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TILINGS OF THE SPHERE BY CONGRUENT QUADRILATERALS II: EDGE COMBINATION a^3b WITH RATIONAL ANGLES

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Abstract. Edge-to-edge tilings of the sphere by congruent quadrilaterals are completely classified in a series of three papers. This second one applies the powerful tool of trigonometric Diophantine equations to classify the case of a^3b-quadrilaterals with all angles being rational degrees. There are 12 sporadic and 3 infinite sequences of quadrilaterals admitting the two-layer earth map tilings together with their modifications, and 3 sporadic quadrilaterals admitting 4 exceptional tilings. Among them only three quadrilaterals are convex. New interesting non-edge-to-edge triangular tilings are obtained as a byproduct.

§1. Introduction

In an edge-to-edge tiling of the sphere by congruent quadrilaterals, the tile can only have four edge arrangements [10], [14]: a^2bc, a^2b^2, a^3b, a^4. Sakano and Akama [13] classified tilings for a^2b^2 and a^4 via Ueno and Agaoka's [15] list of triangular tilings. Tilings for a^2bc are classified in the first paper [10] of this series via the methods in [2], [16]–[18] developed for pentagonal tilings. This second paper classifies tilings for a^3b with all angles being rational multiples of π (such quadrilaterals will be simply called *rational* hereafter). We then classify tilings for a^3b with some irrational angle in the third paper [11] to complete the classification.

Recall that Coolsaet [4] classified convex rational quadrilaterals with three equal sides into 7 infinite classes and 29 sporadic examples. Akama and van Cleemput [1] initiated some explorations of degree 3 vertex types and certain forbidden cases for type a^3b, assuming also convexity.

An a^3b-quadrilateral is given by Figure 1, with normal edge a, thick edge b, and angles $\alpha, \beta, \gamma, \delta$ as indicated. The second picture is the mirror image or flip of the first. The angles determine the orientation. Conversely, the edge lengths and the orientation also determine the angles. So we may present the tiling by shading instead of indicating all angles. Throughout this paper, an a^3b-tiling is always an edge-to-edge tiling of the sphere by congruent simple quadrilaterals in Figure 1, such that all vertices have degree ≥ 3.

The first paper [10] of this series constructed a two-parameter family of *two-layer earth map tilings* by a^2bc-quadrilaterals. The 3D picture in Figure 2 shows an example: One time zone (consisting of two tiles) is outlined by the yellow line, and a cycle of 12 repeating time zones cover the sphere. All a^2-angles appear at the north/south poles. The 24 middle points of all b-edges and c-edges distribute evenly on the equator with spacing $\frac{\pi}{12}$.

We use $\alpha^k\beta^l\gamma^m\delta^n$ to mean a vertex having k copies of α, l copies of β, etc. The angle-wise vertex combination(s), abbreviated as *AVC*, is the collection of all vertices in a tiling.

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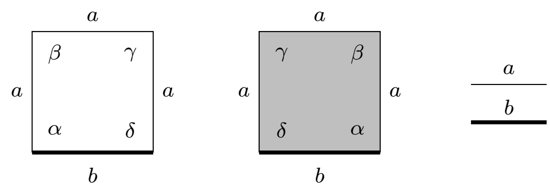

Figure 1.
Quadrilaterals with the edge combination a^3b.

Figure 2.
a^2bc-quadrilateral and a two-layer earth map tiling $T(24\alpha\beta\delta, 2\gamma^{12})$.

Then the notation $T(24\alpha\beta\delta, 2\gamma^{12})$ means the tiling has exactly 24 vertices $\alpha\beta\delta$ and 2 vertices γ^{12}, and is *uniquely* determined by them. In general, there may exist several different tilings with the same set of vertices.

The a^2bc-quadrilateral in Figure 2 reduces to the a^3b-quadrilateral in Figure 1 when $c = a$, and it is natural to expect one-parameter families of two-layer earth map a^3b-tilings. The following main theorem of this paper shows that most rational a^3b-tilings are indeed two-layer earth map tilings.

THEOREM. *There are* 15 *sporadic and* 3 *infinite sequences of rational quadrilaterals which admit a^3b-tilings (Tables 1 and 2). Except the last three sporadic cases, they are all two-layer earth map tilings $T(f\alpha\beta\delta, 2\gamma^{\frac{f}{2}})$ for some even integers $f \geq 6$, together with their modifications when β is an integer multiple of γ. The total number $\mathcal{Q}(f)$ of quadrilaterals in Tables 1 and 2 and their total number $\mathcal{T}(f)$ of different tilings are:*

f	$6, 30$	8	12	16	18	20	36	$12k$	$12k+2$	$12k+4$	$12k+6$	$12k+8$	$12k+10$
k								$2, \geq 4$	≥ 1	≥ 2	≥ 3	≥ 2	≥ 0
$\mathcal{Q}(f)$	4	1	8	4	4	5	5	3	3	3	3	3	3
$\mathcal{T}(f)$	4	1	12	14	6	13	8	6	$k+6$	$k+11$	3	$k+10$	$k+8$

In Tables 1 and 2, the angles and edge lengths are expressed in units of π, and the last column counts all vertices and also all different tilings when they are not uniquely determined by the vertices. All exact and numerical geometric data are provided in the appendix. A rational fraction, such as $\alpha = \frac{2}{9}$, means the precise value $\frac{2\pi}{9}$. A decimal expression, such as $a \approx 0.3918$, means an approximate value $0.3918\pi \leq a < 0.3919\pi$. We put π back in any trigonometric functions to avoid confusion.

Four exceptional tilings for the last three sporadic quadrilaterals in Table 1 ($f = 16, 16, 36, 36$) are shown in Figure 3. The first three tilings have repeated time zones which could be generalized combinatorially. But the quadrilaterals only exist for some particular f due to geometric constraint. We remark that the last tiling ($f = 36$) is the only tiling, among

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Table 1. Fifteen sporadic quadrilaterals and their tilings

f	$(\alpha,\beta,\gamma,\delta),a,b$	Page	All vertices and tilings
6	$(6,3,4,3)/6,1/2,1/6$	21	$6\alpha\beta\delta,2\gamma^3$
	$(1,8,4,3)/6,0.391,1$	27, 27, 30,	
	$(12,4,6,2)/9,0.567,0.174$	18	
12	$(2,10,3,6)/9,0.339,0.532$	27	$12\alpha\beta\delta,2\gamma^6$
	$(1,21,5,8)/15,0.424,0.741$	30	
	$(4,9,5,17)/15,0.424,0.165$	19	
	$(9,28,10,23)/30,0.335,0.415$	11	
	$(3,16,10,41)/30,0.469,0.146$	19	
20	$(5,32,6,23)/30,0.335,0.415$	30	$20\alpha\beta\delta,2\gamma^{10}$
	$(1,16,6,43)/30,0.469,0.273$	19	
30	$(1,42,4,17)/30,0.424,0.549$	30	$30\alpha\beta\delta,2\gamma^{15}$
18	$(3,20,4,13)/18,0.339,0.452$	29	$18\alpha\beta\delta,2\gamma^9$
		29	$16\alpha\beta\delta,2\beta\gamma^4,2\alpha\gamma^5\delta$
		29	$14\alpha\beta\delta,2\alpha^2\gamma\delta^2,4\beta\gamma^4$
16	$(1,4,2,2)/4,1/4,1/2$	34	$8\beta\delta^2,8\alpha^2\beta\gamma,2\gamma^4$: 2 tilings
36	$(5,4,7,3)/9,0.174,0.258$	16	$18\beta\gamma^2,6\alpha^3\delta,6\alpha^2\beta^2,6\alpha\beta\delta^3,2\delta^6$
36	$(15,6,10,7)/18,0.225,0.118$	12	$14\alpha^2\beta,8\alpha\delta^3,10\beta\gamma^3,6\beta^2\gamma\delta^2$

Table 2. Three infinite sequences of quadrilaterals and their tilings

$(\alpha,\beta,\gamma,\delta)$	All vertices and tilings	Page
$(\frac{4}{f},1-\frac{4}{f},\frac{4}{f},1)$	$\forall$ even $f\ge 10: f\alpha\beta\delta,2\gamma^{\frac{f}{2}}$	21
	$f=4k(k\ge 3)$: $(f-2)\alpha\beta\delta,2\alpha\gamma^{\frac{f}{4}-1}\delta,2\beta\gamma^{\frac{f}{4}+1}$	21
	$(f-4)\alpha\beta\delta,2\beta^2\gamma^2,4\alpha\gamma^{\frac{f}{4}-1}\delta$: 2 tilings	21
	$f=12$: $6\alpha\beta\delta,2\beta^3,6\alpha\gamma^2\delta$	21
$(\frac{2}{f},\frac{4f-4}{3f},\frac{4}{f},\frac{2f-2}{3f})$	$\forall$ even $f\ge 6: f\alpha\beta\delta,2\gamma^{\frac{f}{2}}$	23
	$f=6k+4(k\ge 1)$: $(f-2)\alpha\beta\delta,2\beta\gamma^{\frac{f+2}{6}},2\alpha\gamma^{\frac{f-1}{3}}\delta$	25
	$(f-4)\alpha\beta\delta,2\alpha^2\gamma^{\frac{f-4}{6}}\delta^2,4\beta\gamma^{\frac{f+2}{6}}$: $\lfloor\frac{k+2}{2}\rfloor$ tilings	25
	$(f-6)\alpha\beta\delta,2\alpha\delta^3,2\alpha^2\beta\gamma^{\frac{f-4}{6}},4\beta\gamma^{\frac{f+2}{6}}$: 3 tilings	23
$(\frac{2}{f},\frac{2f-4}{3f},\frac{4}{f},\frac{4f-2}{3f})$	$\forall$ even $f\ge 10: f\alpha\beta\delta,2\gamma^{\frac{f}{2}}$	18
	$f=6k+2(k\ge 2)$: $(f-2)\alpha\beta\delta,2\alpha\gamma^{\frac{f-2}{6}}\delta,2\beta\gamma^{\frac{f+1}{3}}$	18
	$(f-4)\alpha\beta\delta,4\alpha\gamma^{\frac{f-2}{6}}\delta,2\beta^2\gamma^{\frac{f+4}{6}}$: $\lfloor\frac{k+3}{2}\rfloor$ tilings	18
	$(f-6)\alpha\beta\delta,2\beta^3\gamma,6\alpha\gamma^{\frac{f-2}{6}}\delta$	18

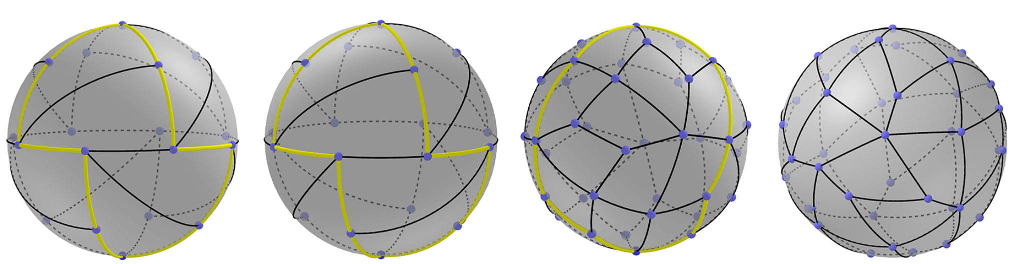

Figure 3.
Four exceptional tilings with $f=16,16,36,36$.

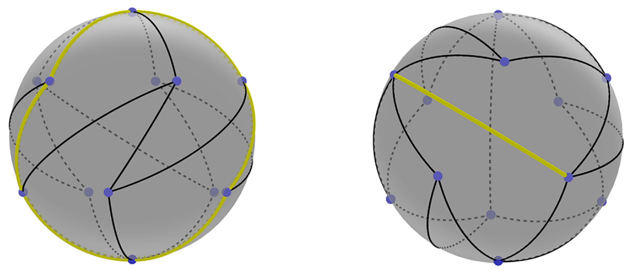

Figure 4.
Two very different tilings of Case $(1,6,2,3)/5$ in Table 2.

