', the derived group of H, are finite groups [**4**, Lemma 1.5, p. 116 and **1**, Lemma 2.1, p. 147].

In Section 1, we answer the question for torsion groups assuming only that G is locally finite (Theorem A), and in Section 3 we give the answer for non torsion groups that are either solvable or FC (Theorem C).

First, we introduce some notation. We will denote $T(G)$ simply by T, and the integer $p \neq 0$ will always denote the characteristic of K. For an element t in a group, we shall say that t is a *p-element* if $o(t)$, the order of t, is a power of p, and that t is a p'-element if $o(t)$ is not divisible by p. Similarly, a group H will be called a p'-group if every element of H is a p'-element.

1. The torsion subgroup of G.

LEMMA 1.1. *If $U^n \subset \xi(U)$ for some n and G has a non central p'-element, then K is finite and the orders of the p'-elements of G are bounded.*

Proof. We shall show first that the orders of the p'-elements of G are bounded.

It is enough to show that if u is a central p'-element of G, then $o(u) \leqq n$. If not, take such a u, with $o(u) > n$. Then $K\langle u\rangle = \bigoplus_i K_i$, a direct sum of fields. For every i, denote by $\pi_i : K\langle u\rangle \to K_i$ the natural projection (that is, if e_i is the unity element of K_i, then $\pi_i(u) = ue_i$). Clearly, at

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least one of the $\pi_i(u)$, say $\pi_1(u)$, has multiplicative order equal to $o(u)$. As a consequence, K_1 has more than n elements.

Now, if t is a non central p'-element of G, we consider $K\langle u, t\rangle = \bigoplus_i K_i[t]$, where $K_i[t]$ denotes the smallest subalgebra of KG that contains K_i and t.

We claim that $K_1[t|$ is not contained in the center of KG. In fact, suppose that $K_1[t]$ is central and let $e = e_1$ be the unity element of K_1. Then et is central. Now take $x \in G$ such that $xtx^{-1} \neq t$. Then, $xetx^{-1} = extx^{-1} = et$.

By considering supports in the last equality we get:

$$u^i xtx^{-1} = u^j t, 0 \leqq i, j < o(u), i \neq j.$$

Hence $xtx^{-1} = u^{j-i}t$.

It follows that: $eu^{j-i}t = et$, or $eu^{j-i} = e$. Then, $(eu)^{j-i} = e$. But $eu = \pi_1(u)$, and hence the multiplicative order of $\pi_1(u)$ divides $|j - i| < o(u)$, a contradiction.

Now, since t is a p'element, we have: $K_1[t] = \bigoplus_i Li$, a direct sum of fields which are Galois extensions of K_1. But $K_1[t]$ is not central, hence one of the L_i, say L_1, is not contained in the center of KG. Let $\bar{L}_1$ be the subfield of L_1 consisting of its central elements, and let $\phi \neq 1$ be an $\bar{L}_1$-automorphism of L_1.

Since $L_1 = K(\zeta)$, with $\zeta^{o(t)} = 1$, we have that $\phi(\zeta) = \zeta^i$, for some i.

Now, take an arbitrary element $k \in \bar{L}_1$. Since $U^n \subset \xi(U)$, we get that $(\zeta + k)^n \in \bar{L}_1$. Then,

$$(\phi(\zeta + k))^n = \phi((\zeta + k)^n) = (\zeta + k)^n,$$

and $\phi(\zeta + k)$ is a root of $X^n - (\zeta + k)^n$; from this we see that

$$\phi(\zeta + k) = \alpha(\zeta + k), \alpha^n = 1, \alpha \neq 1.$$

On the other hand, $\phi(\zeta + k) = \zeta^i + k$, and thus

$$\zeta^i + k = \alpha(\zeta + k).$$

Solving this equation for k, we have that

$$k = \frac{\zeta^i - \alpha\zeta}{\alpha - 1}.$$

Here, only α depends on k. Since α can take at most $n - 1$ values, we see that $|L_1|$, the number of elements of L_1, is at most $n - 1$. But $\bar{L}_1 \supset K_1$, and $|K_1| > n$, a contradiction.

It still remains to prove that K is finite. If not, replace K_1 by K in the proof above. Again, we have that $K_1[t] = K[t]$ is not central, and we can repeat the argument to obtain a contradiction.

LEMMA 1.2. *Assume that $U^n \subset \xi(U)$ for some n. Then there exists a positive integer m, which is a power of p, such that x^m is central in KG, for every nilpotent element x in KG.*

Proof. Let $x \in KG$ be a nilpotent element and let r be such that $x^{p^r} = 0$. Then, $1 + x$ is a p-element of U and by hypothesis $(1 + x)^n \in \xi(U)$. Writing $n = p^a n'$, with $(n', p) = 1$, it is easy to see that x^{p^a} is central in KG.

LEMMA 1.3. *Assume that $U^n \subset \xi(U)$ for some n, and let $n = p^a \cdot n'$, with $(n', p) = 1$. If G has a p-element of order greater than $2p^{3a}$, then $G^{p^a} \subset \xi(G)$.*

Proof. From the proof of Lemma 1.2, we see that x^{p^a} is central for every nilpotent element $x \in KG$. So, set $m = p^a$, and take a p-element $h \in G$ such that $o(h) > 2m^3$. Since $h - 1$ is nilpotent, we have that $(h - 1)^m = h^m - 1$ is central, hence h^m is central.

