Canad. Math. Bull. Vol. **56** (3), 2013 pp. 459–465
http://dx.doi.org/10.4153/CMB-2011-188-x

On Certain Multivariable Subnormal Weighted Shifts and their Duals

Ameer Athavale and Pramod Patil

Abstract. For every subnormal m-variable weighted shift S (with bounded positive weights), there is a corresponding positive Reinhardt measure μ supported on a compact Reinhardt subset of $\mathbb{C}^m$. We show that, for $m \geq 2$, the dimensions of the 1-st cohomology vector spaces associated with the Koszul complexes of S and its dual $\widetilde{S}$ are different if a certain radial function happens to be integrable with respect to μ (which is indeed the case with many classical examples). In particular, S cannot in that case be similar to $\widetilde{S}$. We next prove that, for $m \geq 2$, a Fredholm subnormal m-variable weighted shift S cannot be similar to its dual.

1 Introduction

If $\mathcal{H}$ is a complex infinite-dimensional separable Hilbert space, then we use $\mathcal{B}(\mathcal{H})$ to denote the algebra of bounded linear operators on $\mathcal{H}$. An m-tuple $S = (S_1, \dots, S_m)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an m-tuple $N = (N_1, \dots, N_m)$ of commuting normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that $N_i\mathcal{H} \subset \mathcal{H}$ and $N_i/\mathcal{H} = S_i$ for $1 \leq i \leq m$. Every subnormal operator tuple has a "minimal" normal extension that is unique up to unitary equivalence (see [11]). If $N = (N_1, \dots, N_m)$ (with N_i in $\mathcal{B}(\mathcal{K})$) is the minimal normal extension of a subnormal tuple $S = (S_1, \dots, S_m)$ (with S_i in $\mathcal{B}(\mathcal{H})$), and $\mathcal{H}^\perp \equiv \mathcal{K} \ominus \mathcal{H}$ is the orthocomplement of $\mathcal{H}$ in $\mathcal{K}$, then one defines the dual $\widetilde{S}$ of S to be the subnormal tuple $\widetilde{S} = (\widetilde{S}_1, \dots, \widetilde{S}_m)$, where $\widetilde{S}_i = N_i^*/\mathcal{H}^\perp$.

Suppose $S = (S_1, \dots, S_m)$ is a tuple of commuting operators in $\mathcal{B}(\mathcal{H})$ and $T = (T_1, \dots, T_m)$ a tuple of commuting operators in $\mathcal{B}(\mathcal{J})$. If there exists a bounded linear operator $X\colon \mathcal{H} \to \mathcal{J}$ such that $XS_i = T_iX$ for each i, then X is said to be an *intertwining operator* (for S and T), and we denote this fact by $XS = TX$. The operator tuple S is said to be *similar* (resp. *unitarily equivalent*) *to* T if one can find an invertible (resp. a unitary) intertwining operator for S and T. A subnormal tuple S is said to be *self-dual* if it is unitarily equivalent to its dual.

If $\{e_n = e_{n_1,\dots,n_m}\}_{n\in\mathcal{N}^m}$ with $\mathcal{N} = \{0, 1, 2, \dots\}$ is an orthonormal basis for $\mathcal{H}$ and $\{w_n^{(i)} : n \in \mathcal{N}^m, 1 \leq i \leq m\}$ is a bounded subset of the complex plane $\mathbb{C}$, an m-variable *weighted shift* $T = (T_1, \dots, T_m)$ *acting on* $\mathcal{H}$ is defined through the relations $T_ie_n = w_n^{(i)}e_{n+\epsilon(i)}$, where $\epsilon(i)$ is the m-tuple with 1 in the i-th place and zeros elsewhere. We will always assume that the weights $w_n^{(i)}$ are positive. The various properties of multivariable weighted shifts, and in particular of subnormal weighted shifts, that are relevant here can be found in [6, 7, 12].

Received by the editors January 27, 2011; revised August 11, 2011.
Published electronically November 15, 2011.
AMS subject classification: **47B20**.
Keywords: subnormal, Reinhardt, Betti numbers.