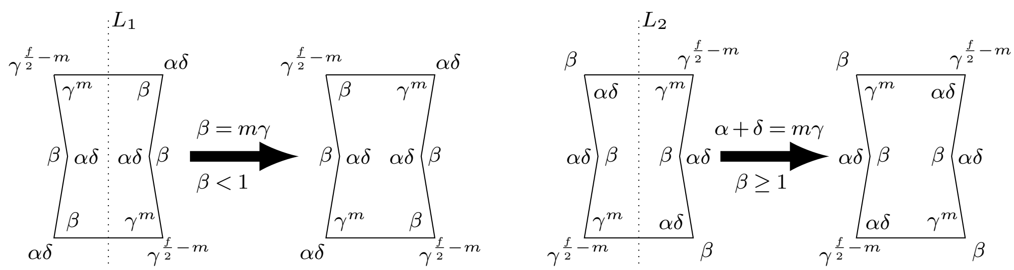

Figure 5.
Two basic flip modifications for certain two-layer earth map tilings.

all edge-to-edge triangular, quadrilateral, pentagonal tilings of the sphere, which has no apparent relation with any platonic solids or earth map tilings.

1.1 Modifications of special two-layer earth map a^3b-tilings

Once all angles are fixed, there are only finitely many combinations of them summing to 2 or form a vertex in the tiling. Then one may apply brute-force trial-and-error to find all tilings. However, the following hindsight can help us to understand most tilings in a constructive way.

It turns out that a two-layer earth map a^3b-tiling $T(f\alpha\beta\delta, 2\gamma^{\frac{f}{2}})$ admits some modification if and only if β is an integer multiple of γ. An authentic 3D picture for a two-layer earth map tiling is shown in the left of Figure 4. The structure of any two-layer earth map tiling is shown in Lemma 2.10. When $\beta = m\gamma < 1$, m continuous time zones ($2m$ tiles) form a dumb-bell like hexagon enclosed by six a-edges in the first picture of Figure 5. Simply flip along the middle vertical line L_1 (or equivalently along the middle horizontal line), and one gets a new tiling of the sphere with different vertices. This is called the first basic flip modification. When $\alpha + \delta = m\gamma \leq 1$, we get the second basic flip modification in the right of Figure 5.

A closer look at the inner and outer sides of this hexagon reveals that these two flips are essentially the same: $\alpha + \delta = m\gamma$ is equivalent to $\beta = (\frac{f}{2} - m)\gamma$, and the sphere is divided by the six a-edges into two complementary hexagons, either of which may be flipped. However, it is more convenient to flip the smaller one so that we can flip several separated regions to get more tilings. So we still use both basic flips in Figure 5 but assuming afterwards that $m \leq \frac{f}{4}$. Case $(3,20,4,13)/18$ of Table 1 and some sub-sequence of each infinite sequence of Table 2 admit two or three basic flips.

Figure 6 shows four different flips of the two-layer earth map tiling in the third case of Table 2 with $f = 14$ tiles. Flipping once, we get the first picture. Flipping twice, we get the

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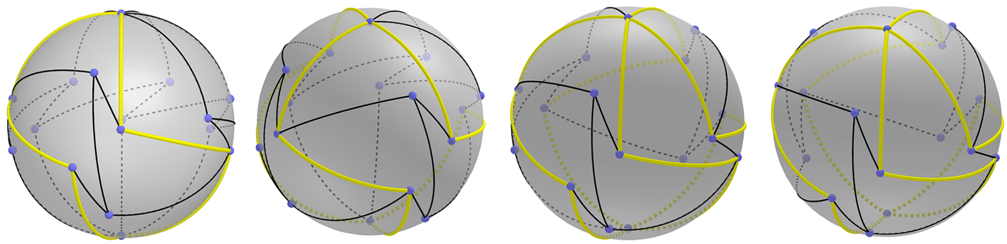

Figure 6.
Four flip modifications for Case $(1,4,2,9)/7$ in Table 2.

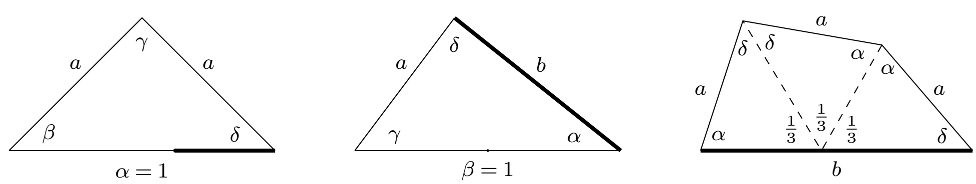

Figure 7.
The degenerate and subdivision ways to get triangular tilings.

second and third pictures when the space between two flips is 0 or 1 time zone. Flipping three times, we get the fourth picture.

For Case $(\frac{2}{f}, \frac{4f-4}{3f}, \frac{4}{f}, \frac{2f-2}{3f})$ with $f = 6k+4\,(k \geq 1)$, there is another kind of modification giving three more tilings, and we will explain it later using Figures 26 and 30. An authentic 3D picture for such a new tiling with $f = 10$ is shown in the right of Figure 4.

1.2 Non-edge-to-edge triangular tilings

When any angle of the quadrilateral is 1, it degenerates to a triangle as shown in the first two pictures of Figure 7. Then the first infinite sequence and two sporadic cases with $f = 6, 16$ produce many new examples of non-edge-to-edge triangular tilings.

The second infinite sequence of quadrilaterals satisfy $\gamma = 2\alpha, \beta = 2\delta$ and can be subdivided into three congruent triangles (observed first in [4]) as shown in the third picture of Figure 7, which also induce new non-edge-to-edge triangular tilings. Note that the sporadic case with $f = 16$ admits such subdivision too, but inducing only some edge-to-edge triangular tiling.

These are new examples, comparing to early explorations of non-edge-to-edge triangular tilings in [5]–[9].

1.3 Outline of the paper

The classification for a^2bc in [10] is mainly the analysis around a special tile. However, a^3b is handled by a new efficient method, different from all methods developed for triangular and pentagonal tilings. While the cost is to solve some trigonometric Diophantine equations, the idea behind this new method is very simple: too many linearly independent vertex types in a tiling would force all angles to be rational, or the vertex types must be very limited. This paper will identify all rational a^3b-quadrilaterals suitable for tiling. Then the third of our series (see [11]) handles the irrational angle case in a fast way due to strong constraints on vertex types.

This paper is organized as follows: Section 2 includes general results from [10] and some technical results specific to a^3b. Section 3 looks for all possible tilings from Coolsaet's list of convex rational a^3b-quadrilaterals. Sections 4 and 6 solve some trigonometric Diophantine equations to identify all *concave* rational a^3b-quadrilaterals suitable for tiling, and then find all of their tilings. Sections 5 and 7 handle two degenerate cases when the quadrilateral becomes some triangle, and thus complete the classification.

§2. Basic facts

We will always express angles in units of π radians for simplicity. So the sum of all angles at a vertex is 2. We present some basic facts and techniques in this section.

Let v, e, f be the numbers of vertices, edges, and tiles in a quadrilateral tiling. Let v_k be the number of vertices of degree k. Euler's formula $v - e + f = 2$ implies (see [10]):

$$f = 6 + \sum_{k=4}^{\infty}(k-3)v_k = 6 + v_4 + 2v_5 + 3v_6 + \dots, \tag{2.1}$$

$$v_3 = 8 + \sum_{k=5}^{\infty}(k-4)v_k = 8 + v_5 + 2v_6 + 3v_7 + \dots. \tag{2.2}$$

So $f \geq 6$ and $v_3 \geq 8$.

LEMMA 2.1 [10, Lem. 2]. *If all tiles in a tiling of the sphere by f quadrilaterals have the same four angles $\alpha, \beta, \gamma, \delta$, then*

$$\alpha + \beta + \gamma + \delta = 2 + \frac{4}{f},$$

ranging in $(2, \frac{8}{3}]$. In particular, no vertex contains all four angles.

LEMMA 2.2 [18, Lem. 3]. *If the quadrilateral in Figure 1 is simple, then $\beta < \gamma$ is equivalent to $\alpha > \delta$.*

LEMMA 2.3. *If the quadrilateral in Figure 1 is simple, then $\beta = \delta$ if and only if $\alpha = 1$. Furthermore, if it is convex with all angles < 1, then $\beta > \delta$ is equivalent to $\alpha < \gamma$, and $\beta < \delta$ is equivalent to $\alpha > \gamma$.*

Proof. If $\alpha = 1$, we get an isosceles triangle in the first picture of Figure 8, thus $\beta = \delta$. If $\beta = \delta$ and $\alpha \neq 1$, then $\angle CBD = \angle BDC$ implies $\angle ABD = \angle ADB$. So we get $a = b$, a contradiction. When the quadrilateral is convex with all angles < 1, the line AC in the second of Figure 8 is inside the quadrilateral, and divides α and γ as $\gamma = \theta + \gamma'$ and $\alpha = \theta + \alpha'$. Then

$$\alpha < \gamma \iff \alpha' < \gamma' \iff a < b.$$

By the same reason, we have $\beta > \delta \iff a < b$. Therefore, $\beta > \delta$ is equivalent to $\alpha < \gamma$. Similarly $\beta < \delta$ is equivalent to $\alpha > \gamma$. ∎

LEMMA 2.4. *If the quadrilateral in Figure 1 is simple, and $\delta \leq 1$, then $2\alpha + \beta > 1$ and $\beta + 2\gamma > 1$.*

Proof. If all angles are < 1, then the quadrilateral is convex and the line AC is inside the quadrilateral in the second picture of Figure 8. Thus $\theta < \alpha, \gamma$. This implies $2\alpha + \beta > 2\theta + \beta > 1$ and $\beta + 2\gamma > \beta + 2\theta > 1$.

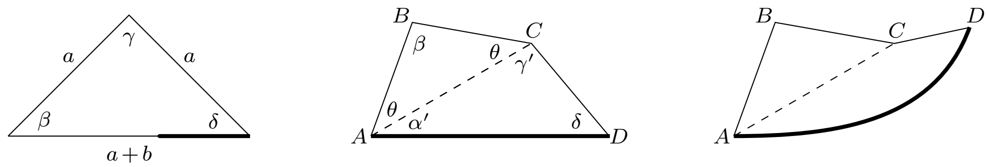

Figure 8.
Proof of Lemmas 2.3 and 2.4.

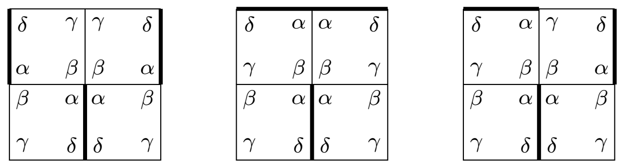

Figure 9.
Different adjacent angle deductions of $\alpha^2\beta^2$.

If $\beta \geq 1$ or both $\alpha,\gamma \geq 1$, then both inequalities certainly hold. If $\delta = 1$, then $\alpha = \gamma$ by Lemma 2.3, and $2\alpha + \beta > 1$ as the angle sum of a triangle. So we only need to consider the following two cases:

1. $\alpha,\beta,\delta < 1$ and $\gamma \geq 1$.
2. $\beta,\gamma,\delta < 1$ and $\alpha \geq 1$.

Case 1 is shown in the third picture of Figure 8, and it suffices to show $2\alpha+\beta > 1$. By $\alpha,\delta < 1$ and $AB = CD = a < 1$, both B and C lie in the interior of the same hemisphere bounded by the great circle $\bigcirc AD$. By $\beta < 1 \leq \gamma$ and $\alpha > \delta$ (Lemma 2.2), the line AC is inside the quadrilateral. Then $\alpha \geq \theta$ as in the second picture of Figure 8, and $2\alpha + \beta > 2\theta + \beta > 1$. Case 2 can be proved similarly. ∎

Lemma 2.5 (Parity Lemma, [18, Lem. 10]). *In an a^3b-tiling, the total number of ab-angles α and δ at any vertex is even.*

Lemma 2.6 (Balance Lemma, [18, Lem. 11]). *In a tiling of the sphere by f congruent tiles, each angle of the tile appears f times in total. In an a^3b-tiling, if either $\alpha^2\cdots$ or $\delta^2\cdots$ is not a vertex, then any vertex either has no α,δ, or is of the form $\alpha\delta\cdots$ with no more α,δ in the remainder.*

The very useful tool *adjacent angle deduction* (abbreviated as *AAD*) has been introduced in [17, Sec. 2.5]. We give a quick review here using Figure 9. Let "$|$" denote an a-edge and "$\pmb{|}$" denote a b-edge. Then we indicate the arrangements of angles and edges by denoting the vertices as $|\alpha\pmb{|}\alpha|\beta|\beta|$. The notation can be reversed, such as $|\alpha\pmb{|}\alpha|\beta|\beta| = |\beta|\beta|\alpha\pmb{|}\alpha|$; and it can be rotated, such as $|\alpha\pmb{|}\alpha|\beta|\beta| = \pmb{|}\alpha|\beta|\beta|\alpha\pmb{|} = |\beta|\alpha\pmb{|}\alpha|\beta|$. We also denote the first vertex in Figure 9 as $\beta|\beta\cdots$, $|\beta|\beta|\cdots,\alpha|\beta\cdots$, $|\beta|\beta|\alpha\pmb{|}\cdots$, and denote the consecutive angle segments as $\beta|\beta$, $|\beta|\beta|,\alpha|\beta$, $|\beta|\beta|\alpha\pmb{|}$.