Set $h' = h^m$, take $x, y \in G$ and consider the nilpotent element $y(h' - 1)$. Again, by Lemma 1.2, we have that

$$(y(h' - 1))^m = y^m(h'^m - 1)$$

is central. Hence:

$$xy^m(h'^m - 1) = y^m(h'^m - 1)x,$$
$$xy^m h'^m - xy^m = y^m h'^m x - y^m x.$$

Since $o(h') > 2m^2$, we know that $h'^m - 1 \neq 0$ and hence we have two elements of G in the support of the above element. If $p \neq 2$, we see immediately that $xy^m = y^m x$, thus $y^m \in \xi(G)$. If $p = 2$, we may have:

$$xy^m = y^m h'^m x,$$
$$xy^m h'^m = y^m x.$$

Using the fact that h'^m is central and replacing $y^m x$ in the first equation by its value in the second one, we get that

$$xy^m = xy^m h'^m h'^m,$$

or $(h')^{2m} = 1$, which contradicts the fact that $o(h') > 2m^2$.

LEMMA 1.4. *Let m be a power of p. If $G^m \subset \xi(G)$ and G contains a normal p-abelian subgroup ϕ such that G/ϕ is a finite p-group, then G is nilpotent.*

Proof. This follows as in [**5**, 6.6, pp. 157–158].

LEMMA 1.5. *Let S be a commutative ring with identity, I a nil ideal of S, of bounded exponent, and Q a finite group. Let $S(Q, \rho, \sigma)$ be a crossed product of Q over S, with an arbitrary factor system ρ and σ such that $\sigma_t(I) \subset I$, for every $t \in Q$. Then, IQ is a nil ideal of $S(Q, \rho, \sigma)$, of bounded exponent.*

Proof. It is immediate to verify that IQ is an ideal of $S(Q, \rho, \sigma)$.

Let $Q = \{t_1, t_2, \ldots, t_n\}$ and choose m such that $s^m = 0$, for every $s \in I$.

Take $r > m(n+1)^2$ and $x = s_1\bar{t}_1 + \ldots + s_n\bar{t}_n$ an arbitrary element of IQ, $s_i \in I$, $1 \leqq i \leqq n$.

We want to prove that $x^r = 0$. It is enough to show that any product of r elements from the set $\{s_1\bar{t}_1, \ldots, s_n\bar{t}_n\}$ is zero. Then let

$$y = s_{i_1}\bar{t}_{i_1} \ldots s_{i_r}\bar{t}_{i_r}$$

be such a product.

It is easy to see that there exists an index j such that $s_{i_j}\bar{t}_{i_j}$ occurs k times in y with $k > m(n+1)$, and we may suppose without loss of generality that $s_{i_j}\bar{t}_{i_j} = s_1\bar{t}_1$.

As the products $s_i\bar{t}_i s_j\bar{t}_j$ still have the form $s\bar{t}$, $s \in I$, $t \in Q$, and $\bar{t}s_1 = \sigma_t(s_1)\bar{t}$, for every $t \in Q$, we can write y in the form

$$y = \left(\prod_{i=1}^{k} z_i\right)\gamma,$$

with

$$z_i \in \{\sigma_t(s_1)|t \in Q\} \cup \{s_1\}, \gamma \in IQ.$$

Since the above set has at most $n + 1$ elements and $k > m(n+1)$, there must exist an index j such that z_j occurs in y more than m times. Now, $z_j \in I$, therefore $z_j^m = 0$, and hence $y = 0$.

LEMMA 1.6. *Let $G = T$, a locally finite group. If $U^n \subset \xi(U)$ for some n, then KT satisfies a polynomial identity.*

Proof. Let m be as in Lemma 1.2. We shall show that KT satisfies a polynomial identity in $2m + 1$ variables. Consider $2m$ arbitrary elements of KT, say $x_1, x_2, \ldots, x_{2m}$. By considering the subgroup generated by the supports of these elements, we may suppose that T is finite.

Denote by $J(KT)$ the Jacobson radical of KT. Then

$$KT/J(KT) = \bigoplus_i M_{n_i}(D_i),$$

a direct sum of full matrix rings over division rings D_i.

Set x' for the image of an element $x \in KT$ under the natural epimorphism $KT \rightarrow KT/J(KT)$. For a given index i, take x_i an arbitrary nilpotent element in $M_{n_i}(D_i)$, and choose any element $y_i \in KT$ such that $(y_i)' = x_i$. Then y_i is nilpotent, since $J(KT)$ is nilpotent because T is finite. By Lemma 1.2, y_i^m is central in KT. Hence $x_i^m = (y_i^m)'$ is a central nilpotent element of $KT/J(K)$, so it must be zero.

Now it is easy to see that the size of the matrices is bounded by m, that is, $n_i \leqq m$, for every i.

On the other hand, given i and $d_i \neq 0$ in D_i, we can choose $u \in U$ such that $u' = d_i$ (see [**5**, Lemma 3.3, p. 179]). As $u^n \in \xi(U)$, d_i^n is central in D_i, and hence D_i is a field, by [**3**, Theorem 3.22, p. 79].

