Bull. Aust. Math. Soc. **96** (2017), 400–411
doi:10.1017/S0004972717000612

THE SHORT RESOLUTION OF A SEMIGROUP ALGEBRA

I. OJEDA and A. VIGNERON-TENORIO✉

(Received 12 March 2017; accepted 4 May 2017; first published online 29 August 2017)

Abstract

This work generalises the short resolution given by Pisón Casares ['The short resolution of a lattice ideal', *Proc. Amer. Math. Soc.* **131**(4) (2003), 1081–1091] to any affine semigroup. We give a characterisation of Apéry sets which provides a simple way to compute Apéry sets of affine semigroups and Frobenius numbers of numerical semigroups. We also exhibit a new characterisation of the Cohen–Macaulay property for simplicial affine semigroups.

2010 *Mathematics subject classification*: primary 13D02; secondary 13P10, 14M25, 68W30.

Keywords and phrases: free resolution, Betti number, semigroup algebra, affine semigroup, Apéry set, Frobenius problem.

1. Introduction

Let $\Bbbk$ be a field and let S be a finitely generated commutative submonoid of $\mathbb{Z}^d$ such that $S \cap (-S) = \{0\}$. There is a large literature on the study and computation of minimal free resolutions of the semigroup algebra $\Bbbk[S] = \bigoplus_{\mathbf{a} \in S} \Bbbk\chi^{\mathbf{a}}$ (see, for example, [12] and the references therein). Most works on this topic consider $\Bbbk[S]$ as a $\Bbbk[\mathbf{X}](:= \Bbbk[X_1, \ldots, X_n])$-module with the structure given by $\Bbbk[\mathbf{X}] \to \Bbbk[S];\ X_i \mapsto \chi^{\mathbf{a}_i}$, where $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a (fixed) system of generators of S.

In [13], Pisón Casares proposed a new and original resolution of $\Bbbk[S]$. She considered $\Bbbk[S]$ as a module over a polynomial ring in fewer variables determined by the extreme rays of the rational cone generated by S and she explicitly constructed a free resolution that she called the *short resolution* of $\Bbbk[S]$. In her construction it is implicitly assumed that the generators of S corresponding to extreme rays are $\mathbb{Z}$-linearly independent. This actually happens when the semigroup is simplicial.

The original aim of this work was to avoid the simplicial hypothesis on the semigroup, by generalising the construction in [13]. However, during the course of the work, we realised that some improvements can be made so that some results and many of the proofs in [13] have been simplified.

Both authors are partially supported by the projects MTM2012-36917-C03-01 and MTM2015-65764-C3-1-P (MINECO/FEDER, UE), National Plan I+D+I. The first author is also partially supported by Junta de Extremadura (FEDER funds) and the second by Junta de Andalucía (group FQM-366).

One of the original contributions of [13] is the explicit computation of test sets for the Apéry sets of affine semigroups, using Gröbner basis techniques with respect to a particular local term order. We improve this approach by obtaining a new and explicit description of the Apéry sets without using the local term orders (Theorem 3.3). This allows us to formulate an easy algorithm for the computation of Apéry sets of affine semigroups and consequently an algorithm to compute the Frobenius number of a numerical semigroup, as described in Section 3. Both algorithms seem to have a good computational behaviour, and, moreover, our construction relies on the computation of just one Gröbner basis with respect to a particular (global) term order.

In Section 4, we give a presentation of any semigroup algebra as a module over a ring in as many variables as the dimension of the cone of the semigroup (Theorem 4.3) without assuming that the semigroup is simplicial. This completes the construction of the short resolution given in [13]. The results of Section 4, combined with our computational description of the Apéry sets, lead to a new characterisation of the Cohen–Macaulayness of simplicial affine semigroups (Corollary 4.7).

Finally, in the last section, we propose a new combinatorial description of the resolution of a semigroup algebra given in [13] and we explicitly determine the isomorphisms from the combinatorial side to the minimal generators for the presentation given in Section 4.

2. Preliminaries

Given a finite subset $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of $\mathbb{Z}^d$, we consider the so-called affine semigroup, the subsemigroup $S = \mathbb{N}\mathbf{a}_1 + \cdots + \mathbb{N}\mathbf{a}_n$ of $\mathbb{Z}^d$ generated by $\mathcal{A}$, where $\mathbb{N}$ denotes the set of nonnegative integers. In particular, S is a finitely generated, cancellative and commutative semigroup with zero element.

Associated to $\mathcal{A}$ is the surjective function

$$\deg_{\mathcal{A}} : \mathbb{N}^n \longrightarrow S; \quad \mathbf{u} = (u_1, \ldots, u_n) \longmapsto \deg_{\mathcal{A}}(\mathbf{u}) = \sum_{i=1}^{n} u_i \mathbf{a}_i.$$

This map is called the factorisation map of S and $\deg_{\mathcal{A}}^{-1}(\mathbf{a})$ is called the set of factorisations of $\mathbf{a} \in S$. Notice that the cardinality of $\deg_{\mathcal{A}}^{-1}(\mathbf{a})$ for $\mathbf{a} \in S$ is not necessarily finite. The necessary and sufficient condition for the finiteness of factorisations is that $S \cap (-S) = 0$ (see [1, Proposition 1.1]); equivalently,

$$u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n = 0 \text{ for } (u_1, \ldots, u_n) \in \mathbb{N}^n \quad \Longrightarrow \quad u_1 = \cdots = u_n = 0. \tag{2.1}$$

Throughout this paper, we will assume that $\mathcal{A}$ satisfies this condition.

Let $\Bbbk$ be a field. The map $\deg_{\mathcal{A}}$ induces the surjective $\Bbbk$-algebra homomorphism

$$\varphi_{\mathcal{A}} : \Bbbk[\mathbb{N}^n] = \Bbbk[X_1, \ldots, X_n] \longrightarrow \Bbbk[S] := \bigoplus_{\mathbf{a} \in S} \Bbbk\chi^{\mathbf{a}};$$
$$\mathbf{X}^{\mathbf{u}} := X_1^{u_1} \cdots X_n^{u_n} \longmapsto \chi^{\deg_{\mathcal{A}}(\mathbf{u})}.$$

Observe that if we consider the grading on $\Bbbk[X_1, \ldots, X_n]$ given by $\deg(X_i) = \mathbf{a}_i$ for $i = 1, \ldots, n$, then $\varphi_{\mathcal{A}}$ is homogeneous of degree zero. Hence, both the toric ideal

https://doi.org/10.1017/S0004972717000612 Published online by Cambridge University Press

$I_{\mathcal{A}} := \ker(\varphi_{\mathcal{A}})$ and the coordinate ring $\Bbbk[S] \cong \Bbbk[X_1, \ldots, X_n]/I_{\mathcal{A}}$ are homogeneous for this S-grading, which is the grading determined by $\mathcal{A}$.

In the following, unless stated otherwise, we set $\deg(X_i) = \mathbf{a}_i$ for $i = 1, \ldots, n$. That is to say, we will consider $\Bbbk[\mathbf{X}]$ multigraded by the semigroup S.