The pictures of Figure 9 have the same vertex $|\alpha\pmb{|}\alpha|\beta|\beta|$, but different arrangements of the four tiles. To indicate the difference, we write ${}^{\lambda}\theta^{\mu}$ to mean λ,μ are the two angles adjacent to θ in the quadrilateral. The first picture has the AAD $|^{\beta}\alpha^{\delta}\pmb{|}^{\delta}\alpha^{\beta}|^{\alpha}\beta^{\gamma}|^{\gamma}\beta^{\alpha}|$. The second and third have the AAD $|^{\beta}\alpha^{\delta}\pmb{|}^{\delta}\alpha^{\beta}|^{\gamma}\beta^{\alpha}|^{\alpha}\beta^{\gamma}|$ and $|^{\beta}\alpha^{\delta}\pmb{|}^{\delta}\alpha^{\beta}|^{\alpha}\beta^{\gamma}|^{\alpha}\beta^{\gamma}|$, respectively. The following useful lemma is from [17, Lem. 10].

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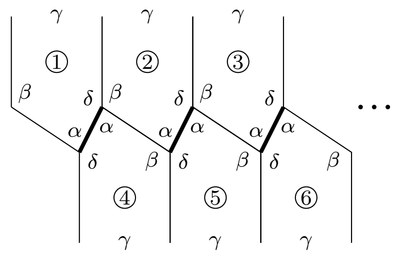

Figure 10.
A two-layer earth map tiling $T(f\alpha\beta\delta, 2\gamma^{\frac{f}{2}})$.

LEMMA 2.7. *Suppose λ and μ are the two angles adjacent to θ in a quadrilateral.*

- *If $\lambda|\lambda\cdots$ is not a vertex, then θ^n has the unique AAD $|^\lambda\theta^\mu|^\lambda\theta^\mu|^\lambda\theta^\mu|\cdots$.*
- *If n is odd, then we have the AAD $|^\lambda\theta^\mu|^\lambda\theta^\mu|$ at θ^n.*

LEMMA 2.8. *There is no tiling of the sphere by congruent quadrilaterals with two angles ≥ 1.*

Proof. If any two angles, say α,β, are greater than or equal to 1, then $\alpha\cdots=\alpha\gamma^x\delta^y(x+y\geq 2)$, $\beta\cdots=\beta\gamma^p\delta^q(p+q\geq 2)$. Given that $\#\alpha=\#\beta=f$, we deduce that $\#\gamma+\#\delta\geq 4f$, which contradicts $\#\gamma+\#\delta=2f$. ∎

PROPOSITION 2.9. *There is no tiling of the sphere by congruent symmetric a^3b-quadrilaterals ($\alpha=\delta$ and $\beta=\gamma$).*

Proof. The convex case with all angles <1 has been proved by Akama and van Cleemput in [1]. If any angle is ≥ 1, we get two angles ≥ 1 by symmetry, then Lemma 2.8 applies. ∎

LEMMA 2.10. *Assume $\gamma^{\frac{f}{2}}$ is a vertex in an a^3b-tiling. If $\beta^2\cdots$ or $\delta^2\cdots$ is not a vertex, and $\beta\delta\cdots=\alpha\beta\delta$, then the tiling must be a two-layer earth map tiling $T(f\alpha\beta\delta,2\gamma^{\frac{f}{2}})$ in Figure 10. In particular, if all β-vertices are $\alpha\beta\delta$, then the tiling must be a two-layer earth map tiling.*

Proof. By Lemma 2.7, when $\beta^2\cdots$ or $\delta^2\cdots$ is not a vertex, we have the unique AAD $\gamma^{\frac{f}{2}}=|^\beta\gamma^\delta|^\beta\gamma^\delta|\cdots$. In Figure 10, $\gamma_1\gamma_2\gamma_3\cdots$ determines T_1,T_2,T_3. Then $\beta_2\delta_1\cdots=\alpha_4\beta_2\delta_1$ determines T_4; $\beta_3\delta_2\cdots=\alpha_5\beta_3\delta_2, \alpha_2\beta_4\delta_5\cdots=\alpha_2\beta_4\delta_5$ determines T_5. The argument started at $\alpha_4\beta_2\delta_1$ can be repeated at $\alpha_5\beta_3\delta_2$. More repetitions give the unique tiling of f tiles with $2\gamma^{\frac{f}{2}}$ and $f\alpha\beta\delta$. ∎

LEMMA 2.11. *In an a^3b-tiling, if $\alpha\geq 1$, then either $\alpha\beta\delta$ or $\alpha\gamma\delta$ is a vertex, and the only other possible vertex with α or δ must be $\alpha\gamma^l\delta$ or $\alpha\beta^l\delta$, respectively, for some $l\geq 2$.*

Proof. $\alpha\geq 1$ implies $\alpha^2\cdots$ is not a vertex. Then Balance Lemma 2.6 and Lemma 2.1 imply that any vertex with α or δ must be of two types $\alpha\beta^l\delta$ or $\alpha\gamma^m\delta$. If there exists only one type, say $\alpha\beta^l\delta$, then $l=1$ by Balance Lemma 2.6. If there exist both types with $l,m\geq 2$, then the only solution satisfying Balance Lemma 2.6 is: $\{\frac{f}{2}\alpha\beta^2\delta,\frac{f}{2}\alpha\gamma^2\delta\}$. This implies $\beta=\gamma$, contradicting Proposition 2.9. Therefore, one of l,m must be 1, and the other must be ≥ 2 since $\beta\neq\gamma$. ∎

LEMMA 2.12. *In an a^3b-tiling, the a-edge and two diagonals are always <1. If both $\beta,\gamma<1$, then $b<1$.*

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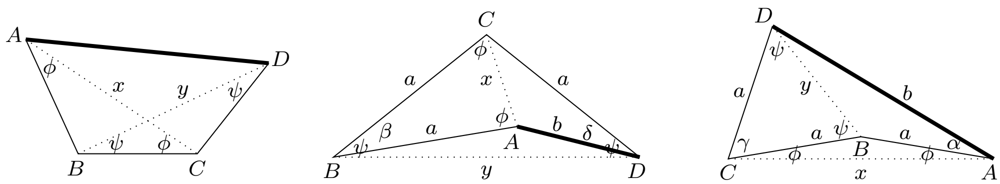

Figure 11.
Proof of Lemmas 2.12 and 2.13.

Proof. By Lemma 2.8, there are just three types of simple quadrilaterals suitable for tiling: convex with all angles <1, $\alpha \geq 1$, or $\beta \geq 1$, as shown in Figure 11. It is clear that $a<1$, otherwise BC and CD would intersect at the antipodal of C, contradicting the simpleness.

For the first two types in Figure 11, both $\beta<1$ and $\gamma<1$, then $a<1$ implies that both A and D lie in the interior of the same hemisphere bounded by the great circle $\bigcirc BC$. Therefore, two diagonals and b-edge are all <1.

For the last type in Figure 11, both $\alpha<1$ and $\delta<1$, then $a<1$ implies that both B and C lie in the interior of the same hemisphere bounded by the great circle $\bigcirc AD$. Therefore, both diagonals are <1. ∎

LEMMA 2.13. *For a^3b-quadrilaterals, the following equations (2.3) and (2.4) always hold, and one of the equations (2.5) or (2.6) must hold.*

$$\begin{aligned}\cos b = & \cos^3 a(1-\cos\beta)(1-\cos\gamma) - \cos^2 a \sin\beta\sin\gamma + \\ & \cos a(\cos\beta+\cos\gamma-\cos\beta\cos\gamma)+\sin\beta\sin\gamma;\end{aligned} \tag{2.3}$$

$$\cos a = \frac{\sin\alpha+\cos\delta\sin\gamma}{2\sin\delta\sin^2\frac{\gamma}{2}} = \frac{\sin\delta+\cos\alpha\sin\beta}{2\sin\alpha\sin^2\frac{\beta}{2}} \quad (\alpha,\delta\neq 1); \tag{2.4}$$

$$\sin(\alpha-\frac{\gamma}{2})\sin\frac{\beta}{2} = \sin\frac{\gamma}{2}\sin(\delta-\frac{\beta}{2}), \tag{2.5}$$

$$\textit{or } \sin(\alpha+\frac{\gamma}{2})\sin\frac{\beta}{2} = -\sin\frac{\gamma}{2}\sin(\delta+\frac{\beta}{2}). \tag{2.6}$$

Proof. The equation (2.3) always holds by the extended cosine law in [17, Lem. 11]. By Lemma 2.8, there are just three types of simple quadrilaterals suitable for tiling: convex with all angles <1, $\alpha\geq 1$, or $\beta\geq 1$, as shown in Figure 11.

For the first type, Lemma 2.12 implies that all edges and diagonals are <1. Therefore, all Coolsaet's assumptions in [4, Th. 2.1] hold and the equation (2.4) was proved there, which further implies either the equation (2.5) or (2.6).

It turns out Coolsaet's proof works for the other two types too. If $\alpha\geq 1$, the sine law $\frac{\sin(2-\alpha)}{\sin y} = \frac{\sin(\psi-\delta)}{\sin a}$ is equivalent to $\frac{\sin\alpha}{\sin y} = \frac{\sin(\delta-\psi)}{\sin a}$. If $\beta\geq 1$, the sine law $\frac{\sin(2-\beta)}{\sin x} = \frac{\sin(\phi)}{\sin a}$ is equivalent to $\frac{\sin\beta}{\sin x} = \frac{\sin(-\phi)}{\sin a}$. Then every step to derive the equation (2.4) is exactly the same as the first type, which further implies either the equation (2.5) or (2.6). ∎

We remark that Coolsaet also showed the equation (2.6) never holds for the first type, but it seems difficult to dismiss (2.6) for the two concave types. It is amazing that all rational solutions (rational multiples of π) to (2.5) or (2.6) can be found via the algebra of cyclotomic fields in Conway–Jones [3], as Coolsaet [4] did for convex a^3b-quadrilaterals using

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Table 3. Nongeneric solutions in Proposition 2.14

x_1	x_2	x_3	x_4
1/21	8/21	1/14	3/14
1/14	5/14	2/21	5/21
4/21	10/21	3/14	5/14
1/20	9/20	1/15	4/15
2/15	7/15	3/20	7/20
1/30	3/10	1/15	2/15
1/15	7/15	1/10	7/30
1/10	13/30	2/15	4/15

x_1	x_2	x_3	x_4
4/15	7/15	3/10	11/30
1/30	11/30	1/10	1/10
7/30	13/30	3/10	3/10
1/15	4/15	1/10	1/6
2/15	$\boxed{7/15}$	1/6	3/10
1/12	5/12	1/10	3/10
1/10	3/10	1/6	1/6

Myerson's Theorem [12] for (2.5). We summarize the algorithm as the following easy-to-use proposition.