Therefore, $KT/J(KT)$ satisfies $S_{2m}(X_1, X_2, \ldots, X_{2m})$, the standard polynomial of degree $2m$ in the non commuting variables $X_1, X_2, \ldots, X_{2m}$. Again, since $J(KT)$ is nilpotent, we can use Lemma 1.2 to obtain, for every $z \in KT$:

$$(S_{2m}(x_1, \ldots, x_{2m}))^m z = z(S_{2m}(x_1, \ldots, x_{2m}))^m.$$

We may now obtain a characterization for U to be of bounded exponent over the center when G is a locally finite group.

THEOREM A. *Let $G = T$, a locally finite group. Then, $U^n \subset \xi(U)$ for some n if and only if the following conditions hold:*
(i) *$T^l \subset \xi(T)$ for some l.*
(ii) *T contains a normal p-abelian subgroup of finite index.*
(iii) *Either every p'-element of T is central or T is of bounded exponent and K is finite.*

Proof. Suppose $U^n \subset \xi(U)$ for some n. Then (i) is trivial and (ii) follows from Lemma 1.6 and a Theorem of Passman [**4**, Corollary 3.10, p. 197].

To prove (iii), assume that not every p'-element of T is central. By Lemma 1.1, K is finite, and for every p'-element $t \in T$, $t^r = 1$, for a suitable r. Now, if $n = p^a \cdot n'$, with $(n', p) = 1$, and T has a p-element of order greater than $2p^{3a}$, then $T^{p^a} \subset \xi(T)$ by Lemma 1.3, and hence every p'-element is central, a contradiction. So, for every p-element $t \in T$, $t^s = 1$, for a suitable s.

Now take $x \in T$ and let T_0 be a normal p-abelian subgroup of index u, as in (ii). Then, $x^u \in T_0$, and we may write: $x^u = yz$, where $y, z \in T_0$, y is a p-element and z is a p'-element. Since T_0/T_0' is abelian, taking $(rs)^{\text{th}}$-powers, we get:

$$x^{urs} \equiv y^{rs}z^{rs} (\text{mod } T_0') \equiv 1 (\text{mod } T_0').$$

But T_0' is finite, so we have that

$$x^{urs|T_0'|} = 1,$$

and (iii) is proved.

Assume now that conditions (i), (ii) and (iii) hold, and let T_0 be a normal p-abelian subgroup of finite index in T, as in (ii), and A the set: $\{t \in T | t \text{ is a } p'\text{-element}\}$.

Suppose first that A is a central subgroup of T. Then, as T is locally finite, it is easy to see that $T = P \times A$, where $P = \{t \in T | t \text{ is a } p\text{-element}\}$ is a subgroup of T.

Considering the subgroup $\phi = T_0 \cdot A$, it is easy to see that T satisfies the conditions of Lemma 1.4 and hence it is nilpotent. Furthermore, T contains a normal p-abelian subgroup ϕ such that T/ϕ is a finite p-group and we conclude from [**5**, Theorem 6.1, p. 155] that KT is Lie m-Engel for a suitable m. Hence, $U^n \subset \xi(U)$ for some n by [**5**, Lemma 4.3, p. 151].

Suppose now that A is a non central subset. By (iii), $T^s = 1$, for some s, and K is finite. Because T_0' is a finite p-group, it is easy to see that

$$P_0 = \{t \in T_0 | t \text{ is a } p\text{-element}\}$$

is a normal subgroup of T.

We claim that $\Delta(T, P_0)$, the kernel of the natural epimorphism $KT \to KT/P_0$, is nil of bounded exponent. Indeed, $\Delta(T_0/T_0', P_0/T_0')$ is nil of bounded exponent because T_0/T_0' is abelian and P_0/T_0' is of bounded exponent. Setting

$$S = KT_0/T_0', Q = (T/T_0')/(T_0/T_0') \simeq T/T_0,$$

we see that KT/T_0' is the crossed product $S(Q, \rho, \sigma)$, with ρ and σ as usual. If $I = \Delta(T_0/T_0', P_0/T_0')$, by Lemma 1.5 we conclude that $IQ = \Delta(T/T_0', P_0/T_0')$ is nil of bounded exponent. Since T_0' is a finite p-group, we see that $\Delta(T_0, T_0')$ is nilpotent and hence it is easy to see that $\Delta(T, P_0)$ is nil of bounded exponent by considering the natural epimorphism $KT \to KT/T_0'$.

Now, T_0/P_0 is a normal subgroup of T/P_0, of finite index, say, r. By [**4**, Lemma 1.10, p. 176], we get that

$$KT/P_0 \subset M_r(KT_0/P_0).$$

Pick now $u \in U$. Considering the subgroup generated by the support of u, we may suppose that T is finite. Hence T_0/P_0 is a finite abelian p'-group, such that $(T_0/P_0)^s = 1$.