The necessary and sufficient condition for the finiteness of factorisations assumed above (see formula 2.1) implies that there exists a minimal S-graded free resolution of $\Bbbk[S]$, which is defined by the property that all the differentials become zero when tensored with $\Bbbk \cong \Bbbk[\mathbf{X}]/\mathfrak{m}$, where $\mathfrak{m} = \langle X_1, \ldots, X_m\rangle$ (see [10, Section 8.3]). This justifies the next definition.

Definition 2.1. The ith (multigraded) Betti number of $\Bbbk[S]$ in degree $\mathbf{a}$ is

$$\beta_{i,\mathbf{a}}(\Bbbk[S]) := \dim_{\Bbbk} \operatorname{Tor}_i^{\Bbbk[\mathbf{X}]}(\Bbbk, \Bbbk[S])_{\mathbf{a}}.$$

3. Computation of Apéry sets

For $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subseteq \mathbb{Z}^d$ satisfying (2.1), define the polyhedral cone of $\mathcal{A}$ by

$$\operatorname{pos}(\mathcal{A}) := \{\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{Q}_{\geq 0}\} \subset \mathbb{Q}^d.$$

Without loss of generality, by relabelling if necessary, we may assume that $\operatorname{pos}(\mathcal{A}) = \operatorname{pos}(\{\mathbf{a}_1, \ldots, \mathbf{a}_r\})$, where $r \leq n$. Thus, in the following we will write $E = \{\mathbf{a}_1, \ldots, \mathbf{a}_r\}$ and $\mathbf{b}_i = \mathbf{a}_{r+i}$, $i = 1, \ldots, s := n - r$, so that

$$\mathcal{A} = E \cup B, \tag{3.1}$$

where $B := \{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$. This is called a convex partition in [4, Definition 4.1].

Observation 3.1. If S is a simplicial semigroup, that is, if $\operatorname{pos}(\mathcal{A})$ can be generated by $\dim_{\mathbb{Q}}(\operatorname{pos}(\mathcal{A}))$ elements of $\mathcal{A}$, we may take $r = \operatorname{rank}(\mathbb{Z}\mathcal{A}) = \dim_{\mathbb{Q}}(\operatorname{pos}(\mathcal{A}))$, where $\mathbb{Z}\mathcal{A}$ denotes the subgroup of $\mathbb{Z}^d$ generated by $\mathcal{A}$. In this case, (3.1) is also called a simplicial partition.

Let S be the semigroup generated by $\mathcal{A}$ and let $\Bbbk[\mathbf{Y}]$ and $\Bbbk[\mathbf{Y}, \mathbf{Z}]$ be the polynomial rings in r and n variables, respectively, over a field $\Bbbk$.

Let $\prec$ be the monomial order on $\Bbbk[\mathbf{Y}, \mathbf{Z}]$ defined as follows: $\mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'} \prec \mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}}$ if and only if the leftmost nonzero entry of $\deg_{\mathcal{A}}(\mathbf{u}, \mathbf{v}) - \deg_{\mathcal{A}}(\mathbf{u}', \mathbf{v}')$ is positive or $\deg_{\mathcal{A}}(\mathbf{u}, \mathbf{v}) = \deg_{\mathcal{A}}(\mathbf{u}', \mathbf{v}')$ and $\mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'} \prec_{\text{revlex}} \mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}}$, where $\prec_{\text{revlex}}$ is a reverse lexicographic ordering on $\Bbbk[\mathbf{Y}, \mathbf{Z}]$ such that $Y_i \prec Z_j$ for each $i = 1, \ldots, r$ and $j = 1, \ldots, s$. By abuse of terminology, we will say that $\prec$ is an S-graded reverse lexicographic monomial order on $\Bbbk[\mathbf{Y}, \mathbf{Z}]$ such that $Y_i \prec Z_j$ for each $i = 1, \ldots, r$ and $j = 1, \ldots, s$.

Set $\varphi_{\mathcal{A}} : \Bbbk[\mathbf{Y}, \mathbf{Z}] \to \Bbbk[S]$; $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}} \mapsto \chi^{\deg_{\mathcal{A}}(\mathbf{v},\mathbf{u})}$ and let $\mathcal{G}_{\prec}(I_{\mathcal{A}})$ be the reduced Gröbner basis of $I_{\mathcal{A}} = \ker(\varphi_{\mathcal{A}})$ with respect to $\prec$. We will write Q for the exponents of the standard monomials in the variables $Z_1, \ldots, Z_s$, that is,

$$Q = \{\mathbf{u} \in \mathbb{N}^s \mid \mathbf{Z}^{\mathbf{u}} \notin \operatorname{in}_{\prec}(I_{\mathcal{A}})\},$$

where $\operatorname{in}_{\prec}(I_{\mathcal{A}})$ denotes the monomial ideal generated by all the leading terms of $I_{\mathcal{A}}$ with respect to $\prec$ (see [10, page 24]).

Proposition 3.2. *The set $\mathcal{Q}$ is finite.*

Proof. Since $\mathbf{b}_j \in \mathrm{pos}(\mathcal{A}) = \mathrm{pos}(\{\mathbf{a}_1, \ldots, \mathbf{a}_r\})$, there exist $u_j, v_{1j}, \ldots, v_{rj} \in \mathbb{N}$ such that $u_j \mathbf{b}_j = \sum_{i=1}^r v_{ij} \mathbf{a}_i$ for $j = 1, \ldots, s$ (that is, $Z_j^{u_j} - Y_1^{v_1} \cdots Y_r^{v_r} \in I_{\mathcal{A}}$ for each j). Therefore, $\mathbf{Z}^{\mathbf{u}'} \in \mathrm{in}_{\prec}(I_{\mathcal{A}})$ for every $\mathbf{u}' \in \mathbb{N}^s$ whose jth coordinate is larger than u_j for some $j = 1, \ldots, s$. □

We recall that the Apéry set, $\mathrm{Ap}(S, E)$, of S relative to E is defined as

$$\mathrm{Ap}(S, E) = \{\mathbf{a} \in S \mid \mathbf{a} - \mathbf{e} \notin S \text{ for all } \mathbf{e} \in E\}.$$

Our main result in this section improves [13, Lemma 1.2]. We explicitly describe a bijection from $\mathcal{Q}$ to $\mathrm{Ap}(S, E)$, in contrast to [13], where the given map is not necessarily bijective, as pointed out in [13, page 1083] after the definition of Ψ_0. Moreover, we consider a global monomial ordering (the least monomial is 1) instead of a local ordering (the biggest monomial is 1), as suggested in [13, top of page 1083]. All this will have important consequences for the forthcoming constructions.

Observe that the natural injection $\iota : \mathbb{N}^s \hookrightarrow \mathbb{N}^n; \mathbf{u} \mapsto (0, \mathbf{u})$ allows us to restrict $\deg_{\mathcal{A}} \circ \iota(-)$ to $\mathcal{Q}$.

Theorem 3.3. *The restriction of* $\deg_{\mathcal{A}} \circ \iota(-)$ *to $\mathcal{Q}$ defines a bijection* $\mathcal{Q} \to \mathrm{Ap}(S, E)$.