PROPOSITION 2.14. *All solutions of* $\sin x_1 \sin x_2 = \sin x_3 \sin x_4$ *with rational angles* $0 \le x_1, x_2, x_3, x_4 \le \frac{1}{2}$ *fall into the following four cases:*

Case 1. $x_1 x_2 = x_3 x_4 = 0$.
Case 2. $\{x_1, x_2\} = \{x_3, x_4\}$.
Case 3. $\{x_1, x_2\} = \{\frac{1}{6}, \theta\}$ *and* $\{x_3, x_4\} = \{\frac{\theta}{2}, \frac{1}{2} - \frac{\theta}{2}\}$, *or* $\{x_3, x_4\} = \{\frac{1}{6}, \theta\}$ *and* $\{x_1, x_2\} = \{\frac{\theta}{2}, \frac{1}{2} - \frac{\theta}{2}\}$, *for some* $0 < \theta \le \frac{1}{2}$.
Case 4. *Up to reordering, all other solutions* x_1, x_2, x_3, x_4 *satisfying* $0 < x_1 < x_3 \le x_4 < x_2 \le \frac{1}{2}$ *are in Table 3.*

REMARK 2.15. We always have $\sin\frac{1}{6}\sin\theta = \sin\frac{\theta}{2}\sin(\frac{1}{2} - \frac{\theta}{2})$. But it is a lengthy computation to get Case 3 when we transform all angles in this formula to the range $(0, 1/2]$ for all possible ranges of θ. We omit the details here.

REMARK 2.16. After Case 1, we can assume all $x_i > 0$. Case 2 and Case 3 have a common solution $\{x_1, x_2\} = \{x_3, x_4\} = \{\frac{1}{6}, \frac{1}{3}\}$.

REMARK 2.17. The 7/15 highlighted in a box in Table 3 was 8/15 in Myerson's original table, which is an obvious typo since $8/15 > 1/2$. This typo remained in [4] but the results there were nonetheless correct.

We will use lemma/proposition n' to denote the use of lemma/proposition n after interchanging $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$.

By Lemma 2.8, the quadrilateral in our tiling can have at most one angle ≥ 1. Up to the symmetry of interchanging $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$, we need only to consider five possibilities: convex (all angles < 1), concave ($\alpha > 1$ or $\beta > 1$), or degenerate ($\alpha = 1$ or $\beta = 1$), which will be discussed in the following sections, respectively.

§3. Convex case $\alpha, \beta, \gamma, \delta < 1$

Coolsaet [4, Th. 3.2] classified simple convex *rational* quadrilateral with three equal sides into 29 sporadic examples in the first column of Table 4 and 7 infinite classes (up to interchanging $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$):

Table 4. All 29 sporadic convex rational a^3b-quadrilaterals

$(\alpha,\beta,\gamma,\delta)$	f	
(29,16,18,23)/42	84	No degree 3 vertex
(31,16,18,23)/42	42	No degree 3 vertex
(35,16,30,17)/42	12	No degree 3 vertex
(37,16,30,17)/42	21/2	f is not even
(35,18,40,17)/42	84/13	f is not even
(11,30,40,7)/42	42	No degree 3 vertex
(29,30,40,23)/42	84/19	f is not even
(49,16,42,17)/60	60	No degree 3 vertex
(53,16,42,17)/60	30	No degree 3 vertex
(21,8,26,7)/30	60	$\alpha\cdots=\alpha\beta^4\delta$
(49,18,56,17)/60	12	No degree 3 vertex
(23,10,28,9)/30	12	
(11,7,9,8)/15	12	No degree 3 vertex
(13,7,9,8)/15	60/7	f is not even
(17,14,28,9)/30	15	f is not even

$(\alpha,\beta,\gamma,\delta)$	f	
(25,16,18,19)/30	20/3	f is not even
(23,16,18,19)/30	15/2	f is not even
(25,16,22,17)/30	6	No $\alpha\cdots$
(27,16,22,17)/30	60/11	f is not even
(23,32,54,13)/60	120	No degree 3 vertex
(31,32,54,19)/60	15	f is not even
(17,16,26,11)/30	12	$\gamma\cdots=\alpha^2\gamma$
(31,36,50,23)/60	12	No degree 3 vertex
(11,9,13,8)/15	60/11	f is not even
(19,18,28,13)/30	20/3	f is not even
(25,18,28,17)/30	30/7	f is not even
(19,42,56,13)/60	24	No degree 3 vertex
(37,42,56,29)/60	60/11	f is not even
(23,22,28,19)/30	15/4	f is not even

1. $\alpha=\gamma$ and $\beta=\delta$ (and all four sides are equal);
2. $\alpha=\delta$ and $\beta=\gamma$;
3. $\alpha=\frac{\gamma}{2}$ and $\delta=\frac{\beta}{2}$, $\alpha,\delta<\frac{1}{2}$;
4. $\alpha=\frac{3\gamma}{2},\beta=\frac{1}{3}$ and $\delta=\frac{2}{3}-\frac{\gamma}{2}$, with $\frac{1}{2}<\gamma<\frac{2}{3}$;
5. $\alpha=\frac{1}{6}+\frac{\gamma}{2},\beta=2\gamma$ and $\delta=\frac{1}{2}+\frac{\gamma}{2}$, with $\frac{1}{3}<\gamma<\frac{1}{2}$;
6. $\alpha=\frac{1}{6}+\frac{\gamma}{2},\beta=2\gamma$ and $\delta=\frac{1}{2}+\frac{3\gamma}{2}=3\alpha$, with $\frac{4}{15}<\gamma<\frac{1}{3}$;
7. $\alpha=\frac{1}{6}+\frac{\gamma}{2},\beta=2-2\gamma$ and $\delta=\frac{3}{2}-\frac{3\gamma}{2}$, with $\frac{1}{2}<\gamma<\frac{5}{6}$.

In fact, Coolsaet assumed additionally that all edges and diagonals are <1, and our Lemma 2.12 shows that such assumptions are satisfied automatically for a^3b-tilings. Cases 1 and 2 are immediately dismissed due to $a\neq b$ and Proposition 2.9. In this section, we will find all possible tilings from Table 4 and from five remaining cases.

3.1 Sporadic cases in Table 4

A quadrilateral is qualified to tile the sphere only if its angle sum is $2+\frac{4}{f}$ for some even integer $f\geq 6$, every angle can be extended to a vertex, and there must also exist degree 3 vertices by the equation (2.2). These basic criteria dismiss most sporadic examples in Table 4, as the details showing in the second and third columns. There are only three subcases left. But $(21,8,26,7)/30$ implies $\alpha\cdots=\alpha\beta^4\delta$ and $(17,16,26,11)/30$ implies $\gamma\cdots=\alpha^2\gamma$, both contradicting Balance Lemma 2.6. So only $(23,10,28,9)/30$ admits a two-layer earth map tiling $T(12\alpha\gamma\delta,2\beta^6)$. In fact the only other possible vertex is $\alpha\beta\delta^3$, but Lemma 2.10′ shows that there is no other tilings. This is $f=12$, $(9,28,10,23)/30$ in Table 1 after interchanging $\alpha\leftrightarrow\delta$ and $\beta\leftrightarrow\gamma$.

3.2 Case 3. $\alpha=\frac{\gamma}{2},\delta=\frac{\beta}{2}$, $\alpha,\delta<\frac{1}{2}$

By Lemma 2.1, we get $\frac{2}{3}<\alpha+\delta\leq\frac{8}{9}$ and $\frac{4}{3}<\beta+\gamma\leq\frac{16}{9}$. Without loss of generality, let $\beta>\gamma$. So we get $\beta>\frac{2}{3}$, and $\delta>\frac{1}{3}$ by Lemma 2.3. By $\beta<1$, we get $\gamma>\frac{1}{3},\alpha>\frac{1}{6}$, and $\delta<\frac{1}{2}$. Let $R(\beta^2\cdots)$ denote the remainder or "$\cdots$" part of the angles at this vertex $\beta^2\cdots$. By $R(\beta^2\cdots)<\beta=2\delta,\gamma=2\alpha$ and Parity Lemma, there is no $\beta^2\cdots$ vertex. Similarly, there is no $\beta\delta^2\cdots$ vertex. By $\alpha<R(\beta\delta\cdots)<3\alpha$ and Parity Lemma, there is no $\beta\delta\cdots$ vertex. By $R(\beta\cdots)<3\gamma$, $\gamma=2\alpha$ and Parity Lemma, we get $\beta\cdots=\beta\gamma^2,\alpha^2\beta\gamma$ or $\alpha^4\beta$. They all satisfy $\#\alpha+\#\gamma\geq 2\#\beta$. If $\alpha^2\beta\gamma$ or $\alpha^4\beta$ is a vertex, then $\#\alpha+\#\gamma>2\#\beta$, contradicting Balance Lemma 2.6. If $\beta\cdots=\beta\gamma^2$, then $\#\gamma>\#\beta$, again a contradiction. We conclude that there is no tiling in this case.

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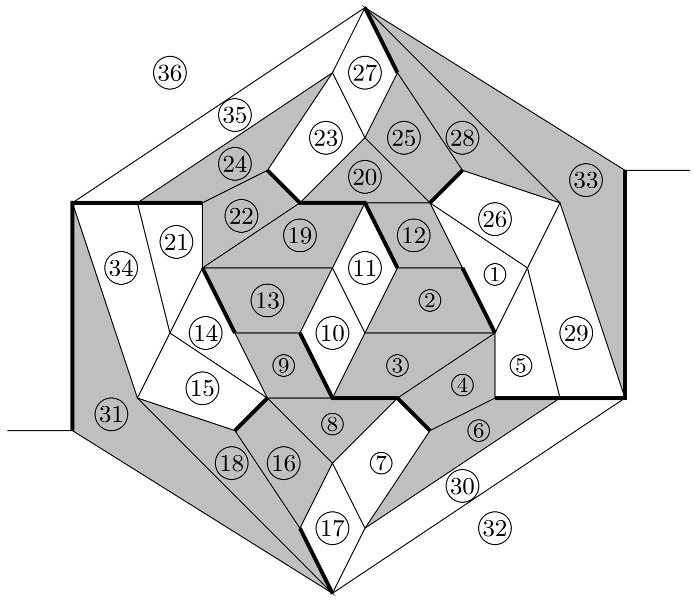

Figure 12.
$T(14\alpha^2\beta, 8\alpha\delta^3, 10\beta\gamma^3, 6\beta^2\gamma\delta^2)$.

3.3 Case 4. $\alpha = \frac{3\gamma}{2}, \beta = \frac{1}{3}, \delta = \frac{2}{3} - \frac{\gamma}{2}, \frac{1}{2} < \gamma < \frac{2}{3}$

We have $\frac{3}{4} < \alpha < 1$ and $\frac{1}{3} < \delta < \frac{5}{12}$. By $R(\gamma\cdots) < 2\alpha$, $0 < R(\alpha\gamma\delta\cdots) < \beta, \gamma, 2\delta$ and Parity Lemma, there is no $\alpha\gamma\cdots$ vertex. By $0 < R(\gamma^3\cdots) < 2\beta, \gamma, 2\delta$ and Parity Lemma, we get $\gamma^3\cdots = \beta\gamma^3$. By $2\beta < R(\gamma^2\cdots) < 3\beta, 3\delta$, $0 < R(\gamma^2\delta^2\cdots) < \beta$ and Parity Lemma, we get $\gamma^2\cdots = \beta\gamma^3$. By $R(\gamma\delta^2\cdots) = 2\beta < 2\delta$, $4\beta < R(\gamma\cdots) < 5\beta$ and Parity Lemma, we get $\gamma\cdots = \beta\gamma^3$ or $\beta^2\gamma\delta^2$. By Balance Lemma, $\beta\gamma^3$ is a vertex. Therefore, $\alpha = \frac{5}{6}, \beta = \frac{1}{3}, \gamma = \frac{5}{9}$ and $\delta = \frac{7}{18}$. Then we get $f = 36$. By Parity Lemma, we get the $AVC \subset \{\alpha^2\beta, \alpha\delta^3, \beta\gamma^3, \beta^2\gamma\delta^2, \beta^6\}$.

If β^6 is a vertex, we have the AAD $\beta^6 = |^{\gamma}\beta^{\alpha}|^{\alpha}\beta^{\gamma}|\cdots$ or $|^{\gamma}\beta^{\alpha}|^{\gamma}\beta^{\alpha}|\cdots$. This gives a vertex $\alpha|\alpha\cdots$ or $\alpha\gamma\cdots$, contradicting the AVC. Then there is only one solution satisfying Balance Lemma 2.6: $\{14\alpha^2\beta, 8\alpha\delta^3, 10\beta\gamma^3, 6\beta^2\gamma\delta^2\}$.