In [4], Conway showed that a subnormal 1-variable weighted shift S of norm 1 is self-dual if and only if the weights of S are either $w_0 = 1, w_1 = 1, w_2 = 1, \ldots$, or $w_0 = \sqrt{1/2}, w_1 = 1, w_2 = 1, \ldots$. Using a result of Curto and Yan in [7], we show that, for $m \geq 2$, an m-variable subnormal weighted shift S (with positive weights) is not similar to its dual if a certain radial function is integrable with respect to the Reinhardt measure associated with S (which is the case with many classical examples). The main idea of the proof is to show that, under the conditions stated, the dimensions of the first cohomology vector spaces associated with the cochain Koszul complexes of the subnormal weighted shift S and its dual $\widetilde{S}$ are different (*cf.* [3, Proposition 2.2]). In the same spirit, we also show that a Fredholm subnormal m-variable weighted shift S cannot be similar to its dual for $m \geq 2$.

2 Preliminaries

Let $e_0 = 1 \in \mathbb{C}$ and let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{C}^m$. By the *exterior algebra* Γ *over* $\mathbb{C}^m$ we understand the vector space direct sum $\Gamma = \Gamma^0 \oplus \cdots \oplus \Gamma^m$, where $\Gamma^0 = \mathbb{C} = \text{lin}\{e_0\}$, where $\Gamma^p = \text{lin}\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 < \cdots < i_p \leq m\}$ $(1 \leq p \leq m)$, and where the multiplication $\wedge$ in Γ is bilinear, associative, and satisfies the relations $1 \wedge \gamma = \gamma \wedge 1 = \gamma$ $(\gamma \in \Gamma)$, $e_i \wedge e_j + e_j \wedge e_i = 0$ $(1 \leq i, j \leq m)$. One can think of Γ as an orthogonal direct sum of Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle_p$ on Γ^p $(p \geq 1)$ defined by $\langle e_{i_1} \wedge \cdots \wedge e_{i_p}, e_{j_1} \wedge \cdots \wedge e_{j_p} \rangle_p = \det(\langle e_{i_k}, e_{j_l} \rangle)$. For $0 \leq i \leq m$, the operators E_i on Γ are defined by $E_i \gamma = e_i \wedge \gamma$ $(\gamma \in \Gamma)$.

Let T be an m-tuple of commuting operators in $\mathcal{B}(\mathcal{J})$. Let $\Gamma(\mathcal{J})$ be the Hilbert space tensor product $\mathcal{J} \otimes \Gamma = \mathcal{J} \otimes \Gamma^0 \oplus \cdots \oplus \mathcal{J} \otimes \Gamma^m$ $(\equiv \Gamma^0(\mathcal{J}) \oplus \cdots \oplus \Gamma^m(\mathcal{J}))$, and let $\partial_T \colon \Gamma(\mathcal{J}) \to \Gamma(\mathcal{J})$ be defined by $\partial_T = \sum_{i=1}^m T_i \otimes E_i$. It is easy to check that $\partial_T^2 = 0$. The Koszul complex $K(T)$ is the cochain complex

$$K(T)\colon 0 \xrightarrow{\partial_{T,-1}} \Gamma^0(\mathcal{J}) \xrightarrow{\partial_{T,0}} \Gamma^1(\mathcal{J}) \xrightarrow{\partial_{T,1}} \cdots \xrightarrow{\partial_{T,m-1}} \Gamma^m(\mathcal{J}) \xrightarrow{\partial_{T,m}} 0,$$

where $\partial_{T,-1}$ and $\partial_{T,m}$ are zero maps and $\partial_{T,p}$ $(0 \leq p \leq m-1)$ are defined by $\partial_{T,p} = \partial_T / \Gamma^p(\mathcal{J})$. The coboundary map $\partial_{T,0}$ is given by $f \mapsto (T_1 f, T_2 f, \ldots, T_m f)$, $f \in \mathcal{J}$. The coboundary map $\partial_{T,1}$ is given by $(f_1, f_2) \mapsto T_1 f_2 - T_2 f_1$ $(f_1, f_2 \in \mathcal{J})$ in case $m = 2$, and by

$$\begin{aligned}(f_1, f_2, \ldots, f_m) \mapsto (T_1 f_2 - T_2 f_1, T_1 f_3 - T_3 f_1, \ldots, T_1 f_m - T_m f_1, T_2 f_3 - T_3 f_2, \ldots,\\ T_{m-1} f_m - T_m f_{m-1})(f_i \in \mathcal{J})\end{aligned}$$

in case $m \geq 3$.