Proof. Let $\mathbf{u} \in \mathcal{Q}$ and set $\mathbf{q} = \deg_{\mathcal{A}}(\iota(\mathbf{u}))$. If $\mathbf{q} \notin \mathrm{Ap}(S, E)$, then there exists $i \in \{1, \ldots, r\}$ such that $\mathbf{q} - \mathbf{a}_i = \sum_{i=1}^r v_i \mathbf{a}_i + \sum_{j=1}^s w_j \mathbf{b}_j \in S$. Thus, we have a binomial $\mathbf{Z}^{\mathbf{u}} - Y_i \mathbf{Y}^{\mathbf{v}} \mathbf{Z}^{\mathbf{w}} \in I_{\mathcal{A}}$. Since the monomials $\mathbf{Z}^{\mathbf{u}}$ and $Y_i \mathbf{Y}^{\mathbf{v}} \mathbf{Z}^{\mathbf{w}}$ are distinct and $Y_i \mathbf{Y}^{\mathbf{v}} \mathbf{Z}^{\mathbf{w}} \prec \mathbf{Z}^{\mathbf{u}}$ because Y_i divides $Y_i \mathbf{Y}^{\mathbf{v}} \mathbf{Z}^{\mathbf{w}}$, we conclude that $\mathbf{u} \notin \mathcal{Q}$, which is a contradiction. Therefore, the image of the restriction of $\deg_{\mathcal{A}} \circ \iota(-)$ to $\mathcal{Q}$ lies in $\mathrm{Ap}(S, E)$.

Consider now $\mathbf{q} \in \mathrm{Ap}(S, E)$. Then $\mathbf{q}$ admits a factorisation $\mathbf{q} = \sum_{i=1}^s v_i \mathbf{b}_i$. The remainder of the division of $\mathbf{Z}^{\mathbf{v}}$ (where $\mathbf{v} = (v_1, \ldots, v_s) \in \mathbb{N}^s$) by $\mathcal{G}_{\prec}(I_{\mathcal{A}})$ is a monomial $\mathbf{Z}^{\mathbf{u}}$ of S-degree $\mathbf{q}$ which does not lie in $\mathrm{in}_{\prec}(I_{\mathcal{A}})$. Hence, $\mathbf{u} \in \mathcal{Q}$ and $\deg_{\mathcal{A}}(\iota(\mathbf{u})) = \mathbf{q}$, which proves the surjectivity of our map.

Finally, in order to prove that $\deg_{\mathcal{A}}(\iota(\mathbf{u})) = \deg_{\mathcal{A}}(\iota(\mathbf{v}))$ implies that $\mathbf{u} = \mathbf{v}$, it suffices to observe that $f := \mathbf{Z}^{\mathbf{u}} - \mathbf{Z}^{\mathbf{v}} \in I_{\mathcal{A}}$. So, if $\mathbf{u} \neq \mathbf{v}$, then $\mathrm{in}_{\prec}(f) = \mathbf{Z}^{\mathbf{u}}$ (or $\mathrm{in}_{\prec}(f) = \mathbf{Z}^{\mathbf{v}}$), that is, $\mathbf{u} \notin \mathcal{Q}$ (or $\mathbf{v} \notin \mathcal{Q}$), which leads us to a contradiction. □

Notice that Theorem 3.3 gives an easy algorithm for the computation of Apéry sets. A similar algorithm, based on a purely semigroup approach, can be found in [9].

In the particular case when S is a numerical semigroup (that is, S is a submonoid of $\mathbb{N}$ with finite complement in $\mathbb{N}$), Theorem 3.3 gives an algorithm for computing the Frobenius number, $g(S)$, of S (that is, the greatest natural number not belonging to S). It suffices to recall the well-known formula due to Apéry (see, for example, [8, Proposition 10.4]) that

$$g(S) = \max\{\mathrm{Ap}(S, \mathbf{a}_1)\} - \mathbf{a}_1$$

and the fact that any nonzero element of a numerical semigroup generates the corresponding polyhedral cone in $\mathbb{Q}$.

https://doi.org/10.1017/S0004972717000612 Published online by Cambridge University Press

Morales and Dung [11] recently gave an algorithm for the computation of the Frobenius number using similar arguments. Similar techniques were used by Einstein *et al.* [7] and Roune [14] to give sophisticated algorithms for the computation of the Frobenius number of a numerical semigroup. However, in these papers the important role of the Apéry sets is not observed.

EXAMPLE 3.4. The following example is taken from [5]. Let $\mathcal{A} = \{8, 11, 18\}$ and let $\prec$ be the S-graded reverse lexicographic monomial order on $\Bbbk[Y, Z_1, Z_2]$ with $Y \prec Z_2 \prec Z_1$. Using Singular [6], we computed the reduced Gröbner basis of $I_{\mathcal{A}} \subseteq \Bbbk[Y, Z_1, Z_2]$ with respect to $\prec$:

$$\mathcal{G}_{\prec}(I_{\mathcal{A}}) = \{Z_1^2 Z_2 - Y^5, Z_1^4 - Z_2^2 Y, Z_2^3 - Z_1^2 Y^4\}.$$

Clearly, $Q = \{1, Z_1, Z_2, Z_1^2, Z_1 Z_2, Z_2^2, Z_1^3, Z_1 Z_2^2\}$ and

$$\mathrm{Ap}(S, \{8\}) = \deg_{\mathcal{A}}(Q) = \{0, 11, 18, 22, 29, 36, 33, 47\}.$$

In this case, the Frobenius number is $47 - 8 = 39$.

The whole process can be automated easily, as the following Singular code shows.

```
LIB "toric.lib";
LIB "general.lib";
intmat A[1][3] = 18,11,8;
ring r = 0, (Z(1..size(A)-1), Y), dp;
ideal i = toric_ideal(A,"hs");
ring s = 0, (Z(1..size(A)-1), Y), wp(A);
ideal i = imap(r,i);
ideal m = lead(std(i));
ideal Q = kbase(std(m+Y));
int n;
intmat Ap[1][size(Q)];
for (n = 1; n <= size(Q); n = n + 1)
    {Ap[1,n] = A*intmat(leadexp(Q[n]));}
int g = sort(intvec(Ap))[1][size(Q)]-A[1,size(A)];
g;
```

The first author and Moreno have written a function in Singular [6] to compute the Apéry set and the Frobenius number of a numerical semigroup. The library is available at http://matematicas.unex.es/~ojedamc/inves/apery.lib. Using this library, we computed the Frobenius number of the numerical semigroup in [11, Example 5.5] in less than 0.6 seconds with an Intel© Core™ i5-2450M CPU @ 2.50 GHz × 4.

Morales provides a program called `Frobenius-public.exe` for computing the Apéry set and the Frobenius number of a numerical semigroup on his web page, https://www-fourier.ujf-grenoble.fr/~morales/. This software uses the algorithms presented in [11]. We have used this program and the library `apery.lib` to compare the computational behaviour of our algorithms and the algorithms in [11]. The

https://doi.org/10.1017/S0004972717000612 Published online by Cambridge University Press

algorithms behave similarly, but for numerical semigroups with large Frobenius number our algorithm may be a little better. For example, the Frobenius number of the semigroup generated by $\{1051, 1071, 1087, 1099, 1129, 1139, 1199, 1207, 1211, 1213, 3331, 4325, 5511, 10311, 11421\}$ is 11703. The program `apery.lib` needs one second to compute its Apéry set and Frobenius number, while `Frobenius-public.exe` needs approximately 15 seconds.