In Figure 12, we have the unique AAD $\beta^2\gamma\delta^2 = |^{\gamma}\delta_1^{\alpha}|^{\alpha}\delta_2^{\gamma}|^{\gamma}\beta_3^{\alpha}|^{\delta}\gamma_4^{\beta}|^{\alpha}\beta_5^{\gamma}|$ which determines T_1, T_2, T_3, T_4, T_5. Then $\alpha_5\beta_4\cdots = \alpha^2\beta$ determines T_6; $\alpha_4\beta_6\cdots = \alpha^2\beta$ determines T_7; $\alpha_1\delta_4\delta_7\cdots = \alpha\delta^3$ determines T_8; $\alpha_8\delta_3\cdots = \alpha\delta^3$ determines T_9, T_{10}. We have $\gamma_2\gamma_3\gamma_{10}\cdots = |\gamma_2|\gamma_3|\gamma_{10}|^{\gamma}\beta_{11}^{\alpha}|$ or $|\gamma_2|\gamma_3|\gamma_{10}|^{\alpha}\beta_{11}^{\gamma}|$. We might as well take $\gamma_2\gamma_3\gamma_{10}\cdots = |\gamma_2|\gamma_3|\gamma_{10}|^{\gamma}\beta_{11}^{\alpha}|$ which determines T_{11}. Similarly, we can determine $T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17}$ and T_{18}. We have $\beta_{10}\gamma_{11}\gamma_{13}\cdots = |\gamma_{11}|\beta_{10}|\gamma_{13}|^{\beta}\gamma_{19}^{\delta}|$ or $|\gamma_{11}|\beta_{10}|\gamma_{13}|^{\delta}\gamma_{19}^{\beta}|$. We might as well take $\beta_{10}\gamma_{11}\gamma_{13}\cdots = |\gamma_{11}|\beta_{10}|\gamma_{13}|^{\beta}\gamma_{19}^{\delta}|$ which determines T_{19}. Similarly, we can determine $T_{20}, T_{21}, \ldots, T_{36}$. For other choices of $\gamma_2\gamma_3\gamma_{10}\cdots$ and $\beta_{10}\gamma_{11}\gamma_{13}\cdots$, we still get this tiling or its equivalent opposite. This is Case $(15,6,10,7)/18$ in Table 1.

3.4 Case 5. $\alpha = \frac{1}{6} + \frac{\gamma}{2}, \beta = 2\gamma, \delta = \frac{1}{2} + \frac{\gamma}{2}, \frac{1}{3} < \gamma < \frac{1}{2}$

We have $\frac{1}{3} < \alpha < \gamma < \frac{1}{2}$ and $\frac{2}{3} < \delta < \beta < 1$. By $R(\beta^2\cdots) < 2\alpha, \beta, 2\gamma, \delta$ and Parity Lemma, we get $\beta^2\cdots = \beta^2\gamma$. By $R(\alpha\beta\delta\cdots) <$ all angles, $0 < R(\alpha^2\beta\cdots) < 2\alpha, 2\gamma$, $2\gamma < R(\beta\cdots) < 4\gamma, 2\delta$ and Parity Lemma, we get $\beta\cdots = \alpha\beta\delta, \beta^2\gamma, \alpha^2\beta\gamma$ or $\beta\gamma^3$. However, $\alpha\beta\delta, \beta^2\gamma$ or $\beta\gamma^3$ implies $f = 9$ or 15, contradicting the fact that f is even. So we have only $\beta\cdots = \alpha^2\beta\gamma$ with

Table 5. The AVC for $\alpha=\frac{7}{12}-\frac{1}{f},\beta=\frac{1}{3}+\frac{4}{f},\gamma=\frac{5}{6}-\frac{2}{f},\delta=\frac{1}{4}+\frac{3}{f}$

f	Vertex
All	$\beta\gamma^2$, $\alpha^3\delta$
20	$\alpha\beta^2\delta$
24	β^4, $\beta\gamma\delta^2$, $\beta\delta^4$
36	$\alpha^2\beta^2$, $\alpha\beta\delta^3$, $\beta^3\delta^2$, δ^6
60	$\beta^3\gamma$, β^5, $\beta^2\delta^4$, $\gamma\delta^4$
84	$\alpha\beta^3\delta$, $\alpha\delta^5$
132	$\beta^4\delta^2$, $\beta\delta^6$

$f=12$. But this again contradicts Balance Lemma 2.6. We conclude that there is no tiling in this case.

3.5 Case 6. $\alpha=\frac{1}{6}+\frac{\gamma}{2},\beta=2\gamma,\delta=\frac{1}{2}+\frac{3\gamma}{2}=3\alpha,\frac{4}{15}<\gamma<\frac{1}{3}$

We have $\frac{3}{10}<\alpha<\frac{1}{3},\frac{8}{15}<\beta<\frac{2}{3}$ and $\frac{9}{10}<\delta<1$. By $R(\delta^2\cdots)<$ all angles, there is no $\delta^2\cdots$ vertex. By $0<R(\alpha^3\delta\cdots)<$ all angles, $0<R(\alpha\beta\delta\cdots)<$ all angles, $2\gamma<R(\alpha\delta\cdots)<3\gamma$ and Parity Lemma, there is no $\delta\cdots$ vertex, a contradiction. We conclude that there is no tiling in this case.

3.6 Case 7. $\alpha=\frac{1}{6}+\frac{\gamma}{2},\beta=2-2\gamma,\delta=\frac{3}{2}-\frac{3\gamma}{2},\frac{1}{2}<\gamma<\frac{5}{6}$

By Lemma 2.1, we have $\alpha=\frac{7}{12}-\frac{1}{f},\beta=\frac{1}{3}+\frac{4}{f},\gamma=\frac{5}{6}-\frac{2}{f}$ and $\delta=\frac{1}{4}+\frac{3}{f}$. So we have $\frac{5}{12}<\alpha<\gamma<\frac{5}{6}$ and $\frac{1}{4}<\delta<\beta<1$.

If $\beta>\gamma$, then we get $6<f<12$. So we have $\frac{5}{12}<\alpha<\frac{1}{2}<\delta<\frac{3}{4}$ and $\frac{1}{2}<\gamma<\frac{2}{3}<\beta<1$. By $R(\beta^2\cdots)<2\alpha,\beta,\gamma,2\delta$ and Parity Lemma, there is no $\beta^2\cdots$ vertex. By $0<R(\alpha^2\beta\cdots)<$ all angles, $R(\alpha\beta\delta\cdots)<$ all angles, $R(\beta\delta^2\cdots)<$ all angles, $R(\beta\cdots)=2\gamma$ and Parity Lemma, we get $\beta\cdots=\alpha\beta\delta,\beta\gamma^2$ or $\beta\delta^2$. Suppose $\alpha\beta\delta$ or $\beta\delta^2$ is a vertex. Then we get $f=\frac{36}{5}$ or $\frac{60}{7}$, a contradiction. So we have $\beta\cdots=\beta\gamma^2$. But this again contradicts Balance Lemma 2.6.

Therefore, $\beta<\gamma$, then we get $f>12$. So we have $\frac{1}{4}<\delta<\frac{1}{2}<\alpha<\frac{7}{12}$ and $\frac{1}{3}<\beta<\frac{2}{3}<\gamma<\frac{5}{6}$.

If $\alpha^k\beta^l\gamma^m\delta^n$ is a vertex, then we have

$$(\tfrac{7}{12}-\tfrac{1}{f})k+(\tfrac{1}{3}+\tfrac{4}{f})l+(\tfrac{5}{6}-\tfrac{2}{f})m+(\tfrac{1}{4}+\tfrac{3}{f})n=2.$$

We also have $\alpha>\frac{1}{2},\beta>\frac{1}{3},\gamma>\frac{2}{3},\delta>\frac{1}{4}$. This implies $k\le 3,l\le 5,m\le 2,n\le 7$. We substitute the finitely many combinations of exponents satisfying the bounds into the equation above and solve for *f*. By the angle values and Parity Lemma, we get all possible *AVC* in Table 5. Its first row "$f=$ all" means that the angle combinations can be vertices for any f; all other rows are mutually exclusive. Note that the $AVC\subset\{\beta\gamma^2,\alpha^3\delta\}$ in the first row admits no solution satisfying Balance Lemma 2.6. All possible tilings based on the other subcases are deduced as follows.

3.6.1. Table 5, $f=20,132$

For $f=20$, the $AVC\subset\{\beta\gamma^2,\alpha^3\delta,\alpha\beta^2\delta\}$ admits no solution satisfying Balance Lemma 2.6.

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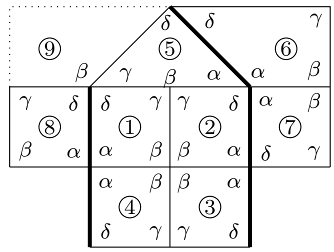

Figure 13.
$f=24$, the $AVC=\{\beta\gamma^2,\alpha^3\delta,\beta^4,\beta\gamma\delta^2\}$ admit no tiling.

For $f=132$, there is no $\alpha\gamma\cdots$ vertex. Then $\beta\delta^6$ cannot be a vertex, since its AAD gives $\alpha\gamma\cdots$. So we get the $AVC\subset\{\beta\gamma^2,\alpha^3\delta,\beta^4\delta^2\}$, which admits no solution satisfying Balance Lemma 2.6.

3.6.2. Table 5, $f=24$

By Table 5, there is no $\alpha\gamma\cdots$ vertex. Then $\beta\delta^4$ cannot be a vertex, since its AAD gives $\alpha\gamma\cdots$. So we get the $AVC\subset\{\beta\gamma^2,\alpha^3\delta,\beta^4,\beta\gamma\delta^2\}$, which admits a unique solution satisfying Balance Lemma 2.6: $\{8\beta\gamma^2,8\alpha^3\delta,2\beta^4,8\beta\gamma\delta^2\}$.

In Figure 13, by the AVC, we know $\alpha\gamma\cdots$ is not a vertex. So we have the AAD $\beta^4=|^{\alpha}\beta_1^{\gamma}|^{\gamma}\beta_2^{\alpha}|^{\alpha}\beta_3^{\gamma}|^{\gamma}\beta_4^{\alpha}|$. This determines T_1,T_2,T_3,T_4. Then $\gamma_1\gamma_2\cdots=\beta_5\gamma_1\gamma_2$ determines T_5. Then $\alpha_5\delta_2\cdots=\alpha_5\alpha_6\alpha_7\delta_2$ determines T_6,T_7. Then $\gamma_5\delta_1\cdots=\beta_9\gamma_5\delta_1\delta_8$ determines T_8. By β_9, we have $\alpha_9\gamma_8\cdots$ or $\alpha_9\delta_5\delta_6\cdots$, contradicting the AVC.

3.6.3. Table 5, $f=60$

By Table 5, there is no $\alpha\gamma\cdots$ vertex. Then β^5 cannot be a vertex, since its AAD gives $\alpha\gamma\cdots$. So we get the $AVC\subset\{\beta\gamma^2,\alpha^3\delta,\beta^3\gamma,\gamma\delta^4,\beta^2\delta^4\}$. By Lemma 2.7, the AAD of $\beta^2\delta^4$ must be $\Vert^{\alpha}\delta^{\gamma}|^{\gamma}\delta^{\alpha}\Vert\cdots$. This gives a vertex $|^{\beta}\gamma^{\delta}|^{\delta}\gamma^{\beta}|\cdots$. By the AVC, we have $|^{\beta}\gamma^{\delta}|^{\delta}\gamma^{\beta}|\cdots=|^{\beta}\gamma^{\delta}|^{\delta}\gamma^{\beta}|^{\alpha}\beta^{\gamma}|$. This gives a vertex $\alpha\beta\cdots$, contradicting the AVC. Therefore, $\beta^2\delta^4$ is not a vertex. Similarly, $\gamma\delta^4$ is not a vertex. Then $\delta\cdots=\alpha^3\delta$, contradicting Balance Lemma 2.6.