For any tuple T of commuting operators T_i in $\mathcal{B}(\mathcal{J})$ and for any $\lambda = (\lambda_1, \ldots, \lambda_m)$ in $\mathbb{C}^m$, we use $T - \lambda$ to denote the tuple $(T_1 - \lambda_1 I_{\mathcal{J}}, \ldots, T_m - \lambda_m I_{\mathcal{J}})$, where $I_{\mathcal{J}}$ stands for the identity operator on the Hilbert space $\mathcal{J}$. The coboundary maps $\partial_{T,p}$ give rise to the cohomology vector spaces

$$H^{(p)}(T) = \frac{\text{Ker}(\partial_{T,p})}{\text{Ran}(\partial_{T,p-1})} (0 \leq p \leq m).$$

The tuple T is said to be *Fredholm* if the *Betti numbers* $\beta_p(T) = \dim(H^{(p)}(T))$ are all finite, and in that case the *Fredholm index* $\operatorname{ind}(T)$ of T is defined to be the Euler characteristic of $K(T)$, viz, $\operatorname{ind}(T) = \sum_{p=0}^{m}(-1)^p\beta_p$. The *essential Taylor spectrum* $\sigma_{ess}(T)$ of T is the set $\sigma_{ess}(T) = \{\lambda \in \mathbb{C}^m : T - \lambda \text{ is not Fredholm}\}$. For T Fredholm, the coboundary maps $\partial_{T,p}$ have closed ranges, and $T = (T_1, \dots, T_m)$ is Fredholm if and only if $T^* = (T_1^*, \dots, T_m^*)$ is. The basic properties of $\sigma_{ess}(T)$ and Fredholm tuples are discussed in [5].

A subset E of $\mathbb{C}^m$ is a *Reinhardt set* if, for every $z = (z_1, \dots, z_m)$ in E and every tuple $(\theta_1, \dots, \theta_m)$ of real θ_i, $e^{i\theta}z \equiv (e^{i\theta_1}z_1, \dots, e^{i\theta_m}z_m)$ lies in E. A *Reinhardt measure* μ is a compactly supported finite positive Borel measure satisfying $\mu(E) = \mu(e^{i\theta}E)$ for every Borel subset E of $\mathbb{C}^m$. An m-variable subnormal weighted shift S (with positive weights) is unitarily equivalent to the tuple $M_z^{(\mu)}$ of multiplications by coordinate functions z_i on $P^2(\mu)$, the closure of m-variable polynomials in $L^2(\mu)$, for some Reinhardt measure μ supported on the Reinhardt set $\operatorname{supp}(\mu)$; in particular, μ satisfies $d\mu(r_1e^{i\theta_1}, \dots, r_me^{i\theta_m}) = d\nu(r_1, \dots, r_m)\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_m}{2\pi}$ with ν being a "radial" measure and θ_i, for any i, being the arc-length measure on the unit circle $\mathbb{T}$ (refer to [7]). The minimal normal extension of $M_z^{(\mu)}$ is the tuple $N_z^{(\mu)}$ of multiplications by z_i on $L^2(\mu)$ and the dual of $M_z^{(\mu)}$ is the tuple of multiplications by the conjugate coordinate functions $\bar{z}_i$ on the Hilbert space $L^2(\mu) \ominus P^2(\mu)$.