4. Pisón's free resolution

We keep the notation of the previous section. Let S_E be the subsemigroup of S generated by E and set

$$\Bbbk[S_E] := \bigoplus_{\mathbf{a}\in S_E} \Bbbk\chi^{\mathbf{a}}.$$

The composition $\Bbbk[\mathbf{Y}] \xrightarrow{\varphi_E} \Bbbk[S_E] \hookrightarrow \Bbbk[S]$ defines a natural structure of a $\Bbbk[\mathbf{Y}]$-module on $\Bbbk[S]$.

Obviously, $\Bbbk[\mathbf{Y}]$ is multigraded by S. So, there exists a minimal S-graded free resolution of $\Bbbk[S]$ as a $\Bbbk[\mathbf{Y}]$-module (see [10, Section 8.3]). In order to compute this resolution effectively, an S-graded presentation of $\Bbbk[S]$ as a $\Bbbk[\mathbf{Y}]$-module is required.

Proposition 4.1. *The set $\{\mathbf{Z}^{\mathbf{u}} \mid \mathbf{u} \in Q\}$ is a minimal system of generators of $\Bbbk[S]$ as a $\Bbbk[\mathbf{Y}]$-module.*

Proof. Since $\Bbbk[S] \cong \Bbbk[\mathbf{Y},\mathbf{Z}]/I_{\mathcal{A}}$, the result follows from the definition of Q. □

Remark 4.2. Note that S is a simplicial semigroup if and only if $I_{S_E} = I_S \cap \Bbbk[\mathbf{Y}] = 0$. In this case, φ_E is an isomorphism. This condition on S is implicitly assumed in [13, Section 1].

In order to give an S-graded presentation of $\Bbbk[S]$ as a $\Bbbk[\mathbf{Y}]$-module in the general setting, we first order Q lexicographically. There is a bijection σ from $\{1, \ldots, \beta_0 := \#Q\}$ to Q. Next, we define the surjective $\Bbbk[\mathbf{Y}]$-module homomorphism

$$\psi_0 : \Bbbk[\mathbf{Y}]^{\beta_0} \longrightarrow \Bbbk[S]$$

with $\psi_0(\varepsilon_i) = \mathbf{Z}^{\sigma(i)}$, $i = 1, \ldots, \beta_0$, where $\{\varepsilon_1, \ldots, \varepsilon_{\beta_0}\}$ is the canonical basis of $\Bbbk[\mathbf{Y}]^{\beta_0}$.

Let $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}} - \mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'}$ be an element of $\mathcal{G}_{\prec}(I_{\mathcal{A}})$ whose leading term is $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}}$ with $\mathbf{v} \neq 0$ and $\mathbf{u} \neq 0$. First of all, we notice that $\mathbf{v}' \neq 0$, which implies that $\mathbf{Z}^{\mathbf{u}}$ and $\mathbf{Z}^{\mathbf{u}'} \in Q$. Moreover, since no variable is a zero divisor modulo $I_{\mathcal{A}}$ (because $I_{\mathcal{A}}$ is a toric ideal), we see that $\mathbf{u} \neq \mathbf{u}'$, because otherwise we would conclude that $\mathbf{Y}^{\mathbf{v}} - \mathbf{Y}^{\mathbf{v}'} \in I_{\mathcal{A}}$, in contradiction with the reducibility of $\mathcal{G}_{\prec}(I_{\mathcal{A}})$. Now, for each $\mathbf{w} \in \mathbb{N}^s$ such that $\mathbf{Z}^{\mathbf{u}+\mathbf{w}} \in Q$, consider the remainder, $\mathbf{Y}^{\mathbf{w}'}\mathbf{Z}^{\mathbf{u}''}$, of $\mathbf{Z}^{\mathbf{u}'+\mathbf{w}}$ on division by $\mathcal{G}_{\prec}(I_{\mathcal{A}})$ (which may be $\mathbf{Z}^{\mathbf{u}'+\mathbf{w}}$ itself) and define the element $\mathbf{f} \in \Bbbk[\mathbf{Y}]^{\beta_0}$ whose $\sigma^{-1}(\mathbf{u}+\mathbf{w})$th and $\sigma^{-1}(\mathbf{u}'')$th coordinates are $\mathbf{Y}^{\mathbf{v}}$ and $-\mathbf{Y}^{\mathbf{v}'+\mathbf{w}'}$, respectively, and zeros elsewhere. Observe that $\psi_0(\mathbf{f}) = 0$. Let

$$\mathcal{M}' = \{\mathbf{f}_1, \ldots, \mathbf{f}_{\beta_0'}\} \subset \Bbbk[\mathbf{Y}]^{\beta_0} \tag{4.1}$$

https://doi.org/10.1017/S0004972717000612 Published online by Cambridge University Press

be the set of elements of $\Bbbk[\mathbf{Y}]^{\beta_0}$ defined as above and let M' be the $\beta_0 \times \beta_0'$ matrix whose columns are $\mathbf{f}_1, \ldots, \mathbf{f}_{\beta_0'}$.

If $I_{S_E} \neq 0$, then there exists $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}} - \mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'} \in \mathcal{G}_{<}(I_{\mathcal{A}})$ whose leading term is $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}}$ with $\mathbf{v} \neq 0$ and $\mathbf{u} = 0$. In particular, $\mathbf{u}' = 0$; otherwise the leading term would be $\mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'}$. This is the case that is not considered in [13]; observe that if S is not simplicial, then $I_{S_E} \neq 0$ and, if S is simplicial and $\mathcal{A} = E \cup B$ is a simplicial partition, then $I_{S_E} = 0$.

Let $\{g_1, \ldots, g_t\} \subset \Bbbk[\mathbf{Y}]$ be a (minimal) system of binomial generators of I_{S_E} and define the $\Bbbk[\mathbf{Y}]$-module generated by the columns of the matrix

$$N = \mathbf{1}_{\beta_0} \otimes (g_1 \ \cdots \ g_t),$$

where $\mathbf{1}_{\beta_0}$ denotes the identity matrix of size β_0 and $\otimes$ denotes the Kronecker product of matrices.

Set $\beta_1 = \beta_0' + t \cdot \beta_0$. Clearly, $M := (M'|N)$ defines a homomorphism of free $\Bbbk[\mathbf{Y}]$-modules, say $\psi_1 : \Bbbk[\mathbf{Y}]^{\beta_1} \to \Bbbk[\mathbf{Y}]^{\beta_0}$, such that $\operatorname{im}(\psi_1) \subseteq \ker(\psi_0)$. If $I_{S_E} = 0$, we take $t = 0$ and $M = M'$ (this is the case in [13, Section 1]).

