AUTOMETRIZED BOOLEAN ALGEBRAS II:

THE GROUP OF MOTIONS OF B

DAVID ELLIS

1. Introduction. The writer [1] has previously examined the fundamental concepts of distance geometry in a Boolean algebra, B, with distance defined[1] by $d(x, y) = xy' + x'y$. Any technical terms from distance geometry which are not defined in this paper may be found in [1]. A Boolean algebra bearing the given distance function is called an *autometrized Boolean algebra.* It is clear that the set of motions B (congruences of B with itself) form a group under substitution. This group we denote by $M(B)$.

M. H. Stone has shown [2] that the point set of any Boolean algebra, B, forms a ring, called the *associated Boolean ring* of B and denoted by $R(B)$, under the operations $a \oplus b = d(a, b)$, $a \otimes b = ab$, the composition of whose additive group, denoted by $G(B)$, is precisely the function $d(a, b)$.

In this paper $M(B)$ is examined more extensively than was done in [1]. It is shown that $M(B)$ and the group of automorphisms of B are subgroups of the group of complementation-preserving bi-uniform mappings of the point set of B onto itself having only the identity mapping in common. The main result is rather surprising although easily obtained, namely: the group of motions of B is isomorphic to the additive group of the associated Boolean ring of B.

2. Preliminary results.

THEOREM 1. *If f is any motion of B and $x \in B$, then $f(x') = f'(x)$.*

Proof. Since f is a motion of B, $d(x, x') = d(f(x), f(x'))$. However,[2] $d(x, x') = 1$. Hence

$$f(x)f'(x') + f'(x)f(x') = 1. \tag{1}$$

Taking meets in (1) with $f(x)$ and $f'(x)$, respectively, one obtains

$$f(x)f'(x') = f(x) \tag{2}$$

$$f'(x)f(x') = f'(x). \tag{3}$$

Complementation and use of DeMorgan's formulas in [3] yields

$$f(x) + f'(x') = f(x). \tag{4}$$

Received January 3, 1950; presented to the American Mathematical Society, February 24, 1950.

[1]We use the following notation for the Boolean operations: $a + b$, ab, a', and $a \subset b$ denote join, meet, complement, and inclusion (in the wide sense), respectively.

[2]We denote the first and last elements of B by 0 and 1, respectively. They are, of course, also the unit elements for addition and multiplication, respectively, in the associated Boolean ring of B.

Then from (2) and (4) it is seen that $f(x) = f'(x')$ so that $f'(x) = f(x')$.

LEMMA 1. *If f is a motion of B sending a,b into $f(a), f(b)$ and $x \in B$, then*

$$(5) \qquad f(x) = f(a)f(b)(abx + a'b'x') + f'(a)f'(b)(a'b'x + abx') + f'(a)f(b)(a'bx + ab'x') + f(a)f'(b)(ab'x + a'bx').$$

Proof. This result was proved in [1].

LEMMA 2. *If f is any motion of B and $x \in B$, then*

$$(6) \qquad f(x) = f'(0)x + f(0)x' = d(f(0), x).$$

Proof. One obtains (6) by setting $a=0$, $b=1$ in (5) and applying Theorem 1.

LEMMA 3. *If $d(a, c) = b$, then $d(a, b) = c$.*

Proof. This result was proved in [1].

COROLLARY. *If f is any motion of B and $x \in B$, then $d(x, f(x)) = d(0, f(0))$. Hence no motion of B other than the identity mapping leaves any point of B fixed and every motion of B is a translation in the sense that it moves each point the same distance.*

Proof. From Lemma 2, $d(f(0), x) = f(x)$ so that by Lemma 3, $f(0) = d(x, f(x))$. However, $d(0, f(0)) = f(0)$.

COROLLARY. *Although the group of motions, $M(B)$, of B and the group of automorphisms of B (B being treated as a Boolean algebra) are subgroups of the group of complementation-preserving bi-uniform mappings of the point set of B onto itself, they have only the identity mapping in common.*

Proof. Every automorphism of B leaves 0 fixed.

LEMMA 4. *If $x, y, z \in B$ then $d(x, d(y, z)) = d(z, d(x, y))$.*

Proof. This may be easily verified by direct expansion or by recalling that $d(x, y)$ is the composition of $G(B)$.

LEMMA 5. *If $a, b \in B$ there is a motion, f, of B with $b = f(a)$; that is, the group of motions of B is simply transitive.*

Proof. This result was proved in [1].

3. Isomorphism of $M(B)$ and $G(B)$.

THEOREM 2. *The groups $M(B)$ and $G(B)$ are isomorphic.*

Proof. Let $a \in B$. Then by Lemma 5, there is an $f \in M(B)$ with $f(0) = a$. This correspondence is one-to-one between $G(B)$ and $M(B)$ since:

(i). Every $a \in G(B)$ (we recall that the point sets of B and of $G(B)$ coincide) corresponds to at least one $f \in M(B)$, by Lemma 5.

(ii). No $a \in G(B)$ may correspond to more than one $f \in M(B)$, by Lemma 2 and the fact (proved in [1]) that any point forms a metric base for B.

(iii). The correspondence exhausts $M(B)$ since for $f \in M(B)$ there is an $a \in G(B)$ with $f(0) = a$.

It remains, then, only to show that if $a, b \in G(B)$ correspond to f and g, respectively, in $M(B)$, then the element of $M(B)$ corresponding to $a \oplus b$ is $f(g)$. To do this, it suffices to show that $f(g(0)) = a \oplus b$. Now

$$f(x) = d(f(0), x) = d(a, x),$$
$$g(x) = d(g(0), x) = d(b, x),$$

by Lemma 2. Hence, by Lemma 4,

$$f(g(x)) = d(f(0), d(g(0), x)) = d(a, d(b, x)) = d(x, d(a, b)).$$

Then

$$f(g(0)) = d(0, d(a, b)) = d(a, b).$$

But $d(a, b) = a \oplus b$, by definition and Theorem 2 is proved.

COROLLARY. *The group of motions of B is isomorphic to the additive group of a Boolean ring with unity and hence is an Abelian group all of whose non-zero elements have order two.*

REFERENCES

[1] David Ellis, *Autometrized Boolean algebras I*, Can. J. Math., vol. 3 (1951), 87-93.
[2] M. H. Stone, *Subsumption of Boolean algebras under the theory of rings*, Proc. Nat. Acad. Sci., vol. 20 (1934), 197-202.

University of Missouri