Remark 2.1 The requirement that the weights $w_n^{(i)}$ associated with an m-variable subnormal weighted shift $S\,(= M_z^{(\mu)})$ be positive guarantees in particular that μ cannot have its support contained in V, the union of the hyperplanes $z_i = 0$. It was shown by Curto and Yan in [7, Proposition 1.14] that, provided a Reinhardt measure μ does not have its support contained in V, the multiplication tuple $S = M_z^{(\mu)}$ on $P^2(\mu)$ satisfies $\sum_{i=1}^{m}\|S_if\|^2 \geq \delta\|f\|^2$ for some positive δ and for all f in $P^2(\mu)$; in particular, $\operatorname{Ran}(\partial_{S,0})$ is closed.

3 Main Results

The argument required to prove Proposition 3.1 is well known (see, for example, [10, Proposition 2.6] or [8, Theorem 2.3]); it depends on the closedness of $\operatorname{Ran}(\partial_{S,0})$ and the exactness of the Koszul complex at the stage $p = 1\ (\leq m-1)$ for the polynomial ring $\mathbb{C}[z]$. In case S is Fredholm, $\operatorname{Ran}(\partial_{S,p})$ is closed for each p and $H^{(p)}(S)$ turns out to be zero-dimensional for any $p \leq m-1$.

Proposition 3.1 *If S is an m-variable subnormal weighted shift acting on $\mathcal{H}$ with $m \geq 2$, then $\beta_1(S) = 0$, that is, $H^{(1)}(S)$ is zero-dimensional.*

Proposition 3.2 *Let S be an m-variable subnormal weighted shift acting on $\mathcal{H}$ with $m \geq 1$ and such that the corresponding Reinhardt measure μ is given by*

$$d\mu(r_1e^{i\theta_1}, \dots, r_me^{i\theta_m}) = d\nu(r_1, \dots, r_m)\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_m}{2\pi}.$$

If $1/(r_1^2 + \cdots + r_m^2)$ is μ-integrable, then $\beta_1(\widetilde{S}) \geq 1$, that is, $H^{(1)}(\widetilde{S})$ has dimension at least 1.

Proof We identify S with $M_z^{(\mu)}$ on $P^2(\mu)$ where the measure μ is as described in the statement of the proposition. It is easy to see that the vector space dimension of $H^{(1)}(\widetilde{S})$ is at least as big as that of $\operatorname{Ker}(\partial_{\widetilde{S},1}) \cap (\operatorname{Ran}(\partial_{\widetilde{S},0}))^{\perp}$ (where $\widetilde{S}$ is the tuple of multiplications by the conjugate coordinate functioins $\bar{z}_i$ on the Hilbert space $L^2(\mu) \ominus P^2(\mu)$). Let $|z| = \sqrt{|z_1|^2 + \cdots + |z_m|^2} = \sqrt{r_1^2 + \cdots + r_m^2}$. The μ-integrability of $1/(r_1^2 + \cdots + r_m^2)$ guarantees that the functions $g_i = \bar{z}_i/|z|^2$ are in $L^2(\mu)$ for $1 \le i \le m$; further, it is easy to see that each g_i is orthogonal to any polynomial in $\mathbb{C}[z]$ and hence lies in $L^2(\mu) \ominus P^2(\mu)$. It further follows from the definition of $\partial_{\widetilde{S},1}$ that the tuple $g = (g_1, \dots, g_m)$ lies in $\operatorname{Ker}(\partial_{\widetilde{S},1})$. We now check that g also lies in $(\operatorname{Ran}(\partial_{\widetilde{S},0}))^{\perp}$. Any element in $\operatorname{Ran}(\partial_{\widetilde{S},0})$ is $(\bar{z}_1 f, \dots, \bar{z}_m f)$ for some f in $L^2(\mu) \ominus P^2(\mu)$, and we have

$$\langle g, (\bar{z}_1 f, \dots, \bar{z}_m f) \rangle = \int_{\operatorname{supp}(\mu)} \bar{f} \, d\mu(z).$$

The last integral, being simply the inner product of the polynomial 1 in $P^2(\mu)$ with $f \in L^2(\mu) \ominus P^2(\mu)$, is zero. Thus g lies in $\operatorname{Ker}(\partial_{\widetilde{S},1}) \cap (\operatorname{Ran}(\partial_{\widetilde{S},0}))^{\perp}$, and this shows that $H^{(1)}(\widetilde{S})$ has dimension at least 1. ■