Theorem 4.3. *With the notation above,* $\operatorname{im}(\psi_1) = \ker(\psi_0)$ *and* $\operatorname{coker}(\psi_1) \cong_{\Bbbk[\mathbf{Y}]} \Bbbk[S]$*, that is,* ψ_1 *is a presentation of* $\Bbbk[S]$ *as a* $\Bbbk[\mathbf{Y}]$*-module.*

Proof. By construction, it suffices to prove that $\operatorname{im}(\psi_1) \supseteq \ker(\psi_0)$.

Let $f_1, \ldots, f_{\beta_0} \in \Bbbk[\mathbf{Y}]$ be such that $\mathbf{f} = (f_1, \ldots, f_{\beta_0})^\top \in \ker(\psi_0)$, where $\top$ denotes the transpose. By hypothesis, $f = \sum_{i=1}^{\beta_0} f_i \mathbf{Z}^{\sigma(i)} \in I_{\mathcal{A}}$. Without loss of generality, we may suppose that $f_i\mathbf{Z}^{\sigma(i)}$ is homogeneous of S-degree $\mathbf{a}$ for every $i = 1, \ldots, \beta_0$.

If $f \neq 0$, then its leading term is $\mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'}$ with $\mathbf{v}' \neq 0$ and $\mathbf{u}' = \sigma(i)$ for some i. Let $g \in \mathcal{G}_{<}(I_{\mathcal{A}})$ be an element whose leading term, $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}}$, divides $\mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'}$.

If $\mathbf{u} \neq 0$, let $\mathbf{w} = \mathbf{u}' - \mathbf{u}$ and consider the element $\mathbf{f}_j \in \mathcal{M}'$ corresponding to g and $\mathbf{w}$. In this case, $\mathbf{f} - \psi_1(\varepsilon_j) := (f_1', \ldots, f_{\beta_0}')^\top \in \ker(\psi_0)$, where ε_j is the jth vector of the canonical basis of $\Bbbk[\mathbf{Y}]^{\beta_1}$, and the leading term of $f' = \sum_{j=1}^{\beta_0} f_j' \mathbf{Z}^{\sigma(i)}$ is less than the leading term of f.

On the other hand, if $\mathbf{u} = 0$, then $g = \mathbf{Y}^{\mathbf{v}} - \mathbf{Y}^{\mathbf{v}''} \in I_{S_E}$. Therefore, $g = \sum_{j=1}^{t} h_j g_j$. Let H be the $t \times \beta_0$ matrix whose ith column is $(h_1 \cdots h_t)^\top$ and define $\mathbf{h}_i = \binom{\mathbf{0}}{\operatorname{vec}(H)} \in \Bbbk[\mathbf{Y}]^{\beta_1}$, where $\operatorname{vec}(-)$ denotes the vectorisation operator and $\mathbf{0}$ is a vector of zeros. Clearly, $\mathbf{f} - \psi_1(\mathbf{h}_i) = (f_1', \ldots, f_{\beta_0}')^\top \in \ker(\psi_1)$ and the leading term of $f' = \sum_{j=1}^{\beta_0} f_j' \mathbf{Z}^{\sigma(i)}$ is less than the leading term of f.

Repeat the process on $(f_1', \ldots, f_{\beta_0}')^\top$ and so on. In each step the leading term of the corresponding polynomial in $\Bbbk[\mathbf{Y}, \mathbf{Z}]$ decreases, so the process must terminate. □

Recall that an S-graded free resolution of $\operatorname{coker}(\varphi_1)$ as a $\Bbbk[\mathbf{Y}]$-module is an acyclic complex of length $t \leq r$, namely

$$\mathcal{P} : \Bbbk[\mathbf{Y}]^{\beta_t} \xrightarrow{\psi_t} \cdots \longrightarrow \Bbbk[\mathbf{Y}]^{\beta_1} \xrightarrow{\psi_1} \Bbbk[\mathbf{Y}]^{\beta_0} \longrightarrow \operatorname{coker}(\psi_1),$$

where the maps are all homogeneous of S-degree 0. Since, by Theorem 4.3, $\operatorname{coker}(\psi_1) \cong_{\Bbbk[\mathbf{Y}]} \Bbbk[S]$, we call $\mathcal{P}$ a *Pisón's free resolution* of $\Bbbk[S]$. Both the isomorphism and ψ_1 are given explicitly, so this resolution can be effectively computed.

https://doi.org/10.1017/S0004972717000612 Published online by Cambridge University Press

Corollary 4.4. *With the notation above, if $\mathbb{Z}B$, the subgroup of $\mathbb{Z}^d$ generated by B, is contained in $S_E \cup (-S_E)$, then the ith map in the Pisón's free resolution of $\Bbbk[S]$ can be taken to be the direct sum of $\#\mathcal{Q}$ copies of the ith map in a minimal free resolution of $\Bbbk[S_E]$ for every $i > 0$.*

Proof. We claim that the set $\mathcal{M}'$ defined in (4.1) is empty. Otherwise, there exists $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}} - \mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'} \in \mathcal{G}_{\prec}(I_{\mathcal{A}})$ whose leading term is $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}}$ with $\mathbf{v} \neq 0$ and $\mathbf{u} \neq 0$. Since $\sum_{i=1}^{s}(u_i - u_i')\mathbf{b}_i \in \mathbb{Z}B$, by hypothesis, there exist $\mathbf{w}_i \in \mathbb{N}$, $i = 1, \ldots, r$, such that $\sum_{i=1}^{s}(u_i - u_i')\mathbf{b}_i = \pm \sum_{i=1}^{r} w_i\mathbf{e}_i$. Therefore, either $\sum_{i=1}^{s} u_i\mathbf{b}_i + \sum_{i=1}^{r} w_i\mathbf{e}_i = \sum_{i=1}^{s} u_i'\mathbf{b}_i$ or $\sum_{i=1}^{s} u_i\mathbf{b}_i = \sum_{i=1}^{s} u_i'\mathbf{b}_i + \sum_{i=1}^{r} w_i\mathbf{e}_i$, that is, either $\mathbf{Z}^{\mathbf{u}'} - \mathbf{Y}^{\mathbf{w}}\mathbf{Z}^{\mathbf{u}} \in I_{\mathcal{A}}$ or $\mathbf{Z}^{\mathbf{u}} - \mathbf{Y}^{\mathbf{w}}\mathbf{Z}^{\mathbf{u}'} \in I_{\mathcal{A}}$, in contradiction with the reducibility of $\mathcal{G}_{\prec}(I_{\mathcal{A}})$. □

The above condition is not necessary, as the following example shows.

