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MAXIMUM MODULUS THEOREMS AND SCHWARZ LEMMATA FOR SEQUENCE SPACES

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1. **Introduction**. In this note, we prove analogues of the classical maximum modulus theorem and Schwarz lemma, for sequence spaces. We begin by stating these two results in a convenient way; that is for the unit disk and functions of bound one.

MAXIMUM MODULUS THEOREM. *If $f(z)$ is analytic in the disk $|z|<1$, continuous for $|z|\leq 1$ and satisfies $|f(z)|\leq 1$ on $|z|=1$, then $|f(z)|\leq 1$ for $|z|<1$.*

SCHWARZ LEMMA. *If f satisfies the conditions of the maximum modulus theorem and, in addition, satisfies $f(0)=0$, then either*

(*a*) $|f(z)| < |z|$ *for* $z \neq 0$ *and* $|f'(0)| < 1$, *or*

(*b*) $f(z) = cz$ *where c is a constant with* $|c| = 1$.

In what follows, we write $f \in MM$ if f satisfies the conditions of the maximum modulus theorem, and we write $f \in SL$ if f satisfies the conditions of the Schwarz lemma.

Further, we shall assume, whenever $x=\{x_k\}$ is a sequence of complex numbers, that $f(x)=\{f(x_k)\}$.

2. **The sequence space** s. Let s be the space of all sequences of complex numbers with

$$\|x\|_s=\sum_{k=1}^{\infty} 2^{-k}\frac{|x_k|}{1+|x_k|}.$$

Clearly $\|x\|_s\leq 1$ for all $x \in s$, and so the following result is immediate:

THEOREM 1. *If $f \in MM$ and $x \in s$ with $\|x\|_s\leq 1$, then $f(x) \in s$ and $\|f(x)\|_s\leq 1$.*

3. **The sequence spaces** m, c, **and** c_0. Let m be the space of bounded sequences with $\|x\|_m=\sup_k |x_k|$ finite; let c be the subspace of m of convergent sequences with $\|x\|_c=\|x\|_m$; and let c_0 be the subspace of c of null sequences with $\|x\|_{c_0}=\|x\|_m$.

THEOREM 2. *If $f \in MM$ and $x \in m$ with $\|x\|_m\leq 1$, then $f(x) \in m$ and $\|f(x)\|_m\leq 1$.*

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Proof. $\|x\|_m \leq 1$ implies that $|x_k| \leq 1$ and so that $|f(x_k)| \leq 1$. Thus $\|f(x)\|_m = \sup_k |f(x_k)| \leq 1$.

The same argument holds with c in place of m.

THEOREM 3. *If $f \in SL$ and $x \in c_0$ with $\|x\|_{c_0} \leq 1$, then $f(x) \in c_0$ and $\|f(x)\|_{c_0} \leq \|x\|_{c_0}$.*

This follows from the Schwarz lemma in the same way that theorem 2 follows from the maximum modulus theorem.

4. **The sequence spaces l_p.** For $p>0$, we write $x \in l_p$ if $\|x\|_{l_p} = (\sum_{k=1}^\infty |x|^p)^{1/q}$ is finite, where $q=1$ whenever $0<p<1$ and $q=p$ whenever $p \geq 1$.

THEOREM 4. *If $f \in SL$ and $x \in l_p$ with $\|x\|_{l_p} \leq 1$, then $f(x) \in l_p$ and $\|f(x)\|_{l_p} \leq \|x\|_{l_p}$.*

Proof. Since $f(0)=0$, write $f(z)=zg(z)$. It follows that $g \in MM$. Thus,

$$\begin{aligned}(\|f(x)\|_{l_p})^q &= \sum_{k=1}^\infty |f(x_k)|^p = \sum_{k=1}^\infty |x_k g(x_k)|^p \\ &\leqq \sum_{k=1}^\infty |x_k|^p \qquad (\text{since } g(x_k) \leqq 1) \\ &= (\|x\|_{l_p})^q.\end{aligned}$$

5. **The sequence space bv_0.** We write $x \in bv_0$ if $x \in c_0$ and $\|x\|_{bv_0} = \sum_{k=1}^\infty |x_k - x_{k+1}|$ is finite.

Suppose that $f(z) = \sum_{n=1}^\infty b_n z^n$. If $f \in MM$ or $f \in SL$, the radius of convergence of the McLaurin series representing f is at least one, and since f is continuous for $|z| \leq 1$, we have that $\sum_{n=1}^\infty b_n = f(1)$, provided that $\sum_{n=1}^\infty b_n$ is convergent. (See [3]).

LEMMA. *If $x \in bv_0$ and $f(z) = z^{p+1}$ $(p \in N)$, then $f(x) \in bv_0$ and $\|f(x)\|_{bv_0} \leq f(\|x\|_{bv_0})$.*