Examples 3.3 (a) Consider the reproducing kernel Hilbert spaces $\mathcal{H}(m; n)$ ($n \in \mathbb{N}, n \ge m$) corresponding to the positive definite kernels

$$\kappa(m; n) = \frac{1}{(1 - \bar{z}_1 w_1 - \cdots - \bar{z}_m w_m)^n} (n \in \mathbb{N}, n \ge m)$$

on $\mathbb{B}^{2m} \times \mathbb{B}^{2m}$, where $\mathbb{B}^{2m}$ is the open unit ball in $\mathbb{C}^m$ centered at the origin. Modulo constants, $\kappa(m; m)$ and $\kappa(m; m+1)$ are the reproducing kernels for the Hardy space of $\mathbb{B}^{2m}$ and the Bergman space of $\mathbb{B}^{2m}$, respectively. We observe that the spaces $\mathcal{H}(m; n)$ are spaces $P^2(\mu_{m;n})$ with the Reinhardt measures $\mu_{m;n}$ having their support in the closure of $\mathbb{B}^{2m}$; indeed, modulo constants, $\mu_{m;n}$ can be described as follows: $\mu_{m;m}$ is the Haar measure on $\mathbb{S}^{2m-1}$, the topological boundary of $\mathbb{B}^{2m}$; $\mu_{m;m+1}$ is the volumetric measure ν in $\mathbb{C}^m = \mathbb{R}^{2m}$ restricted to $\mathbb{B}^{2m}$; $\mu_{m;n}$ ($n > m+1$) is the measure $(1 - r_1^2 - \cdots - r_m^2)^{n-m-1} \mu_{m;m+1}$ (refer to [1]). In the case of $P^2(\mu_{m;m})$, the radial function $1/(r_1^2 + \cdots + r_m^2)$ reduces to the constant function 1. Also, in polar coordinates one has $d\mu_{m;m+1}(z) = r_1 dr_1 d\theta_1 \dots r_m dr_m d\theta_m$ so that the $\mu_{m;n}$-integrability of $1/(r_1^2 + \cdots + r_m^2)$, for $m \ge 2$ and $n \ge m+1$, is an easy consequence of the inequality

$$\frac{r_1 r_2 \dots r_m}{r_1^2 + \cdots + r_m^2} \le 1$$

(which holds everywhere except at the origin). Thus, the conclusions of Propositions 3.1 and 3.2 hold for the m-tuples $S = M_z^{(\mu_{m;n})}$ in case $m \ge 2$.

(b) Consider the reproducing kernel Hilbert spaces $\mathcal{H}(m; n_1, \dots, n_m)$ ($n_i \ge 1$) corresponding to the positive definite kernels

$$\kappa(m; n_1, \dots, n_m) = \prod_{i=1}^{m} \frac{1}{(1 - \bar{z}_i w_i)^{n_i}} (n_i \in \mathbb{N}, n_i \ge 1)$$

on $\mathbb{D}^m \times \mathbb{D}^m$, where $\mathbb{D}^m$ is the open unit polydisk in $\mathbb{C}^m$ centered at the origin. Modulo constants, $\kappa(m; 1, \ldots, 1)$ and $\kappa(m; 2, \ldots, 2)$ are the reproducing kernels for the Hardy space of $\mathbb{D}^m$ and the Bergman space of $\mathbb{D}^m$, respectively. We observe that the spaces $\mathcal{H}(m; n_1, \ldots, n_m)$ are spaces $P^2(\mu_{m;n_1,\ldots,n_m})$ with the Reinhardt measures $\mu_{m;n_1,\ldots,n_m}$ having their support in the closure of $\mathbb{D}^m$ and being the products of appropriate measures $\mu_{1;n}$ (refer to (a) above). In the case of $P^2(\mu_{m;1,\ldots,1})$, the radial function $1/(r_1^2 + \cdots + r_m^2)$ reduces to the constant function $1/m$. The $\mu_{m;n_1,\ldots,n_m}$-integrability of $1/(r_1^2 + \cdots + r_m^2)$, for $m \geq 2$ and $(n_1, \ldots, n_m) \neq (1, \ldots, 1)$, is clear in view of our discussion in part (a) above. Thus, the conclusions of Propositions 3.1 and 3.2 hold for the m-tuples $S = M_z^{(\mu_{m;n_1,\ldots,n_m})}$ in case $m \geq 2$.