Example 4.5. Let

$$A = \begin{pmatrix} 3 & 1 & 1 & 1 & 2 & 4 & 1 \\ 1 & 3 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 3 & 1 & 4 & 2 & 2 \\ 1 & 1 & 1 & 3 & 0 & 0 & 2 \end{pmatrix}$$

and consider the subsemigroup S of $\mathbb{N}^4$ generated by the columns, $\mathbf{a}_1, \ldots, \mathbf{a}_6$ and $\mathbf{b}$, of A. Set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_6, \mathbf{b}\}$ and $E = \{\mathbf{a}_1, \ldots, \mathbf{a}_6\}$. Clearly, $\mathrm{pos}(\mathcal{A}) = \mathrm{pos}(E)$ and $\mathbb{Z}B = \mathbb{Z}\mathbf{b} \not\subseteq S_E \cup (-S_E)$. The ideal $I_{\mathcal{A}} \subseteq \Bbbk[Y_1, \ldots, Y_6, Z]$ is equal to $\langle Z^2 - Y_3Y_4\rangle + I_{S_E}$. Therefore, $\mathcal{Q} = \{0, 1\}$,

$$M = \begin{pmatrix} g_1 & g_2 & g_3 & g_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_1 & g_2 & g_3 & g_4 \end{pmatrix},$$

where

$$\{g_1 = Y_1Y_5 - Y_3Y_6, g_2 = Y_1Y_3^3 - Y_2Y_4Y_5^2, g_3 = Y_1^2Y_3^2 - Y_2Y_4Y_5Y_6, g_4 = Y_1^3Y_3 - Y_2Y_4Y_6^2\}$$

is a minimal system of generators of I_{S_E}. We conclude that the Pisón's free resolution of $\Bbbk[S]$ is

$$0 \to \Bbbk[\mathbf{Y}]^2 \xrightarrow{\phi_3\oplus\phi_3} \Bbbk[\mathbf{Y}]^8 \xrightarrow{\phi_2\oplus\phi_2} \Bbbk[\mathbf{Y}]^8 \xrightarrow{\phi_1\oplus\phi_1} \Bbbk[\mathbf{Y}]^2 \xrightarrow{\psi_0} \Bbbk[S],$$

where

$$0 \to \Bbbk[\mathbf{Y}] \xrightarrow{\phi_3} \Bbbk[\mathbf{Y}]^4 \xrightarrow{\phi_2} \Bbbk[\mathbf{Y}]^4 \xrightarrow{\phi_1} \Bbbk[\mathbf{Y}] \xrightarrow{\varphi_E} \Bbbk[S_E],$$

is a minimal free resolution of $\Bbbk[S_E]$.

Lemma 4.6. *With the notation above,* $\mathrm{depth}_{\Bbbk[\mathbf{Y},\mathbf{Z}]}(\Bbbk[S]) = \mathrm{depth}_{\Bbbk[\mathbf{Y}]}(\Bbbk[S])$.

Proof. Since we are assuming that $S \cap (-S) = 0$, both $\Bbbk[\mathbf{Y}, \mathbf{Z}]$ and $\Bbbk[\mathbf{Y}]$ can be regarded as local rings with maximal ideals $\langle Y_1, \ldots, Y_r, Z_1, \ldots, Z_s\rangle$ and $\langle Y_1, \ldots, Y_r\rangle$, respectively, because of the grading given by the semigroup S. Clearly, the natural projection $\Bbbk[\mathbf{Y}, \mathbf{Z}] \to \Bbbk[\mathbf{Y}]$ is an homomorphism of local rings. So, our claim follows from [3, Exercise 1.2.26(b)]. □

COROLLARY 4.7. *With the notation above, if S is a simplicial semigroup, then $\Bbbk[S]$ is Cohen–Macaulay if and only if the generators of* $\mathrm{in}_{\prec}(I_S)$ *do not depend on* $Y_1, \dots, Y_r$.

PROOF. Since S is simplicial, we may assume that $\dim(\mathrm{pos}(\mathcal{A})) = r$ (see Observation 3.1), so the Krull dimension of $\Bbbk[S]$ equals r (see the proof of [10, Proposition 7.5]). Therefore, $\Bbbk[S]$ is Cohen–Macaulay if and only if $\mathrm{depth}_{\Bbbk[\mathbf{Y},\mathbf{Z}]}(\Bbbk[S]) = r$. Now, since $\mathrm{depth}_{\Bbbk[\mathbf{Y},\mathbf{Z}]}(\Bbbk[S]) = \mathrm{depth}_{\Bbbk[\mathbf{Y}]}(\Bbbk[S])$ by Lemma 4.6 and $\Bbbk[\mathbf{Y}] = \Bbbk[S_E]$ because $I_{S_E} = 0$ (see Remark 4.2), from the Auslander–Buchbaum formula it follows that $\Bbbk[S]$ is Cohen–Macaulay if and only if the projective dimension of $\Bbbk[S]$ as a $\Bbbk[\mathbf{Y}]$-module is 0. Equivalently,

$$\psi_0 : \Bbbk[\mathbf{Y}]^{\#Q} \cong_{\Bbbk[\mathbf{Y}]} \Bbbk[S],$$

which means that $\mathrm{in}_{\prec}(I_S)$ is minimally generated in $\Bbbk[\mathbf{Z}]$, as we can deduce from our construction. □

EXAMPLE 4.8. Let

$$\mathcal{A} = \{(6,1), (6,2), (6,3), (7,2), (7,3), (8,2), (8,3), (9,3), (10,3)\} \subset \mathbb{Z}^2$$

and $\Bbbk[\mathbf{Y},\mathbf{Z}] = \Bbbk[Y_1, Z_1, Y_2, Z_2, \dots, Z_7]$. Let $\prec$ be the S-graded reverse lexicographic term ordering on $\Bbbk[\mathbf{Y},\mathbf{Z}]$ such that $Y_1 \prec Y_2 \prec Z_1 \prec \cdots \prec Z_7$. The computation of the minimal system of generators of $\mathrm{in}_{\prec}(I_{\mathcal{A}})$ can be done with Singular [6].

```
LIB "toric.lib";
option(redSB);
intmat A[2][9] = 6,6,6,7,7,8,8,9,10,
                 1,3,2,2,3,2,3,3,3;
intmat B[9][9] = 6,6,6,7,7,8,8,9,10,
                 1,3,2,2,3,2,3,3,3,
                -1,0,0,0,0,0,0,0,0,
                 0,-1,0,0,0,0,0,0,0,
                 0,0,0,0,0,0,0,0,1,
                 0,0,0,0,0,0,0,1,0,
                 0,0,0,0,0,0,1,0,0,
                 0,0,0,0,0,1,0,0,0,
                 0,0,0,0,1,0,0,0,0;
ring r = 0, (Y1,Y2,Z1,Z2,Z3,Z4,Z5,Z6,Z7), dp;
ideal i = toric_ideal(A,"hs");
ring s = 0, (Y1,Y2,Z1,Z2,Z3,Z4,Z5,Z6,Z7), M(B);
ideal i = imap(r,i);
i = groebner(i);
ideal m = lead(i);
```

Now, since $\mathrm{in}_{\prec}(I_{\mathcal{A}}) = \langle Z_1^2, Z_1Z_2, Z_1Z_3, Z_2^2, Z_1Z_4, Z_2Z_3, Z_1Z_5, Z_3^2, Z_2Z_4, Z_2Z_5, Z_1Z_6, Z_3Z_5, Z_4^2, Z_2Z_6, Z_1Z_7, Z_5^2, Z_3Z_6, Z_2Z_7, Z_3Z_7, Z_4Z_7, Z_6^2, Z_5Z_7, Z_6Z_7, Z_7^2, Z_4Z_5Z_6\rangle$, by Corollary 4.7, we conclude that the semigroup algebra of the subsemigroup of $\mathbb{Z}^2$ generated by $\mathcal{A}$ is Cohen–Macaulay.

https://doi.org/10.1017/S0004972717000612 Published online by Cambridge University Press

As a consequence of Corollary 4.7, we obtain a formula for the Castelnouvo–Mumford regularity of I_S in terms of the set Q when S is a simplicial semigroup and $\Bbbk[S]$ is Cohen–Macaulay.