Therefore, $KT_0/P_0 = \bigoplus_i K_i$, a direct sum of fields, all of them contained in $K(\zeta)$, with $\zeta^s = 1$. Hence,

$$M_r(KT_0/P_0) = M_r(\bigoplus_i K_i) = \bigoplus_i M_r(K_i),$$

and we have that $KT/P_0 \subset \bigoplus_i M_r(F_i)$, with $F_i = K(\zeta)$, for every i. Set S for $\bigoplus_i M_r(F_i)$ and u' for the image of u by the composition map of the natural epimorphism $KT \to KT/P_0$ followed by the inclusion $KT/P_0 \to S$. As K is finite, the group of nonsingular matrices of $M^r(K(\zeta))$ is finite, say of order q, depending on r and s only. So, $u'^q = 1$ and we get:

$$u^q = 1 + \delta, \delta \in \Delta(T, P_0).$$

As $\Delta(T, P_0)$ is nil of bounded exponent, we can take m a power of p such that $x^m = 0$ for all $x \in \Delta(T, P_0)$. Now we can conclude that

$$u^{qm} = 1 + \delta^m = 1 \in \xi(U).$$

COROLLARY. *Let $G = T$, a locally finite group, and assume that the set of all p-elements of T is not of bounded exponent. Then the following conditions are equivalent*:

(i) $U^n \subset \xi(U)$ *for some* n.
(ii) KT *is Lie* m*-Engel for some* m.

Proof. First suppose that $U^n \subset \xi(U)$ for some n. By the preceeding theorem, we get that (i), (ii) and (iii) hold.

Furthermore, every p'-element of T is central by Lemma 1.3. Follow now the "only if" part of the proof of the theorem to conclude that KT is Lie m-Engel for some m.

By [**5**, Lemma 4.3, p. 151], the converse is obvious.

2. A certain nil ideal of KG. In this section, G will be either a solvable or an FC-group. As we mentioned in the introduction, if $G/\xi(G)$ is torsion, then we can conclude that T is a locally finite subgroup of G and G' is contained in T.

We shall denote by A the set of all p'-elements of G and by P the set of all p-elements of G.

LEMMA 2.1. *Suppose that* $U^n \subset \xi(U)$ *for some* n *and* G *has an element of infinite order. Then, every idempotent of* KG *is central.*

Proof. See [**1**, Lemma 2.4, p. 148].

LEMMA 2.2. *Suppose that* $U^n \subset \xi(U)$ *for some* n *and* G *has an element of infinite order. Then:*

(i) A *is an abelian subgroup of* G.
(ii) *If* A *is non central, then* K *is finite and for every* $x \in G$ *and every* $t \in A$ *there exists an integer* r *such that* $xtx^{-1} = t^{p^r}$, *and* $(K:\mathbf{F}_p)|r$.
(iii) P *is a subgroup of* G.
(iv) $T = P \times A$.

Proof. For the proof of (i) see [**1**, Corollary 2.5, p. 148].

To prove (ii) we notice that if A is non central, then K is finite by Lemma 1.1.

Now, take $x \in G$ and $t \in A$ such that $xtx^{-1} \neq t$. We have that $K\langle t\rangle = \bigoplus_i k_i$, a direct sum of fields such that at least one of them, say K_1, is of the form $K_i = K(\zeta)$, where ζ is a root of unity whose order is equal to the order of t, and the natural projection $K\langle t\rangle \to K_1$ maps t on ζ.

Since, by Lemma 2.1, every idempotent is central, we must have $xtx^{-1} = t^i$ for some i (this can be seen by considering the idempotent $e = (o(t))^{-1}\ (1 + t + \ldots + t^{o(t)-1})$). Hence, conjugation by x defines an automorphism ϕ of K_1. By the above, $\phi(\zeta) = \zeta^i$.

On the other hand, since K_1 is finite, ϕ is a power of the Frobenius automorphism of K_1, F, given by:

$$F(k) = k^p, \text{ for all } k \in K_1.$$

If $\phi = F^r$, we have that

$$\phi(\zeta) = \zeta^{p^r} = \zeta^i,$$

from which we conclude that $o(t) = o(\zeta)$ divides $p^r - i$. Then,

$$p^r \equiv i \pmod{o(t)} \text{ and } xtx^{-1} = t^i = t^{p^r}.$$

Furthermore, as every element of K is fixed by ϕ, we have that $k^{p^r} = k$ for every $k \in K$, and hence K is contained in a field with p^r elements, that is: $(K:\mathbf{F}_p)|r$.

For (iii) and (iv), we observe first that every p-element commutes with every p'-element. If not, then by (ii) K is finite. Now take $\pi \in P$ and $t \in A$ such that $\pi t \pi^{-1} \neq t$ and proceed as in [**1**, 3.2, p. 152] to conclude that this implies the existence of a non central idempotent, which contradicts Lemma 2.1.

As T is locally finite, the proof of (iii) and (iv) is now trivial.

LEMMA 2.3. *Let A_1 be an abelian p'-subgroup of G, and K a finite field. If, for every $t \in A_1$ and every $x \in G$, there exists an integer r such that $xtx^{-1} = t^{p^r}$, and $(K:\mathbf{F}_p)|r$, then every idempotent of KA_1 is central in KG.*

Proof. Let $e \in KA_1$ be an idempotent, and let $x \in G$. By considering the subgroup generated by the support of e, we may suppose that A_1 is finite.

Let $A_1 = \langle t_1 \rangle \times \ldots \times \langle t_s \rangle$, a direct product of cyclic groups. It is easy to see that we may choose an integer r such that $xt_ix^{-1} = t_i^{p^r}$, for every i.