3.6.4. Table 5, $f=84$

By Table 5, we get the $AVC\subset\{\beta\gamma^2,\alpha^3\delta,\alpha\beta^3\delta,\alpha\delta^5\}$, which admits a unique solution satisfying Balance Lemma 2.6: $\{42\beta\gamma^2,20\alpha^3\delta,14\alpha\beta^3\delta,10\alpha\delta^5\}$. Since $\alpha\gamma\cdots$ is not a vertex, we have the AAD $\alpha\beta^3\delta=|^{\beta}\alpha^{\delta}\Vert^{\alpha}\delta^{\gamma}|^{\gamma}\beta^{\alpha}|^{\alpha}\beta^{\gamma}|^{\gamma}\beta^{\alpha}|$. This gives a vertex $|^{\delta}\gamma^{\beta}|^{\beta}\gamma^{\delta}|\cdots$. By the AVC, we have $|^{\delta}\gamma^{\beta}|^{\beta}\gamma^{\delta}|\cdots=|^{\delta}\gamma^{\beta}|^{\beta}\gamma^{\delta}|^{\gamma}\beta^{\alpha}|$. This gives a vertex $\gamma\delta\cdots$, contradicting the AVC.

3.6.5. Table 5, $f=36$

By Table 5, there is no $\alpha\gamma\cdots$ vertex. Then $\beta^3\delta^2$ cannot be a vertex, since its AAD gives $\alpha\gamma\cdots$. So we get the $AVC\subset\{\beta\gamma^2,\alpha^3\delta,\alpha^2\beta^2,\alpha\beta\delta^3,\delta^6\}$, and $\alpha=\frac{5}{9},\beta=\frac{4}{9},\gamma=\frac{7}{9},\delta=\frac{1}{3}$. If δ^6 is not a vertex, there is only one solution satisfying Balance Lemma 2.6: $\{18\beta\gamma^2,6\alpha^3\delta,4\alpha^2\beta^2,10\alpha\beta\delta^3\}$.

In Figure 14, we have the unique AAD $\alpha^3\delta=\Vert\alpha_1|\alpha_2\Vert\alpha_3|\delta_4\Vert$ which determines T_1,T_2,T_3,T_4. Then $\beta_3\gamma_4\cdots=\beta_3\gamma_4\gamma_5$. By γ_5, $\gamma_3\cdots=\beta_5\gamma_3\cdots$ determines T_5. Then $\beta_1\beta_2\cdots=\alpha_6\alpha_7\beta_1\beta_2$

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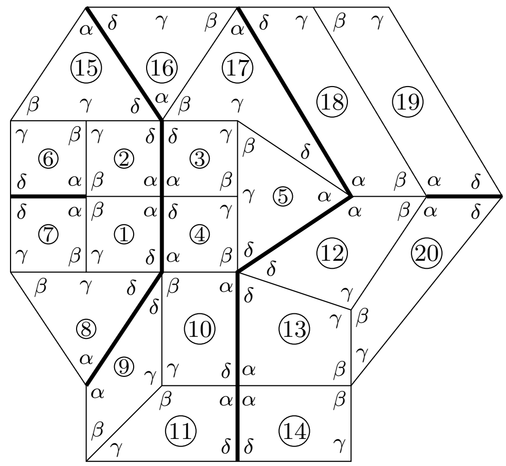

Figure 14.
$f = 36$, the $AVC = \{\beta\gamma^2, \alpha^3\delta, \alpha^2\beta^2, \alpha\beta\delta^3\}$ admit no tiling.

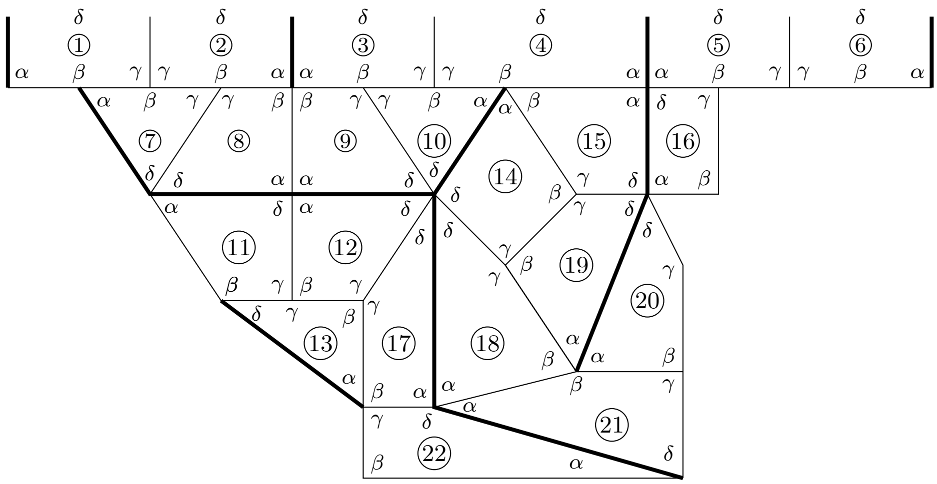

Figure 15.
Case $\alpha_2\alpha_3\cdots = \alpha^2\beta^2$ admits no tiling.

determines T_6, T_7; $\beta_7\gamma_1\cdots = \beta_7\gamma_1\gamma_8$. By γ_8, $\gamma_7\cdots = \beta_8\gamma_7\cdots$ determines T_8. Then $\alpha_4\delta_1\delta_8\cdots = \alpha_4\beta_{10}\delta_1\delta_8\delta_9$ determines T_9. By β_{10}, $\gamma_9\cdots = \beta_{11}\gamma_9\gamma_{10}$ determines T_{10}. By β_{11}, $\delta_{10}\cdots = \alpha_{11}\delta_{10}\cdots$ determines T_{11}. Then $\alpha_{10}\beta_4\delta_5\cdots = \alpha_{10}\beta_4\delta_5\delta_{12}\delta_{13}$ determines T_{12}, T_{13}; $\alpha_{11}\alpha_{13}\delta_{10}\cdots = \alpha_{11}\alpha_{13}\alpha_{14}\delta_{10}$ determines T_{14}; $\beta_6\gamma_2\cdots = \beta_6\gamma_2\gamma_{15}$. By γ_{15}, $\gamma_6\cdots = \beta_{15}\gamma_6\cdots$ determines T_{15}. Then $\delta_2\delta_3\delta_{15}\cdots = \alpha_{16}\beta_{17}\delta_2\delta_3\delta_{15}$ determines T_{16}; $\beta_5\gamma_3\cdots = \beta_5\gamma_3\gamma_{17}$ determines T_{17}; $\alpha_5\alpha_{12}\delta_{17}\cdots = \alpha_5\alpha_{12}\alpha_{18}\delta_{17}$ determines T_{18}; $\beta_{12}\beta_{18}\cdots = \alpha_{19}\alpha_{20}\beta_{12}\beta_{18}$ determines T_{19}; $\beta_{20}\gamma_{12}\gamma_{13}\cdots = \beta_{20}\gamma_{12}\gamma_{13}$ determines T_{20}. Then we get $\beta_{13}\beta_{14}\gamma_{20}\cdots$, contradicting the *AVC*.

Therefore, δ^6 is a vertex. We have the unique AAD for $\delta^6 = \Vert\delta_1|\delta_2\Vert\cdots$ which determines $T_1, T_2, T_3, T_4, T_5, T_6$. Then $\gamma_1\gamma_2\cdots = \beta_7\gamma_1\gamma_2$ determines T_7. So $\alpha_2\alpha_3\cdots = \alpha^2\beta^2$ or $\alpha^3\delta$, shown in Figures 15 and 16, respectively.

In Figure 15, $\alpha_2\alpha_3\cdots = \alpha_2\alpha_3\beta_8\beta_9$. Then $\beta_2\gamma_7\cdots = \beta_2\gamma_7\gamma_8$ determines T_8. By β_9, $\alpha_8\cdots = \alpha_8\alpha_9\cdots$ determines T_9. Then $\gamma_3\gamma_4\cdots = \beta_{10}\gamma_3\gamma_4, \beta_3\gamma_9\cdots = \beta_3\gamma_9\gamma_{10}$ determine

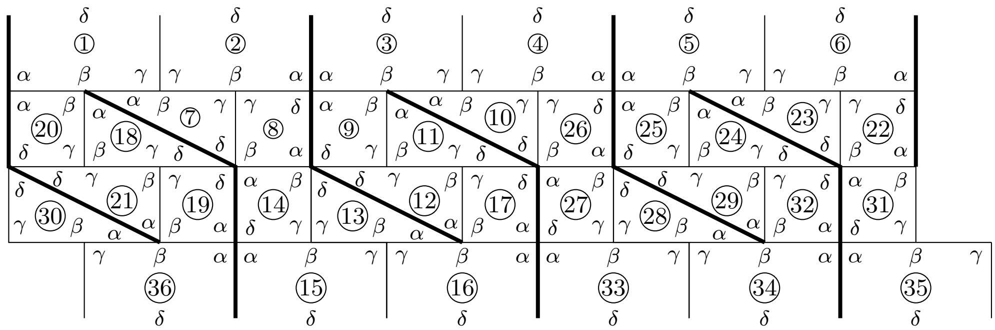

Figure 16.
Case $\alpha_2\alpha_3\cdots=\alpha^3\delta$ admits $T(18\beta\gamma^2,6\alpha^3\delta,6\alpha^2\beta^2,6\alpha\beta\delta^3,2\delta^6)$.

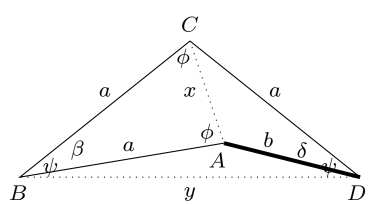

Figure 17.
a^3b-quadrilateral with $\alpha>1,\beta,\gamma,\delta<1$.

T_{10}; $\|\alpha_8|\alpha_9\|\cdots=\alpha_8\alpha_9\alpha_{12}\delta_{11}$ determines T_{11},T_{12}; $\beta_{12}\gamma_{11}\cdots=\beta_{12}\gamma_{11}\gamma_{13}$. By γ_{13}, $\gamma_{12}\cdots=\beta_{13}\gamma_{12}\cdots$ determines T_{13}. We have $\alpha_{10}\beta_4\cdots=\alpha^2\beta^2$ or $\alpha\beta\delta^3$. If $\alpha_{10}\beta_4\cdots=\alpha\beta\delta^3$, then we get $\alpha_4\alpha_5\gamma\cdots$, contradicting the *AVC*. Therefore, $\alpha_{10}\beta_4\cdots=\alpha_{10}\alpha_{14}\beta_4\beta_{15}$. This determines T_{14}. By β_{15}, $\alpha_4\alpha_5\cdots=\alpha_4\alpha_5\alpha_{15}\delta_{16}$ determines T_{15},T_{16}; $\delta_9\delta_{10}\delta_{12}\delta_{14}\cdots=\delta_9\delta_{10}\delta_{12}\delta_{14}\delta_{17}\delta_{18}$ determines T_{17},T_{18}; $\beta_{14}\gamma_{15}\cdots=\beta_{14}\gamma_{15}\gamma_{19}$, $\gamma_{14}\gamma_{18}\cdots=\beta_{19}\gamma_{14}\gamma_{18}$ determine T_{19}; $\alpha_{16}\delta_{15}\delta_{19}\cdots=\alpha_{16}\delta_{15}\delta_{19}\delta_{20}\cdots$ determines T_{20}; $\alpha_{19}\alpha_{20}\beta_{18}\cdots=\alpha_{19}\alpha_{20}\beta_{18}\beta_{21}$. By β_{21}, $\alpha_{17}\alpha_{18}\cdots=\alpha_{17}\alpha_{18}\alpha_{21}\delta_{22}$ determines T_{21},T_{22}. Then we get $\alpha_{13}\beta_{17}\gamma_{22}\cdots$, contradicting the *AVC*.