Proof. We are given that $\sum_{k=1}^\infty |x_k - x_{k+1}| < \infty$. Thus $y_n = \sum_{k=n}^\infty |x_k - x_{k+1}| \to 0$ as $n \to \infty$. Note also that $y_n - y_{n+1} = |x_n - x_{n+1}|$ and that $y_n = \sum_{k=n}^\infty |x_k - x_{k+1}| \geq |\sum_{k=n}^\infty (x_k - x_{k+1})| = |x_n|$. Thus

$$\begin{aligned}\|f(x)\|_{bv_0} &= \sum_{k=1}^\infty |x_k^{p+1} - x_{k+1}^{p+1}| \leq \sum_{k=1}^\infty |x_k - x_{k+1}| \sum_{r=0}^p |x_k|^{(p-r)} |x_{k+1}|^r \\ &\leq \sum_{k=1}^\infty (y_k - y_{k+1}) \sum_{r=0}^p y_k^{p-r} y_{k+1}^r = \sum_{k=1}^\infty (y_k^{p+1} - y_{k+1}^{p+1}) = y_1^{p+1} = (\|x\|_{bv_0})^{p+1} \\ &= f(\|x\|_{bv_0}).\end{aligned}$$

THEOREM 5. *If $f \in SL$ with $\sum_{n=1}^\infty |b_n| \leq 1$, and $x \in bv_0$ with $\|x\|_{bv_0} \leq 1$, then $f(x) \in bv_0$ and $\|f(x)\|_{bv_0} \leq \|x\|_{bv_0}$.*

Proof. Since $x_k \to 0$ and $f(0)=0$, it follows that $f(x_k)\to 0$. Using the lemma above, it follows that

$$\|f(x)\|_{bv_0} = \sum_{k=1}^{\infty} |f(x_k)-f(x_{k+1})| = \sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} b_n(x_k^n - x_{k+1}^n) \right|$$

$$\leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |b_n| \cdot |x_k^n - x_{k+1}^n| = \sum_{n=1}^{\infty} |b_n| \sum_{k=1}^{\infty} |x_k^n - x_{k+1}^n|$$

$$\leq \sum_{n=1}^{\infty} |b_n| \cdot (\|x\|_{bv_0})^n \leq \|x\|_{bv_0} \sum_{n=1}^{\infty} |b_n| \leq \|x\|_{bv_0}.$$

To illustrate how close the conditions of theorem 5 are to being necessary, consider the following example [1]*:

Suppose that $0<\alpha<3-2\sqrt{2}$, and set

$$f(z) = \frac{z(z+1)(z-\alpha)}{2(1-\alpha)} = \sum_{n=1}^{3} b_n z^n.$$

It is readily shown that $|f(z)|\leq 1$ on $|z|=1$, and that $\sum_{n=1}^{3} |b_n| = 1/(1-\alpha) > 1$. Let $x=\{1, \alpha, \alpha/2, 0, 0, \ldots\}$ so that $\|x\|_{bv_0}=1$ and $\|f(x)\|_{bv_0}=1+2\,|f(\alpha/2)|>1$.

6. **The sequence spaces** hbv_0 **and** Hbv_0. Many complex analysts hold the view that Euclidean distance is not the best distance function when working in the unit disk, and prefer to use a hyperbolic distance such as

$$D(z, w) = \left| \frac{w-z}{1-\bar{z}w} \right| \quad \text{or} \quad d(z, w) = \tfrac{1}{2} \log \frac{1+D(z, w)}{1-D(z, w)}. \qquad \text{[See e.g. 2.]}$$

For $f \in MM$, both these distances have the property that they are "distance decreasing": that is

$$D(f(z), f(w)) \leq D(z, w) \quad \text{and} \quad d(f(z), f(w)) \leq d(z, w).$$

We use these hyperbolic distances to define hyperbolic bounded variation sequence spaces as follows: we write $x \in hbv_0$ [resp. Hbv_0] if $x \in c_0$ and $\|x\|_{hbv_0} = \sum_{k=1}^{\infty} d(x_k, x_{k+1})$ [resp. $\|x\|_{Hbv_0} = \sum_{k=1}^{\infty} D(x_k, x_{k+1})$] is finite.

Because of the distance decreasing property, it is easy to show the following result:

THEOREM 6. *If* $f \in SL$ *and* $x \in hbv_0$ *with* $\|x\|_{hbv_0} \leq 1$, *then* $f(x) \in hbv_0$ *with* $\|f(x)\|_{hbv_0} \leq \|x\|_{hbv_0}$, *and the same result with* Hbv_0 *replacing* hbv_0.

7. **The sequence spaces** bs **and** cs. We write $x \in bs$ if $\|x\|_{bs} = \sup_n |\sum_{k=1}^{n} x_k|$ is finite; we write $x \in cs$ if $x \in bs$ and $\sum_{k=1}^{\infty} x_k$ is convergent and set $\|x\|_{cs} = \|x\|_{bs}$.

* It is clear that we cannot hope for necessary and sufficient conditions since we may choose x with $\|x\|_{bv_0}$ to be as small as we please. However, whether the condition $\sum_{n=1}^{\infty} |b_n| \leq 1$ is a necessary condition if we insist that $\|x\|_{bv_0} = 1$, is an open question.

We cannot prove theorems for these sequence spaces as the following examples show: Let $f(z)=z^2$.

(1) If $x=\{(-1)^k\}$, then $x \in bs$ with $\|x\|_{bs}=1$ but $f(x) \notin bs$.

(2) If $x=\{(-1)^k/k\}$, then $x \in cs$ with $\|x\|_{cs}=1$ but $f(x) \in cs$ with $\|f(x)\|_{cs}=\pi^2/6>1$.

References

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2. W. A. Veech, *A second course in complex analysis*, W. A. Benjamin Inc., New York, 1967.
3. A. Zygmund, *Trigonometric Series*, 2nd Edition, Cambridge, 1959, p. 300, Theorem 1.14.

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