We have that $e = f(t_1, \ldots, t_s)$, where $f(X_1, \ldots, X_s)$ is a polynomial in the commuting variables $X_1, \ldots, X_s$, with coefficients in K. Conjugating by x, we have:

$$xex^{-1} = f(t_1^{p^r}, \ldots, t_s^{p^r}).$$

But by hypothesis every element $k \in K$ satisfies $k^{p^r} = k$, therefore this is true for the coefficients of f. Hence

$$f(t_1^{p^r}, \ldots, t_s^{p^r}) = (f(t_1, \ldots, t_1))^{p^r}$$

and

$$xex^{-1} = e^{p^r} = e,$$

as we wished to prove.

We can now give a partial characterization for U to be of bounded exponent over the center, when G is non torsion.

THEOREM B. *Suppose that G has an element of infinite order. Then $U^n \subset \xi(U)$ for some n if and only if the following conditions hold:*

(i) *$G^l \subset \xi(G)$ for some l.*

(ii) *A is an abelian subgroup of G and, if A is non central, then K is finite, A is of bounded exponent and for every $t \in A$ and every $x \in G$ there exists an integer r such that $xtx^{-1} = t^{p^r}$, where $(K:\mathbf{F}_p)|r$.*

(iii) *P is a subgroup of G contained in the centralizer of A.*

(iv) *There exists an integer m, which is a power of p, such that x^m is central in KG, for every $x \in \Delta(G, P)$.*

Proof. Suppose first that $U^n \subset \xi(U)$ for some n. (i) is trivial, (ii) follows from Lemma 1.1 and Lemma 2.2, and (iii) follows from Lemma 2.2.

To prove (iv), let $x \in \Delta(G, P)$. We may suppose that G is finitely generated and hence P is a finite normal subgroup of G. Therefore, x is nilpotent and we can apply Lemma 1.2 to obtain the result.

Suppose now conditions (i) to (iv) hold, and pick $u \in U$. Again we may suppose that G is finitely generated and hence T is finite.

We observe that if $KA = \bigoplus_i K_i$, a direct sum of fields, then $K(A \times P) = \bigoplus_i K_iP$.

Consider now the natural epimorphism $KG \rightarrow KG/P$, with kernel $\Delta(G, P)$.

Setting S' for the image of a subset S of KG under this epimorphism, we have:

$$(K(A \times P))' = \oplus K_i'$$

where each K_i' is a field. Furthermore,

$$T(G/P) = (A \times P)/P,$$

and hence

$$KT(G/P) = \bigoplus_i K_i'.$$

Since, by condition (ii) and Lemma 2.3, every idempotent of KA is central in KG, and $G' \subset T$, we may apply [**5**, Lemma 3.22, p. 194], and u can be written in the form:

$$(*) \quad u = \sum_i f_i g_i + \delta, f_i \in K_i, g_i \in G, \delta \in \Delta(G, P).$$

Suppose first that A is central. Taking the $(l)^{\text{th}}$-power in (*), we have:

$$u^l = \sum_i f_i^l g_i^l + \delta', \delta' \in \Delta(G, P).$$

Now, $\sum_i f_i^l g_i^l$ is central, and it is sufficient to apply condition (iv).

Suppose now that K is finite and A is of bounded exponent s. Then, $K_i \subset K(\zeta)$, with $\zeta^s = 1$, for all i.

Computing $(l)^{\text{th}}$-powers in (*), we obtain:

$$u^l = \sum_i f_i' g_i^{\,l} + \delta', f_i' \in K_i, g_i^{\,l} \in \xi(G), \delta' \in \Delta(G, P).$$

Since $K_i \subset K(\zeta)$, for all i, we have that $f_i'^{\,r} = 1$ for every $f_i' \neq 0$ and a suitable r which depends on K and s only.

Taking $(r)^{\text{th}}$-powers above, we have:

$$u^{lr} = \sum_i g_i^{\,lr} + \delta'', \delta'' \in \Delta(G, P).$$

Now, $\sum_i g_i^{\,lr}$ is central and it suffices to use condition (iv).

COROLLARY. *Suppose that the set of all p-elements of G is not of bounded exponent. Then the following conditions are equivalent:*

(i) *$U^n \subset \xi(U)$ for some n.*

(ii) *KG is Lie l-Engel for some l.*

Proof. Suppose first $U^n \subset \xi(U)$ for some n. By the corollary to Theorem A, KT is Lie m-Engel for some m and hence T is nilpotent.

As $G' \subset T$, we can conclude that G is solvable (even if it were an FC-group).

By Lemma 1.3, $G^{p^a} \subset \xi(G)$ for some a and by [**1**, Lemma 2.2, p. 148] G' is a p-group. Hence, as we noted before, $\Delta(G, G')$ is a nil ideal.

Given $x, y \in KG$, $xy - yx \in \Delta(G, G')$. By (iv) of Theorem B, for every $z \in KG$ we have that

$$(xy - yx)^m z = z(xy - yx)^m.$$

Hence, KG satisfies a polynomial identity. By Passman's theorem [**2**, Theorem 1.1, p. 142], setting ϕ for the FC-subgroup of G, we have: $|G/\phi| < \infty$, $|\phi'| < \infty$.

Furthermore, it is easy to see that both are p-groups. By Lemma 1.4, G is nilpotent. We conclude from [**5**, Theorem 6.1, p. 155] that KG is Lie l-Engel for some l.

As we noted before, the converse is immediate by [**5**, Lemma 4.3, p. 151].