In Figure 16, $\alpha_2\alpha_3\cdots=\alpha^3\delta$. We have $\beta_2\gamma_7\cdots=\beta_2\gamma_7\gamma_8$. By γ_8, $\alpha_2\alpha_3\cdots=\alpha_2\alpha_3\alpha_9\delta_8$ determines T_8,T_9. Then $\beta_3\beta_9\cdots=\alpha_{10}\alpha_{11}\beta_3\beta_9$ determines T_{10},T_{11}; $\beta_{11}\gamma_9\cdots=\beta_{11}\gamma_9\gamma_{12}$. By γ_{12}, $\gamma_{11}\cdots=\beta_{12}\gamma_{11}\cdots$ determines T_{12}. Then $\alpha_8\delta_9\delta_{12}\cdots=\alpha_8\beta_{14}\delta_9\delta_{12}\delta_{13}$ determines T_{13}. By β_{14}, $\gamma_{13}\cdots=\beta_{15}\gamma_{13}\gamma_{14}$ determines T_{14}. By β_{15}, $\delta_{14}\cdots=\alpha_{15}\delta_{14}\cdots$ determines T_{15}. We have $\alpha_{12}\alpha_{13}\cdots=\alpha^3\delta$ or $\alpha^2\beta^2$. If $\alpha_{12}\alpha_{13}\cdots=\alpha^3\delta$, then we get $\beta_{13}\gamma_{15}\beta\cdots$ or $\beta_{12}\gamma_{11}\beta\cdots$, contradicting the *AVC*. Therefore, $\alpha_{12}\alpha_{13}\cdots=\alpha^2\beta^2$. Then $\beta_{13}\gamma_{15}\cdots=\beta_{13}\gamma_{15}\gamma_{16},\beta_{12}\gamma_{11}\cdots=\beta_{12}\gamma_{11}\gamma_{17}$ determine T_{16},T_{17}. The argument started at T_7 can be repeated at T_{10}. Two repetitions give a unique tiling $T(18\beta\gamma^2,6\alpha^3\delta,6\alpha^2\beta^2,6\alpha\beta\delta^3,2\delta^6)$. This is Case $(5,4,7,3)/9$ in Table 1.

§4. Concave case $\alpha>1$

An a^3b-quadrilateral with $\alpha>1,\beta,\gamma,\delta<1$ is shown in Figure 17, where $\phi=\angle ACB=\angle BAC$ and $\psi=\angle BDC=\angle CBD$. We first prove some basic facts. Recall that Lemma 2.4 implies $\beta+2\gamma>1$.

LEMMA 4.1. *In an a^3b-tiling with $\alpha>1$, we have $a>b$, $\alpha>1>\gamma>\beta>\delta$, $\gamma>\frac{1}{3}$ and $\delta<\frac{1}{2}$.*

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Proof. $\alpha > 1 > \gamma$ implies $\angle CAD = \alpha - \phi > \gamma - \phi = \angle ACD$. So $a > b$. Then $\angle ABD < \angle ADB$. By $\angle CBD = \angle BDC$, we get $\beta > \delta$. By Lemma 2.2, we have $\beta < \gamma$. By Lemma 2.4, we have $\beta + 2\gamma > 1$, so $\gamma > \frac{1}{3}$.

If $\delta \ge \frac{1}{2}$, by $\alpha > 1 > \gamma > \beta > \delta$, the sum of α with any two angles is > 2 and there is no $\alpha\cdots$ vertex, a contradiction. ∎

LEMMA 4.2. *In an a^3b-tiling with $\alpha > 1$, if $\alpha\beta\delta$ appears, then $\gamma = \frac{2}{3}, \frac{1}{2}$ or $\frac{2}{5}$.*

Proof. By $\alpha + \beta + \delta = 2$, we get $\gamma = \frac{4}{f}$ with $f \ge 6$ being even. By Lemma 4.1, $\gamma > \frac{1}{3}$. So $f < 12$. Then $f = 6, 8, 10$ and $\gamma = \frac{2}{3}, \frac{1}{2}$ or $\frac{2}{5}$. ∎

To find rational a^3b-quadrilaterals by solving (2.5) or (2.6) via Proposition 2.14, we have to transform $\alpha - \frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \delta - \frac{\beta}{2}, \alpha + \frac{\gamma}{2}, -\delta - \frac{\beta}{2}$ to the range $[0, \frac{1}{2}]$. For (2.5), by Lemma 4.1, we have $\frac{1}{2} < \alpha - \frac{\gamma}{2} < 2$, $0 < \frac{\beta}{2}, \frac{\gamma}{2} < \frac{1}{2}$, $-\frac{1}{2} < \delta - \frac{\beta}{2} < \frac{1}{2}$. For (2.6), by Lemma 4.1, we have $0 < \frac{\beta}{2}, \frac{\gamma}{2} < \frac{1}{2}, 0 < \delta + \frac{\beta}{2} < 1$, which implies $\sin(\alpha + \frac{\gamma}{2}) < 0$. By $1 < \alpha + \frac{\gamma}{2} < \frac{5}{2}$, we get $1 < \alpha + \frac{\gamma}{2} < 2$. Thus, we have to consider the following seven choices:

$$\{x_1, x_2, x_3, x_4\} = \begin{cases} \{1 - \alpha + \frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \delta - \frac{\beta}{2}\}, \\ \{-1 + \alpha - \frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, -\delta + \frac{\beta}{2}\}, \\ \{2 - \alpha + \frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, -\delta + \frac{\beta}{2}\}, \\ \{-1 + \alpha + \frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \delta + \frac{\beta}{2}\}, \\ \{-1 + \alpha + \frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, 1 - \delta - \frac{\beta}{2}\}, \\ \{2 - \alpha - \frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \delta + \frac{\beta}{2}\}, \\ \{2 - \alpha - \frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, 1 - \delta - \frac{\beta}{2}\}. \end{cases} \tag{4.1}$$

We will match these choices with four cases of solutions in Proposition 2.14 as follows.

4.1 Case 1: $x_1x_2 = x_3x_4 = 0$

By $-\frac{1}{2} < \delta - \frac{\beta}{2} < \frac{1}{2}$, $\frac{1}{2} < \alpha - \frac{\gamma}{2} < 2$ and $0 < \delta + \frac{\beta}{2} < 1$, the only solution of $x_1x_2 = x_3x_4 = 0$ for (4.1) comes from $\alpha - \frac{\gamma}{2} = 1$ and $\delta - \frac{\beta}{2} = 0$. By Lemma 2.11, we know that $\alpha\beta\delta$ or $\alpha\gamma\delta$ is a vertex.

If $\alpha\beta\delta$ is a vertex, we get three subcases by Lemma 4.2:

1. $\alpha = \frac{4}{3}, \beta = \frac{4}{9}, \gamma = \frac{2}{3}, \delta = \frac{2}{9}$ (Case $(12,4,6,2)/9$ in Table 1).
2. $\alpha = \frac{5}{4}, \beta = \gamma = \frac{1}{2}, \delta = \frac{1}{4}$.
3. $\alpha = \frac{6}{5}, \beta = \frac{8}{15}, \gamma = \frac{2}{5}, \delta = \frac{4}{15}$.

For the second and third subcases, we have $\beta \ge \gamma$, contradicting $\beta < \gamma$ in Lemma 4.1. In the first subcase, we have $\alpha\cdots = \alpha\beta\delta$ or $\alpha\delta^3$. By $\#\alpha = \#\delta$, we get $\alpha\cdots = \alpha\beta\delta$. There is only one solution satisfying Balance Lemma 2.6: $\{6\alpha\beta\delta, 2\gamma^3\}$, and it gives a two-layer earth map tiling by Lemma 2.10.

If $\alpha\gamma\delta$ is a vertex, then we get $\alpha = \frac{4}{3} - \frac{2}{3f}, \beta = \frac{4}{f}, \gamma = \frac{2}{3} - \frac{4}{3f}, \delta = \frac{2}{f}$. By Lemma 4.1, $\frac{4}{f} = \beta < \gamma = \frac{2}{3} - \frac{4}{3f}$. This implies $f > 8$. By the angle values, Parity Lemma and Lemma 2.11, we get the

$$AVC \subset \{\alpha\gamma\delta, \beta\gamma^3, \alpha\beta^{\frac{f-2}{6}}\delta, \beta^{\frac{f+4}{6}}\gamma^2, \beta^{\frac{f+1}{3}}\gamma, \beta^{\frac{f}{2}}\}.$$

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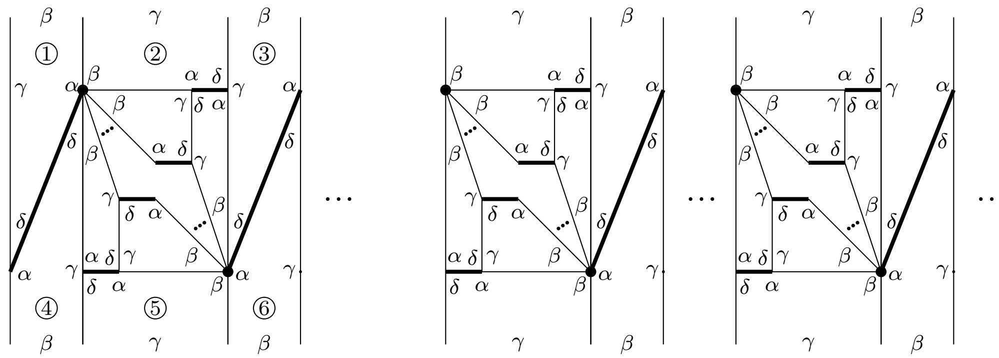

$T(6k\alpha\gamma\delta, 2\alpha\beta^k\delta, 2\beta^{2k+1}\gamma)$

$\{(6k-2)\alpha\gamma\delta, 4\alpha\beta^k\delta, 2\beta^{k+1}\gamma^2\}$

Figure 18.
Tilings by flipping once, or twice with different spacing.

When $f = 6k(k \geq 2)$ or $6k+4(k \geq 1)$, we have the $AVC \subset \{\alpha\gamma\delta, \beta\gamma^3, \beta^{\frac{f}{2}}\}$, and the only solution satisfying Balance Lemma 2.6 is $\{f\alpha\gamma\delta, 2\beta^{\frac{f}{2}}\}$ which gives a two-layer earth map tiling by Lemma 2.10 $'$.

When $f = 6k+2$ $(k \geq 2)$, we have $\gamma = k\beta$, $\alpha + \delta = (2k+1)\beta$ and the

$$AVC \subset \{\alpha\gamma\delta, \beta\gamma^3, \alpha\beta^k\delta, \beta^{k+1}\gamma^2, \beta^{2k+1}\gamma, \beta^{\frac{f}{2}}\}.$$

By the AVC, we know $\alpha^2\cdots, \delta^2\cdots$, and $\alpha|\delta\cdots$ are not vertices. So we have the unique AAD for any $\beta^x\gamma^y = |^\beta\gamma^\delta|^\gamma\beta^\alpha|\cdots|^\beta\gamma^\delta|^\gamma\beta^\alpha|$. We will discuss all possible β-vertices in any tiling as follows.

If $\beta^{\frac{f}{2}}$ appears, the tiling is a two-layer earth map tiling by Lemma 2.10 $'$.

If $\beta^{2k+1}\gamma$ appears ($\beta^{\frac{f}{2}}$ never appears), then $R(\gamma_2\cdots) = \beta^{2k+1}$ in the first picture of Figure 18 and this β^{2k+1} determines $2k+1$ time zones ($4k+2$ or $\frac{2f+2}{3}$ tiles). Then $R(\alpha_1\delta_4\cdots) = \beta^k$ and this β^k determines k time zones ($2k$ or $\frac{f-2}{3}$ tiles). We obtain a unique tiling $T(6k\alpha\gamma\delta, 2\alpha\beta^k\delta, 2\beta^{2k+1}\gamma)$ which can be viewed as the first flip modification of the two-layer earth map tilings.

If $\beta^{k+1}\gamma^2$ appears ($\beta^{2k+1}\gamma, \beta^{\frac{f}{2}}$ never appear), the tilings are shown in the second picture of Figure 18. Depending on the space between two flips, there are $\lfloor\frac{k+3}{2}\rfloor$ or $\lfloor\frac{f+16}{12}\rfloor$ different tilings with the same set of vertices.

If $\beta\gamma^3$ appears ($\beta^{k+1}\gamma^2, \beta^{2k+1}\gamma, \beta^{\frac{f}{2}}$ never appear), the tiling is shown in Figure 19. We obtain a unique tiling $T((6k-4)\alpha\gamma\delta, 2\beta\gamma^3, 6\alpha\beta^k\delta)$ which can be obtained by applying the first flip modification three times.

All of the above tilings belong to the third infinite sequence in Table 2 after interchanging $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$ to keep consistent the AVC for two-layer earth map tilings.

If the $AVC \subset \{\alpha\gamma\delta, \alpha\beta^k\delta\}$, there is no solution satisfying Balance Lemma 2.6.