Examples 3.4 Let $m \geq 2$ and let K be any compact Reinhardt subset of the closure of $\mathbb{D}^m$ with $\nu(K) > 0$, where ν is the volumetric measure in $\mathbb{C}^m = \mathbb{R}^{2m}$. If $w(r_1, \ldots, r_m)$ is any bounded positive Borel function of r_i $(0 \leq r_i \leq 1)$ and if $\nu_{K,w}$ is the measure $w(r_1, \ldots, r_m)\nu/K$, then the conclusions of Propositions 3.1 and 3.2 hold for $S = M_z^{(\nu_{K,w})}$. We note that, for $m \geq 2$, $M_z^{(\mu_{m;n})}$ $(n \geq m + 1)$ and $M_z^{(\mu_{m;n_1,\ldots,n_m})}$ $(n_i \geq 2)$ of Examples 3.3 are special cases of $S = M_z^{(\nu_{K,w})}$.

Examples 3.5 Let $m \geq 2$ and let K be any compact Reinhardt subset of $\mathbb{C}^m$ such that K is not contained in the union of the hyperplanes $z_i = 0$, K is the support of a positive Reinhardt measure μ, and K does not include the origin $\mathbf{0}$ of $\mathbb{C}^m$. Then $1/(r_1^2 + \cdots + r_m^2)$ is clearly μ-integrable, and the conclusions of Propositions 3.1 and 3.2 hold for $S = M_z^{(\mu)}$.

Proposition 3.6 *Let S be an m-variable subnormal weighted shift acting on $\mathcal{H}$ with $m \geq 2$ and such that the corresponding Reinhardt measure μ is given by*

$$d\mu(r_1 e^{i\theta_1}, \ldots, r_m e^{i\theta_m}) = d\nu(r_1, \ldots, r_m) \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_m}{2\pi}.$$

If $1/(r_1^2 + \cdots + r_m^2)$ is μ-integrable, then S is not similar to its dual $\widetilde{S}$.

Proof If there were to exist an invertible intertwining operator X for S and $\widetilde{S}$ so that $XS = \widetilde{S}X$, then the map $\phi\colon H^{(1)}(S) \to H^{(1)}(\widetilde{S})$ given by

$$\phi((f_1, \ldots, f_m) + \operatorname{Ran}(\partial_{S,0})) = (Xf_1, \ldots, Xf_m) + \operatorname{Ran}(\partial_{\widetilde{S},0}),$$

where $(f_1, \ldots, f_m) \in \operatorname{Ker}(\partial_{S,1})$, would be a well-defined linear isomorphism of $H^{(1)}(S)$ onto $H^{(1)}(\widetilde{S})$, and that would force $\beta_1(S) = \beta_1(\widetilde{S})$. This is not possible in view of Propositions 3.1 and 3.2. ■

Remark 3.7 If an m-tuple S of operators S_i in $\mathcal{B}(\mathcal{H})$ is similar to an m-tuple of operators T_i in $\mathcal{B}(\mathcal{J})$, then one can check that $H^{(p)}(S)$ and $H^{(p)}(T)$ are isomorphic vector spaces for $0 \leq p \leq m$, and thus the vector space dimensions of $H^{(p)}(S)$ and $H^{(p)}(T)$ must be the same; we considered the case $p = 1$ in Proposition 3.6. In particular, if S is Fredholm and T is not, then S cannot be similar to T.

https://doi.org/10.4153/CMB-2011-188-x Published online by Cambridge University Press

It is clear from Proposition 3.6 that, in case $m \geq 2$, none of the subnormal m-tuples S discussed in Examples 3.3, 3.4, and 3.5 are similar to their duals. In fact, in light of the results obtained in [2, 3], it can be checked that, in case $m \geq 2$, none of the subnormal m-tuples discussed in Example 3.3 are even *quasisimilar* to their duals (refer to [2, Theorems 2 and 3] and to [3, Proposition 2.2]).