3. Nil augmentation ideals of bounded exponent. We shall now discuss condition (iv) of Theorem B.

In all this section, except in Theorem C, G will be either a solvable or an FC-group, such that T is a locally finite subgroup of G, and $G' \subset T$.

Furthermore, we shall assume that T has the form: $T = P \times A$, where

$$A = \{t \in G | t \text{ is a } p'\text{-element}\}$$

is an abelian subgroup of G and

$$P = \{t \in G | t \text{ is a } p\text{-element}\}$$

is a subgroup of G, of bounded exponent.

We introduce some notation. Given a group H, $\phi(H)$ will denote the FC-subgroup of H. The group $\phi(G)$ will be denoted simply by ϕ.

LEMMA 3.1. *Suppose that there exists an integer m, which is a power of p, such that x^m is central in KG for all $x \in \Delta(G, P)$. Then G contains normal subgroup H, of finite index, containing ϕ, such that $H' \cap P$ is finite.*

Proof. By [**2**, Theorem 1.1, p. 142], if KG satisfies a polynomial identity, then $|G/\phi| < \infty$, $|\phi'| < \infty$. Since $G' \subset P \times A$, and $(\phi \cdot A)/A \subset \phi(G/A)$, it is enough to prove that $K(G/A)$ satisfies a polynomial identity.

In the group ring $K(G/A)$, the ideal $\Delta(G/A, (P \times A)/A)$ is the image of $\Delta(G, P)$ by the natural epimorphism $KG \rightarrow K(G/A)$ and hence it also satisfies the hypothesis.

As $(G/A)' \subset (P \times A)/A$, we have that

$$\Delta(G/A, (G/A)') \subset \Delta(G/A, (P \times A)/A)$$

and it follows easily, as in the proof of the corollary to Theorem B, that $K(G/A)$ satisfies the polynomial $(XY - YX)^m Z - Z(XY - YX)^m$.

In the following lemmas, H will be a normal subgroup of G, of finite index, containing ϕ, as in the previous lemma, P_0 will be $H' \cap P$ (hence a finite group) and P_1 will be $H \cap P$.

LEMMA 3.2. *Suppose that there exists a positive integer m, which is a power of p, such that x^m is central in KG, for all x in $\Delta(G, P)$. Then, $\Delta(G, P)$ is nil of bounded exponent.*

Proof. We observe first that it is enough to prove that $\Delta(H, P_1)$ is nil of bounded exponent. In fact, suppose that this were proved. Given $x \in \Delta(G, P)$, then x^m is a central element. But every central element of $\Delta(G, P)$ is contained in $\Delta(G, P) \cap K\phi$, which is contained in $\Delta(H, P_1)$. Then $\Delta(G, P)$ is nil of bounded exponent.

Let us prove now that $\Delta(H, P_1)$ is nil of bounded exponent. As P_0 is finite, $\Delta(H, P_0)$ is nilpotent and hence it suffices to prove that $\Delta(H/P_0, P_1/P_0)$ is nil of bounded exponent.

We have that $(H/P_1)'$ is a p'-group, P_1/P_0 is a central subgroup of H/P_0, and for all $x \in \Delta(H/P_0, P_1/P_0)$, x^m is central in KH/P_0. Furthermore, all hypothesis on G carry on to H/P_0. Therefore, in order to prove that $\Delta(H/P_0, P_1/P_0)$ is nil of bounded exponent, we may replace H/P_0 by H and P_1/P_0 by P_1 and assume in addition that H' is a p'-group and P_1 is central.

Now, we shall see which is the form of a central element of KH that belongs to $\Delta(H, P_1)$.

Let S' be a transversal of $T(H)$ in H. Observing that $T(H) = P_1 \times A_1$, where

$$A_1 = \{t \in H | t \text{ is a } p'\text{-element}\},$$

it is easy to see that

$$S = \{ab | a \in A_1, b \in S'\}$$

is a transversal of P_1 in H.

Now, let $x \in \phi(H)$, $y \in H$, $x = hab$, $h \in P_1$, $ab \in S$. Since

$$y(ab)y^{-1}(ab)^{-1} \in H' \subset A_1$$

and P_1 is central, yxy^{-1} has the form:

$$yxy^{-1} = ha'b, \ a' \in A_1.$$

So denoting by $\gamma(x)$ the sum of the elements of the conjugacy class of x, we have that

$$\gamma(hab) = h(a_1 + \ldots + a_r)b, \ a_j \in A_1, 1 \leqq j \leqq r.$$

Therefore, we may write every central element $z \in KH$ in the form:

$$(1) \quad z = \sum_i \alpha_i b_i,$$

where $\alpha_i \in KT(H)$, $\alpha_i b_i$ is central in KH for all i, and the b_i are distinct elements of S'.

We claim that $\alpha_i b_i = b_i \alpha_i$, for all i. In fact, as $\alpha_i b_i$ is central, we have that

$$\alpha_i b_i = b_i(\alpha_i b_i)b_i^{-1} = b_i \alpha_i.$$

If in addition $z \in \Delta(H, P_1)$, then

$$\alpha_i \in \Delta(T(H), P_1), \text{ for all } i.$$

In fact, denote by x' the image of an element $x \in KH$ by the natural epimorphism $KH \rightarrow KH/P_1$. Since $z \in \Delta(H, P_1)$,

$$z' = \sum_i \alpha_i' b_i' = 0.$$

But the b_i' are distinct elements of a basis of the $K(T(H)/P_1)$-module KH/P_1. Hence $\alpha_i' = 0$, for all i, which means that

$$\alpha_i \in \Delta(H, P_1) \cap KT(H) = \Delta(T(H), P_1).$$

Therefore every central element z of $\Delta(H, P_1)$ has the form (1), and furthermore $\alpha_i \in \Delta(T(H), P_1)$, for all i.