Corollary 4.9. *With the notation above, if S is a simplicial semigroup, $\Bbbk[S]$ is Cohen–Macaulay and I_S is homogeneous for the standard grading, then the Castelnouvo–Mumford regularity of I_S is*

$$\operatorname{reg}(I_S) = \max\Big\{\sum_{i=1}^{r} u_i \;\Big|\; \deg_{\mathcal{A}}(\mathbf{u}) \in Q\Big\}.$$

Proof. By the proof of Corollary 4.7, $\Bbbk[\mathbf{Y}]^{\#Q} \cong_{\Bbbk[\mathbf{Y}]} \Bbbk[S]$. Now the corollary is a particular case of [2, Theorem 16]. □

5. Combinatorial description

We end this paper by giving a new combinatorial description of Pisón's resolution. Again, we keep the notation of the previous sections.

Let

$$\mathcal{P} : \Bbbk[\mathbf{Y}]^{\beta_t} \xrightarrow{\psi_t} \cdots \longrightarrow \Bbbk[\mathbf{Y}]^{\beta_1} \xrightarrow{\psi_1} \Bbbk[\mathbf{Y}]^{\beta_0} \xrightarrow{\psi_0} \Bbbk[S]$$

be an S-graded free resolution of $\Bbbk[S]$ as a $\Bbbk[\mathbf{Y}]$-module, that is, a Pisón's free resolution of $\Bbbk[S]$. Let $\mathfrak{m}_E$ be the irrelevant ideal of $\Bbbk[\mathbf{Y}]$, $M_i = \ker(\psi_i)$, $i = 0, \ldots, t$, and $W_i(\mathbf{a}) = (M_i/\mathfrak{m}_E M_i)_{\mathbf{a}}$, with $\mathbf{a} \in S$. Since $W_i(\mathbf{a}) \cong \operatorname{Tor}_i^{\Bbbk[\mathbf{Y}]}(\Bbbk, \Bbbk[S])_{\mathbf{a}}$, the ith Betti number of $\Bbbk[S]$ of degree $\mathbf{a}$ is $\dim_{\Bbbk}(W_i(\mathbf{a}))$. Thus,

$$\beta_i = \sum_{\mathbf{a}\in S} \dim_{\Bbbk}(W_i(\mathbf{a})) \quad \text{for } i = 0, \ldots, t.$$

The abstract simplicial complexes

$$T_{\mathbf{a}} = \Big\{F \subseteq E \,\Big|\, \mathbf{a} - \sum_{\mathbf{e}\in F} \mathbf{e} \in S\Big\}$$

were introduced in [4] and used in [13] to describe the combinatorics of $\mathcal{P}$. The following result holds without assuming that S is simplicial.

Proposition 5.1 [13, Proposition 2.1]. *For every $\mathbf{a} \in S$ and $i \in \{0, \ldots, t\}$,*

$$\widetilde{H}_i(T_{\mathbf{a}}) \cong W_i(\mathbf{a}),$$

where $\widetilde{H}_i(-)$ denotes the ith reduced homology $\Bbbk$-vector space of $T_{\mathbf{a}}$.

Given $\mathbf{a} \in S$, we define

$$C_{\mathbf{a}} = \{\mathbf{Y}^{\mathbf{v}} \in \Bbbk[\mathbf{Y}] \mid \deg_{\mathcal{A}}((\mathbf{v}, \mathbf{u})) = \mathbf{a} \text{ for some } \mathbf{u} \in Q\}.$$

Let $\Gamma_{\mathbf{a}}$ be the abstract simplicial complex with vertex set $C_{\mathbf{a}}$ defined by

$$\Gamma_{\mathbf{a}} = \{F \subseteq C_{\mathbf{a}} \mid \gcd(F) \neq 1\}.$$

THEOREM 5.2. *For every* $\mathbf{a} \in S$ *and* $i = \{0, \ldots, t\}$,

$$\widetilde{H}_i(\Gamma_{\mathbf{a}}) \cong \widetilde{H}_i(T_{\mathbf{a}}).$$

PROOF. First, we claim $F \in T_{\mathbf{a}}$ if and only if there exists $\mathbf{Y}^{\mathbf{v}} \in C_{\mathbf{a}}$ with $\operatorname{supp}(\mathbf{Y}^{\mathbf{v}}) \supseteq F$. Indeed, if $F \in T_{\mathbf{a}}$, then $\mathbf{a} - \sum_{\mathbf{e} \in F} \mathbf{e} \in S$, that is, there exists $\mathbf{Y}^{\mathbf{v}'}$ with $\deg_{\mathcal{A}}((\mathbf{v}', \mathbf{u})) = \mathbf{a}$ for some $\mathbf{u} \in \mathbb{N}^s$. Now, by taking the remainder of $\mathbf{Z}^{\mathbf{u}}$ on division by $\mathcal{G}_{<}(I_{\mathcal{A}})$, we obtain a monomial $\mathbf{Y}^{\mathbf{w}}\mathbf{Z}^{\mathbf{u}'}$ with $\mathbf{u}' \in Q$. Therefore, $\mathbf{Y}^{\mathbf{v}'+\mathbf{w}} \in C_{\mathbf{a}}$ and $\operatorname{supp}(\mathbf{Y}^{\mathbf{v}'+\mathbf{w}}) \supseteq \operatorname{supp}(\mathbf{Y}^{\mathbf{v}'}) \supseteq F$. Clearly, by definition, the opposite implication is true.

Now, for each $\mathbf{Y}^{\mathbf{v}} \in C_{\mathbf{a}}$, define the simplicial complex $K_{\mathbf{v}} = \mathcal{P}(\operatorname{supp}(\mathbf{Y}^{\mathbf{v}}))$ to be the full subcomplex of $T_{\mathbf{a}}$ whose vertex set is $\operatorname{supp}(\mathbf{Y}^{\mathbf{v}})$.