Proposition 3.8 *Let S be an m-variable subnormal weighted shift acting on $\mathcal{H}$ with $m \geq 2$. If S is Fredholm, then S is not similar to its dual $\widetilde{S}$.*

Proof We identify S with $M_z^{(\mu)}$ for an appropriate μ. Consider, as in [9], the following exact sequence of complexes:

$$0 \longrightarrow K(S) \longrightarrow K(N) \longrightarrow K(\widetilde{S}^*) \longrightarrow 0,$$

where $\widetilde{S}^*$ is $((\widetilde{S}_1)^*, \dots, (\widetilde{S}_m)^*)$, where the arrow between $K(S)$ and $K(N)$ is induced by the inclusion of $P^2(\mu)$ into $L^2(\mu)$, and where the arrow between $K(N)$ and $K(\widetilde{S}^*)$ is induced by the orthogonal projection of $L^2(\mu)$ onto $L^2(\mu) \ominus P^2(\mu)$. This short exact sequence gives rise to the long exact sequence of cohomology

$$\begin{aligned} 0 \to H^{(0)}(S) \to H^{(0)}(N) \to H^{(0)}(\widetilde{S}^*) \to H^{(1)}(S) \to H^{(1)}(N) \to H^{(1)}(\widetilde{S}^*) \to H^{(2)}(S) \to \cdots \\ \to H^{(m-1)}(S) \to H^{(m-1)}(N) \to H^{(m-1)}(\widetilde{S}^*) \to H^{(m)}(S) \to H^{(m)}(N) \to H^{(m)}(\widetilde{S}^*) \to 0. \end{aligned}$$

Assume that S is similar to $\widetilde{S}$. Then Remark 3.7 yields $\beta_p(S) = \beta_p(\widetilde{S})$ for all p. The ranges of the coboundary maps $\partial_{S,p}$ are closed so that, by our comments preceding Proposition 3.1, one has $\beta_p(S) = 0$ for $0 \leq p \leq m-1$. Since $\beta_1(\widetilde{S}^*) = \beta_{m-1}(\widetilde{S})$ (refer to [3]), we have in effect $\dim(H^{(1)}(\widetilde{S}^*)) = \beta_1(\widetilde{S}^*) = 0$. Combined with $\dim(H^{(1)}(S)) = \beta_1(S) = 0$, that leads to $\dim(H^{(1)}(N)) = 0$ in light of the long exact sequence given above. If the origin $\mathbf{0}$ were to be an atom of the measure μ, then the tuple $f = (f_1, \dots, f_m)$, with each f_i being 1 at $\mathbf{0}$ and 0 elsewhere, would be in $\operatorname{Ker}(\partial_{N,1})$ but not in $\operatorname{Ran}(\partial_{N,0})$, thereby forcing $H^{(1)}(N)$ to be nontrivial. Thus $\mathbf{0}$ is not an atom of μ, and that obviously forces $H^{(0)}(N)$ to be trivial. Further, it is easy to see that $H^{(m)}(S) = \frac{\mathcal{H}}{S_1\mathcal{H} + \cdots + S_m\mathcal{H}}$ is one-dimensional. In light of the long exact sequence given above, we are then led to the contradiction $0 = \beta_0(N) = \beta_0(\widetilde{S}^*) = \beta_m(\widetilde{S}) = \beta_m(S) = 1$. Hence S cannot be similar to $\widetilde{S}$. ∎

Remark 3.9 If S and $\widetilde{S}$ are both Fredholm subnormal m-tuples, then one has the formula $\operatorname{ind}(S) = (-1)^{m+1} \operatorname{ind}(\widetilde{S})$ (refer to [9]). In view of Remark 3.7, it then follows that a Fredholm subnormal m-tuple S with non-zero Fredholm index cannot be similar to its dual $\widetilde{S}$ if m is even. The analysis of Proposition 3.8, which occurs in the context of weighted shifts, is based on examining individual Betti numbers (rather than Fredholm indices) and holds for even m as well as for odd $m > 1$.