Call s the exponent of P_1. Since $T(H)$ is abelian, computing the $(s)^{\text{th}}$-power of z in (1) we have that:

$$z^s = \sum_k (\alpha_i b_i)^s = \sum_i \alpha_i^s b_i^s = 0.$$

We have proved that $z^s = 0$, for a fixed s and every central element z of $\Delta(H, P_1)$. As every $x \in \Delta(H, P_1)$ is such that x^m is central, we can conclude that $\Delta(H, P_1)$ is nil of bounded exponent.

LEMMA 3.3. *If $\Delta(H, P_1)$ is nil of bounded exponent, then either P_1 is finite or H contains a characteristic p-abelian subgroup of finite index.*

Proof. Suppose that P_1 is infinite. By Passman [**4**, Corollary 3.10, p. 197], it is enough to prove that KH satisfies a polynomial identity. Since $\Delta(H, P_0)$ is nilpotent it suffices to prove that KH/P_0 satisfies a polynomial identity.

We observe that all the hypotheses of the lemma carry on to KH/P_0. In fact, $\Delta(H/P_0, P_1/P_0)$ is nil of bounded exponent and P_1/P_0 is infinite (because P_0 is finite). Furthermore, $(H/P_0)'$ is a p'-group and P_1/P_0 is central in H/P_0.

Therefore, in order to prove that KH/P_0 satisfies a polynomial identity we may replace H/P_0 by H and assume in addition that H' is a p'-group and P_1 is a central subgroup of H.

Now, pick $n > 0$ such that $x^n = 0$ for all $x \in \Delta(H, P_1)$. We want to show that there exist elements $x_1, x_2, \ldots, x_n$ in $\Delta(P_1) = \Delta(P_1, P_1)$ such that $x_i^2 = 0$, for all i, but $x_1x_2 \ldots x_n \neq 0$. In fact, as P_1 is abelian of bounded exponent, it is a direct product of cyclic groups [**2**, Theorem 11.2, p. 44]. P_1 is infinite and hence the number of cyclic groups is infinite. Choose $h_1, \ldots, h_n$ generators in different cyclic subgroups and set $m_i = o(h_i) - 1$. The elements $x_i = (h_i - 1)^{m_i}$ are easily verified to satisfy the required conditions.

Take now $S = \{g_i\}_{i \in I}$ a transversal of P_1 in H, and elements $g_1, \ldots, g_n \in S$. Then, the element

$$\alpha = g_1x_1 + \ldots + g_nx_n$$

belongs to $\Delta(H, P)$, hence $\alpha^n = 0$.

Since the x_i are central and $x_i^2 = 0$, computing α^n we get:

$$\alpha^n = F(g_1, \ldots, g_n)x_1x_2 \ldots x_n,$$

where $F(X_1, \ldots X_n)$ is a polynomial in the non-commuting variables $X_1, X_2, \ldots, X_n$, namely:

$$F(X_1, \ldots, X_n) = \sum X_{\sigma(1)} \ldots X_{\sigma(n)},$$

the sum running over all $\sigma \in S_n$, the symmetric group on n elements.

We want to show that $F(g_1, \ldots, g_n) = 0$. First we make some observations. If

$$g_{\sigma(1)}g_{\sigma(2)} \ldots g_{\sigma(n)} \equiv g_{\tau(1)}g_{\tau(2)} \ldots g_{\tau(n)} \bmod (P_1)$$

for $\sigma, \tau \in S_n$, then

$$g_{\sigma(1)}g_{\sigma(2)} \dots g_{\sigma(n)} = ag_{\tau(1)}g_{\tau(2)} \dots g_{\tau(n)},$$

for some $a \in P_1$.

On the other hand, since

$$g_i g_j = \alpha(i, j) g_j g_i,$$

where $\alpha(i, j) \in H'$, we can write the product $g_{\sigma(1)}g_{\sigma(2)} \dots g_{\sigma(n)}$ in the form

$$bg_{\tau(1)}g_{\tau(2)} \dots g_{\tau(n)} \text{ for some } b \in H'.$$

Therefore, we obtain: $a = b$, with $a \in P_1$ and $b \in H'$, which is a p'-group. Thus $a = b = 1$.