By the claim above, $T_{\mathbf{a}} = \bigcup_{\mathbf{Y}^{\mathbf{v}} \in C_{\mathbf{a}}} K_{\mathbf{v}}$. So, by definition, $\mathcal{K}^{\mathbf{a}} := \{K_{\mathbf{v}} \mid \mathbf{Y}^{\mathbf{v}} \in C_{\mathbf{a}}\}$ is a cover of $T_{\mathbf{a}}$. Moreover, since $\bigcap_{i=1}^{q} K_{\mathbf{v}_i} \neq \emptyset$ if and only if $\gcd(\mathbf{Y}^{\mathbf{v}_1}, \ldots, \mathbf{Y}^{\mathbf{v}_q}) \neq 1$, by definition again, $\Gamma_{\mathbf{a}}$ is the nerve of $\mathcal{K}^{\mathbf{a}}$. (For the definitions of cover and nerve, see [10, page 94].) Finally, since each nonempty finite intersection, $\bigcap_{i=1}^{q} K_{\mathbf{v}_i}$, is a full simplex, it is acyclic. Thus, by the nerve lemma (see [10, Lemma 5.36]), we conclude that $\widetilde{H}_i(\Gamma_{\mathbf{a}}) \cong \widetilde{H}_i(T_{\mathbf{a}})$. □

From the proof of Theorem 5.2, it follows that if $\Gamma_{\mathbf{a}}$ is disconnected, we may choose $\mathbf{Y}^{\mathbf{v}}, \mathbf{Y}^{\mathbf{v}'} \in C_{\mathbf{a}}$ in different connected components of $\Gamma_{\mathbf{a}}$ so that $\mathbf{Y}^{\mathbf{v}}\mathbf{Z}^{\mathbf{u}} - \mathbf{Y}^{\mathbf{v}'}\mathbf{Z}^{\mathbf{u}'} \in I_{\mathcal{A}}$ for some $\mathbf{u}$ and $\mathbf{u}' \in Q$. Now, with the same notation as in Section 2, suppose that the $\sigma^{-1}(\mathbf{u})$th and $\sigma^{-1}(\mathbf{u}')$th coordinates of $\mathbf{f} \in \Bbbk[\mathbf{Y}]^{\beta_0}$ are $\mathbf{Y}^{\mathbf{v}}$ and $-\mathbf{Y}^{\mathbf{v}'}$, respectively, and all other coordinates are zero. The case $\mathbf{u} = \mathbf{u}'$ is not avoided. In this case, by construction, the only nonzero entry of $\mathbf{f}$ is a minimal generator of I_{S_E} in position $\sigma^{-1}(\mathbf{u})$. Putting this together with the construction of the presentation of $\Bbbk[S]$ as a $\Bbbk[\mathbf{Y}]$-module given in Section 2 shows that the isomorphisms $\widetilde{H}_0(\Gamma_{\mathbf{a}}) \cong W_0(\mathbf{a})$ are explicitly described for every $\mathbf{a} \in S$.

Finally, we give the explicit relation between the Betti numbers of the S-graded minimal free resolution of $\Bbbk[S]$ and Pisón's free resolution of $\Bbbk[S]$. Recall that $\beta_{i,\mathbf{a}}(I_{\mathcal{A}}) = \beta_{i+1,\mathbf{a}}(\Bbbk[S])$ for every $i \geq 0$.

COROLLARY 5.3. *If* $\bar{\beta}_{i,\mathbf{a}}(I_{\mathcal{A}})$ *and* $\beta_{i,\mathbf{a}}(\Bbbk[S])$ *respectively denote the ith Betti number of* $I_{\mathcal{A}} \subseteq \Bbbk[\mathbf{Y}, \mathbf{Z}]$ *and the ith Betti number of* $\Bbbk[S]$ *as a* $\Bbbk[\mathbf{Y}]$*-module, both in degree* $\mathbf{a}$*, then*

$$\bar{\beta}_{i,\mathbf{a}}(I_{\mathcal{A}}) = 0 \Longrightarrow \beta_{i-\#F,\mathbf{a}-\sum_{j \in F \subseteq B} \mathbf{b}_j}(\Bbbk[S]) = 0$$

for every $F \subseteq B$ *with* $\#F \leq i + 1$.

PROOF. For each $l \geq 0$, let $D(l) = \{\mathbf{a}' \in S \mid \dim \widetilde{H}_l(T_{\mathbf{a}}) = \beta_{l,\mathbf{a}'}(\Bbbk[S]) \neq 0\}$ and, for each $l \geq 0$, let $C_i = \{\mathbf{a} \in S \mid \mathbf{a} - \sum_{j \in F} \mathbf{b}_j \in D(i - \#F)$ for some $F \subseteq B$ with $\#F \leq i + 1\}$. By [4, Proposition 3.3] and [10, Theorem 9.2], if $\bar{\beta}_{i,\mathbf{a}}(I_{\mathcal{A}}) = 0$, then $\mathbf{a} \notin C_i$ for any $i \geq 0$ and our claim follows. □

Acknowledgements

We thank Marcel Morales, Antonio Campillo and the anonymous referees for helpful comments and suggestions.

https://doi.org/10.1017/S0004972717000612 Published online by Cambridge University Press

References

[1] E. Briales, A. Campillo, C. Marijuán and P. Pisón, 'Minimal systems of generators for ideals of semigroups', *J. Pure Appl. Algebra* **124** (1998), 7–30.

[2] E. Briales, A. Campillo, P. Pisón and A. Vigneron, 'Simplicial complexes and syzygies of lattice ideals', in: *Symbolic Computation: Solving Equations in Algebra, Geometry, and Engineering (South Hadley, MA, 2000)*, Contemporary Mathematics, 286 (American Mathematical Society, Providence, RI, 2001), 169–183.

[3] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge Studies in Advanced Mathematics, 39 (Cambridge University Press, New York, 1993).

[4] A. Campillo and P. Gimenez, 'Syzygies of affine toric varieties', *J. Algebra* **225** (2000), 142–161.

[5] T. Cortadellas Benítez, R. Jafari and S. Zarzuela Armengou, 'On the Apéry sets of monomial curves', *Semigroup Forum* **86**(2) (2013), 289–320.

[6] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, 'Singular 4-0-2—A computer algebra system for polynomial computations', 2015, http://www.singular.uni-kl.de.

[7] D. Einstein, D. Lichtblau, A. Strzebonski and S. Wagon, 'Frobenius numbers by lattice point enumeration', *Integers* **7** (2007), A15, 63 pages.

[8] P. A. García-Sánchez and J. C. Rosales, *Finitely Generated Commutative Monoids* (Nova Science, Commack, NY, 1999).

[9] G. Marquez-Campos, I. Ojeda and J. M. Tornero, 'On the computation of the Apéry set of numerical monoids and affine semigroups', *Semigroup Forum* **91**(1) (2015), 139–158.

[10] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics, 227 (Springer, New York, 2005).

[11] M. Morales and N. T. Dung, 'Gröbner basis, a "pseudo-polynomial" algorithm for computing the Frobenius number', Preprint, 2015, arXiv:1510.01973 [math.AC].

[12] I. Ojeda and A. Vigneron-Tenorio, 'Simplicial complexes and minimal free resolution of monomial algebras', *J. Pure Appl. Algebra* **214** (2010), 850–861.

[13] P. Pisón Casares, 'The short resolution of a lattice ideal', *Proc. Amer. Math. Soc.* **131**(4) (2003), 1081–1091.

[14] B. H. Roune, 'Solving thousand-digit Frobenius problems using Gröbner bases', *J. Symbolic Comput.* **43**(1) (2008), 1–7.

I. OJEDA, Departamento de Matemáticas,
Universidad de Extremadura, E-06071 Badajoz, España
e-mail: ojedamc@unex.es

A. VIGNERON-TENORIO, Departamento de Matemáticas,
Universidad de Cádiz, E-11405 Jerez de la Frontera (Cádiz), España
e-mail: alberto.vigneron@uca.es