Finally, let us comment on the special case $m = 2$. Since $\operatorname{supp}(\mu)$ is not contained in the union of the hyperplanes $z_1 = 0$ and $z_2 = 0$ (see Remark 2.1), there is at least one point $(p, q) \in \operatorname{supp}(\mu)$ $(\subset \mathbb{C}^2)$ such that $p \neq 0, q \neq 0$. Then, as follows from the Maximum Modulus Theorem, the set $\{(z_1, z_2) : |z_1| \leq |p|, |z_2| \leq |q|\}$ is contained in the polynomial convex hull $\widehat{\operatorname{supp}(\mu)}$ of $\operatorname{supp}(\mu)$; in particular, the origin $\mathbf{0} = (0,0)$ of $\mathbb{C}^2$ does not belong to the topological boundary of $\widehat{\operatorname{supp}(\mu)}$ and the pair $S = (S_1, S_2)$ is Fredholm in view of [7, Theorem 3.5(iii)], which asserts that $\sigma_{ess}(S)$ is the topological boundary of $\widehat{\operatorname{supp}(\mu)}$. The authors do not know whether a suitable analog of [7, Theorem 3.5(iii)] is available for $m \geq 3$ to enable one to deduce that, even for $m \geq 3$, every subnormal m-variable weighted shift with bounded positive weights is Fredholm.

Acknowledgements Thanks to a suggestion by the referee, the authors were able to shorten the original proof of Proposition 3.8.

References

[1] A. Athavale, *Model theory on the unit ball in $\mathbb{C}^m$.* J. Operator Theory **27**(1992), no. 2, 347–358.
[2] ———, *Quasisimilarity-invariance of joint spectra for certain subnormal tuples.* Bull. London Math. Soc. **40**(2008), no. 5, 759–769. http://dx.doi.org/10.1112/blms/bdn054
[3] A. Athavale and P. Patil, *On the duals of Szegö and Cauchy tuples.* Proc. Amer. Math. Soc. **139**(2011), no. 2, 491–498. http://dx.doi.org/10.1090/S0002-9939-2010-10482-3
[4] J. B. Conway, *The dual of a subnormal operator.* J. Operator Theory **5**(1981), no. 2, 195–211.
[5] R. E. Curto, *Fredholm and invertible n-tuples of operators. The deformation problem.* Trans. Amer. Math. Soc. **266**(1981), no. 1, 129–159.
[6] R. E. Curto and N. Salinas, *Spectral properties of cyclic subnormal m-tuples.* Amer. J. Math. **107**(1985), no. 1, 113–138. http://dx.doi.org/10.2307/2374459
[7] R. E. Curto and K. Yan, *The spectral picture of Reinhardt measures.* J. Func. Anal. **131**(1995), no. 2, 279–301. http://dx.doi.org/10.1006/jfan.1995.1090
[8] J. Eschmeier, *Grothendieck's comparison theorem and multivariable Fredholm theory.* Arch. Math. **92**(2009), no. 5, 461–475.
[9] J. Gleason, *On a question of Ameer Athavale.* Irish Math. Soc. Bull. **48**(2002), 31–33.
[10] J. Gleason, S. Richter, and C. Sundberg, *On the index of invariant subspaces in spaces of analytic functions in several complex variables.* J. Reine Angew. Math. **587**(2005), 49–76. http://dx.doi.org/10.1515/crll.2005.2005.587.49
[11] T. Itô, *On the commutative family of subnormal operators.* J. Fac. Sci. Hokkaido Univ. **14**(1958), 1-15.
[12] N. Jewell and A. R. Lubin, *Commuting weighted shifts and analytic function theory in several variables.* J. Operator Theory **1**(1979), no. 2, 207–223.

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India
e-mail: athavale@math.iitb.ac.in pramodp@math.iitb.ac.in