We have shown that

$$g_{\sigma(1)}g_{\sigma(2)} \dots g_{\sigma(n)} \equiv g_{\tau(1)}g_{\tau(2)} \dots g_{\tau(n)} \pmod{P_1}$$

if and only if

$$g_{\sigma(1)}g_{\sigma(2)} \dots g_{\sigma(n)} = g_{\tau(1)}g_{\tau(2)} \dots g_{\tau(n)}.$$

We now define an equivalence relation $\sim$ in S_n setting: for $\sigma, \tau \in S_n$, $\sigma \sim \tau$ if and only if

$$g_{\sigma(1)} \dots g_{\sigma(n)} = g_{\tau(1)} \dots g_{\tau(n)}.$$

We denote by $S_1, S_2, \dots, S_t$ the equivalence classes of this relation. Choosing, for each i, an element $\sigma_i \in S_i$, it follows from the above that we may write:

$$F(g_1, \dots, g_n) = g_{\sigma_1(1)} \dots g_{\sigma_1(n)}|S_1| + \dots + g_{\sigma_t(1)} \dots g_{\sigma_t(n)}|S_t|.$$

Now, since

$$g_{\sigma_i(1)} \dots g_{\sigma_i(n)} \not\equiv g_{\sigma_j(1)} \dots g_{\sigma_j(n)} \pmod{P_1}$$

if $i \neq j$, from $F(g_1, \dots, g_n)x_1x_2 \dots x_n = 0$, we get:

$$g_{\sigma_i(1)} \dots g_{\sigma_i(n)}|S_i|x_1x_2 \dots x_n = 0, \text{ for all } i.$$

But $g_{\sigma_i(1)} \dots g_{\sigma_i(n)}$ is invertible and $x_1x_2 \dots x_n \neq 0$; hence this can happen only if $|S_i| = 0$, for all i (that is, $p \mid |S_i|$, for all i).

Therefore, $F(g_1, \dots, g_n) = 0$, for arbitrary elements $g_1, \dots, g_n \in S$.

Now, KH is a left module over the central subalgebra KP having S as a basis, and $F(X_1, \dots, X_n)$ is a multilinear polynomial. By [**4**, Lemma 1.2, p. 171], F is a polynomial identity for KH.

COROLLARY. *If $\Delta(G, P)$ is nil of bounded exponent, then either P is finite or G contains a normal p-abelian subgroup of finite index.*

Proof. Suppose that P is infinite, and take H as in Lemma 3.1. Since $|G/H| < \infty$, we have that $|PH/H| < \infty$ and hence $|P/P_1| < \infty$. Therefore, P_1 must be infinite. By Lemma 3.3, H contains a characteristic p-abelian subgroup of finite index, and thus the result follows.

LEMMA 3.4. *If G contains a normal p-abelian subgroup of finite index, then $\Delta(G, P)$ is nil of bounded exponent.*

Proof. Let L be such a subgroup of G. We have that L/L' is abelian; hence $\Delta(L/L', P \cap L/L')$ is nil of bounded exponent.

Setting

$$S = KL/L', Q = (G/L')/(L/L') \cong G/L,$$

we see that $K(G/L')$ is the crossed product $S(Q, \rho, \sigma)$, with ρ and σ as usual.

If $I = \Delta(L/L', P \cap L/L')$, applying Lemma 1.5 we conclude that $IQ = \Delta(G/L', (P \cap L)/L')$ is nil of bounded exponent. Since L' is a finite p-group, $\Delta(G, L')$ is nilpotent and hence $\Delta(G, P \cap L)$ is nil of bounded exponent.

Now, let us consider the natural epimorphism

$$\Phi: KG \rightarrow K(G/(P \cap L).$$

We have that

$$\Phi(\Delta(G, P)) = \Delta(G/(P \cap L), P/(P \cap L)).$$

But $P/(P \cap L) \cong PL/L$, and PL/L is a finite p-group since it is contained in G/L. Therefore, there exists an integer n such that

$$\Delta(G, P)^n \subset \Delta(G, P \cap L).$$

Since we have shown that this ideal is nil of bounded exponent, the lemma is proved.

We can now give a complete answer to the initial question.

THEOREM C. *Suppose that G is non-torsion and either solvable or FC. Then, $U^n \subset \xi(U)$ for some n if and only if either KG is Lie m-Engel for some m or the following conditions hold:*

(i) *$G^l \subset \xi(G)$ for some l.*

(ii) *A is an abelian subgroup of G and, if A is non central, then K is finite, A is of bounded exponent and for every $x \in G$ and every $t \in A$ there exists an integer r such that $xtx^{-1} = t^{p^r}$, where $(K:\mathbf{F}_p)|r$.*

(iii) *P is a subgroup of G of bounded exponent, contained in the centralizer of A and, if P is not finite, then G contains a normal p-abelian subgroup of finite index.*

Proof. Suppose that $U^n \subset \xi(U)$ for some n and that KG is not Lie m-Engel.

By Theorem B, the conditions (i) and (ii) hold, and by (iii) of Theorem B, P is a subgroup of G contained in the centralizer of A. If P is not of bounded exponent, by the corollary to Theorem B, KG is Lie m-Engel, for some m, a contradiction. Hence, P is of bounded exponent.

Also, since condition (iv) of Theorem B holds, we have that $\Delta(G, P)$ is nil of bounded exponent by Lemma 3.2. By the corollary to Lemma 3.3, the remainder of condition (iii) holds.

Suppose now KG is Lie m-Engel for some m. Then, as we have noted before, $U^n \subset \xi(U)$ for a suitable n [see **5**, Lemma 4.3. p. 151].

Finally, suppose that conditions (i), (ii) and (iii) hold. If P is finite, then $\Delta(G, P)$ is nilpotent and condition (iv) of Theorem B holds. If G contains a normal p-abelian subgroup of finite index, then using Lemma 3.4, again condition (iv) of Theorem B holds. Since conditions (i), (ii) and (iii) of this theorem are verified, then there exists an n such that $U^n \subset \xi(U)$.

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