4.2 Case 2: $\{x_1, x_2\} = \{x_3, x_4\}$

By Proposition 2.9, $\beta \neq \gamma$. So the only possibility is that $x_1 = x_3$ and $x_2 = x_4$ in (4.1). After an easy check of the seven choices in (4.1), only the last one might hold: $2 - \alpha - \frac{\gamma}{2} = \frac{\gamma}{2}$ and $\frac{\gamma}{2} = 1 - \delta - \frac{\beta}{2}$. Then we get $\alpha + \beta + \gamma + \delta = 3 > \frac{8}{3}$, a contradiction.

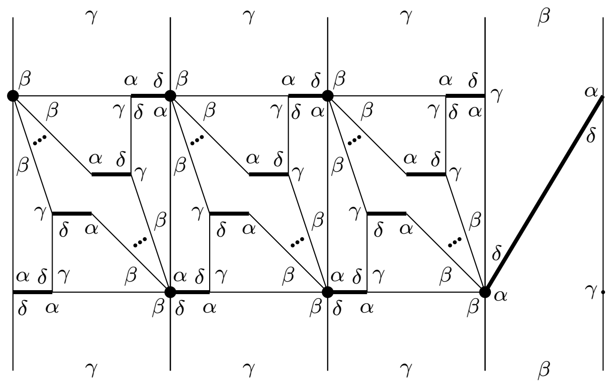

Figure 19.
$T((6k-4)\alpha\gamma\delta, 2\beta\gamma^3, 6\alpha\beta^k\delta)$ obtained by flipping three times.

4.3 Case 3: $\{x_1,x_2\} = \{\frac{1}{6},\theta\}$ and $\{x_3,x_4\} = \{\frac{\theta}{2},\frac{1}{2}-\frac{\theta}{2}\}$, or $\{x_3,x_4\} = \{\frac{1}{6},\theta\}$ and $\{x_1,x_2\} = \{\frac{\theta}{2},\frac{1}{2}-\frac{\theta}{2}\}$, for some $0<\theta\le\frac{1}{2}$

In the seven choices of (4.1), if $\frac{\gamma}{2}=\frac{1}{6}$, then $\gamma=\frac{1}{3}$, contradicting Lemma 4.1; if $\frac{\gamma}{2}=\frac{\theta}{2}$ and $\frac{\beta}{2}=\theta$, then $\gamma<\beta$, contradicting Lemma 4.1. Therefore, we only have $5\times 7 = 35$ options to consider. It turns out 27 of these options are dismissed by Lemmas 2.1 and 4.1. We list the corresponding details in the right-hand column of Table 6. The remaining eight options are summarized as the following six subcases:

1. $\alpha = 1+\frac{\gamma}{2}$, $\beta=\frac{1}{3}$, $\delta=\frac{1}{3}-\frac{\gamma}{2}$, $\frac{1}{3}<\gamma<\frac{2}{3}$;
2. $\alpha = 2-\frac{3\gamma}{2}$, $\beta=\frac{1}{3}$, $\delta=\frac{1}{3}-\frac{\gamma}{2}$, $\frac{1}{3}<\gamma<\frac{2}{3}$;
3. $\alpha = \frac{3\gamma}{2}$, $\beta=\frac{1}{3}$, $\delta=\frac{2}{3}-\frac{\gamma}{2}$, $\frac{2}{3}<\gamma<1$;
4. $\alpha = \frac{5}{6}+\frac{\gamma}{2}$, $\beta=2-2\gamma$, $\delta=\frac{3}{2}-\frac{3\gamma}{2}$, $\frac{5}{6}\le\gamma<1$;
5. $\alpha = \frac{1}{2}+\frac{3\gamma}{4}$, $\beta=\frac{\gamma}{2}$, $\delta=\frac{1}{6}+\frac{\gamma}{4}$, $\frac{2}{3}<\gamma\le\frac{4}{5}$;
6. $\alpha = \frac{3}{2}-\frac{\gamma}{4}$, $\beta=\frac{\gamma}{2}$, $\delta=\frac{1}{6}-\frac{\gamma}{4}$, $\frac{1}{3}<\gamma<\frac{2}{3}$.

For the first, second, and sixth subcases, we have $\beta,\gamma<R(\alpha\delta\cdots)$; for the third, fourth, and fifth subcases, we have $\beta<R(\alpha\delta\cdots)<\gamma$. So neither $\alpha\beta\delta$ nor $\alpha\gamma\delta$ is a vertex, contradicting Lemma 2.11.

4.4 Case 4: $\{x_1,x_2,x_3,x_4\}$ are in Table 3.

There are $8\times 7\times 15 = 840$ subcases to consider, but most are ruled out by violating $2>\alpha>1>\gamma>\beta>\delta>0$, $\gamma>\frac{1}{3}$, $\delta<\frac{1}{2}$ or f being even integer. Such computations can be carried out efficiently by any spreadsheet program. Only 29 subcases are left in Table 7. But 26 of them are ruled out by Lemma 2.11: there is neither $\alpha\beta\delta$ nor $\alpha\gamma\delta$. There are only three subcases left: $(17,5,9,4)/15$, $(41,10,16,3)/30$, $(43,6,16,1)/30$. They all imply $\alpha\cdots=\alpha\gamma\delta$ by the angle values and Parity Lemma. There is only one solution satisfying Balance Lemma 2.6: $\{f\alpha\gamma\delta, 2\beta^{\frac{f}{2}}\}$, and it gives three two-layer earth map tilings in Table 1 after interchanging $\alpha\leftrightarrow\delta$ and $\beta\leftrightarrow\gamma$ by Lemma 2.10.

§5. Degenerate case $\alpha=1$

If $\alpha=1$, the quadrilateral degenerates to an isosceles triangle in Figure 20.

Then $\beta=\delta$, and Lemma 2.2 implies $\beta<\gamma$. By Lemma 2.11, exactly one of $\alpha\beta\delta$ or $\alpha\gamma\delta$ must be a vertex in any spherical tiling by congruent such quadrilaterals.

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Table 6. Case $\{x_1,x_2\}=\{\frac{1}{6},\theta\}$ *and* $\{x_3,x_4\}=\{\frac{\theta}{2},\frac{1}{2}-\frac{\theta}{2}\}$ *or* $\{x_3,x_4\}=\{\frac{1}{6},\theta\}$ *and* $\{x_1,x_2\}=\{\frac{\theta}{2},\frac{1}{2}-\frac{\theta}{2}\}$*, for some* $0<\theta\le\frac{1}{2}$

θ	$\frac{1}{6}$	$\frac{\theta}{2}$	$\frac{1}{2}-\frac{\theta}{2}$	α	β	γ	δ	$\alpha>1>\gamma>\beta>\delta$ (Lemma 4.1)
$1-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$1-\frac{\theta}{2}$	$\frac{1}{3}$	θ	$\frac{2}{3}-\frac{\theta}{2}$	$\alpha<1$
$-1+\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$1+\frac{3\theta}{2}$	$\frac{1}{3}$	θ	$-\frac{1}{3}+\frac{\theta}{2}$	$\delta<0$
$2-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$2-\frac{\theta}{2}$	$\frac{1}{3}$	θ	$-\frac{1}{3}+\frac{\theta}{2}$	$\delta<0$
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$1+\frac{\theta}{2}$	$\frac{1}{3}$	θ	$\frac{1}{3}-\frac{\theta}{2}$	$\surd$ **Subcase** 1
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$1+\frac{\theta}{2}$	$\frac{1}{3}$	θ	$\frac{1}{3}+\frac{\theta}{2}$	$\beta<\delta$
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$2-\frac{3\theta}{2}$	$\frac{1}{3}$	θ	$\frac{1}{3}-\frac{\theta}{2}$	$\surd$ **Subcase** 2
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$2-\frac{3\theta}{2}$	$\frac{1}{3}$	θ	$\frac{1}{3}+\frac{\theta}{2}$	$\beta<\delta$

θ	$\frac{1}{6}$	$\frac{1}{2}-\frac{\theta}{2}$	$\frac{\theta}{2}$	α	β	γ	δ	$\alpha>1>\gamma>\beta>\delta$ (Lemma 4.1)
$1-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{3}{2}-\frac{3\theta}{2}$	$\frac{1}{3}$	$1-\theta$	$\frac{1}{6}+\frac{\theta}{2}$	$\surd$ **Subcase** 3
$-1+\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{3}{2}+\frac{\theta}{2}$	$\frac{1}{3}$	$1-\theta$	$\frac{1}{6}-\frac{\theta}{2}$	$\delta>0\Rightarrow\theta<\frac{1}{3}$ but $\alpha+\beta+\gamma+\delta\le\frac{8}{3}\Rightarrow\theta\ge\frac{1}{3}$
$2-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{5}{2}-\frac{3\theta}{2}$	$\frac{1}{3}$	$1-\theta$	$\frac{1}{6}-\frac{\theta}{2}$	$\alpha<2\Rightarrow\theta>\frac{1}{3}$ but $\delta>0\Rightarrow\theta<\frac{1}{3}$
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$\frac{1}{2}+\frac{3\theta}{2}$	$\frac{1}{3}$	$1-\theta$	$-\frac{1}{6}+\frac{\theta}{2}$	$\surd$ **Subcase** 2
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$\frac{1}{2}+\frac{3\theta}{2}$	$\frac{1}{3}$	$1-\theta$	$\frac{5}{6}-\frac{\theta}{2}$	$\beta<\delta$
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$\frac{3}{2}-\frac{\theta}{2}$	$\frac{1}{3}$	$1-\theta$	$-\frac{1}{6}+\frac{\theta}{2}$	$\surd$ **Subcase** 1
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$\frac{3}{2}-\frac{\theta}{2}$	$\frac{1}{3}$	$1-\theta$	$\frac{5}{6}-\frac{\theta}{2}$	$\beta<\delta$

$\frac{1}{6}$	θ	$\frac{1}{2}-\frac{\theta}{2}$	$\frac{\theta}{2}$	α	β	γ	δ	$\alpha>1>\gamma>\beta>\delta$ (Lemma 4.1)
$1-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{4}{3}-\frac{\theta}{2}$	2θ	$1-\theta$	$\frac{3\theta}{2}$	$\surd$ **Subcase** 4
$-1+\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{5}{3}-\frac{\theta}{2}$	2θ	$1-\theta$	$\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
$2-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{7}{3}-\frac{\theta}{2}$	2θ	$1-\theta$	$\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$\frac{2}{3}+\frac{\theta}{2}$	2θ	$1-\theta$	$-\frac{\theta}{2}$	$\delta<0$
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$\frac{2}{3}+\frac{\theta}{2}$	2θ	$1-\theta$	$1-\frac{3\theta}{2}$	$\alpha<1$
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$\frac{4}{3}+\frac{\theta}{2}$	2θ	$1-\theta$	$-\frac{\theta}{2}$	$\delta<0$
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$\frac{4}{3}+\frac{\theta}{2}$	2θ	$1-\theta$	$1-\frac{3\theta}{2}$	$\alpha+\beta+\gamma+\delta=\frac{10}{3}>\frac{8}{3}$

$\frac{\theta}{2}$	$\frac{1}{2}-\frac{\theta}{2}$	θ	$\frac{1}{6}$	α	β	γ	δ	$\alpha>1>\gamma>\beta>\delta$ (Lemma 4.1)
$1-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$1+\frac{\theta}{2}$	$1-\theta$	2θ	$\frac{2}{3}-\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
$-1+\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$1+\frac{3\theta}{2}$	$1-\theta$	2θ	$\frac{1}{3}-\frac{\theta}{2}$	$\beta<\gamma\Rightarrow\theta>\frac{1}{3}$ but $\alpha+\beta+\gamma+\delta\le\frac{8}{3}\Rightarrow\theta\le\frac{1}{6}$
$2-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$2+\frac{\theta}{2}$	$1-\theta$	2θ	$\frac{1}{3}-\frac{\theta}{2}$	$\alpha>2$
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$1-\frac{\theta}{2}$	$1-\theta$	2θ	$-\frac{1}{3}+\frac{\theta}{2}$	$\alpha<1$
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$1-\frac{\theta}{2}$	$1-\theta$	2θ	$\frac{1}{3}+\frac{\theta}{2}$	$\alpha<1$
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$2-\frac{3\theta}{2}$	$1-\theta$	2θ	$-\frac{1}{3}+\frac{\theta}{2}$	$\delta<0$
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$2-\frac{3\theta}{2}$	$1-\theta$	2θ	$\frac{1}{3}+\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta=\frac{10}{3}>\frac{8}{3}$

$\frac{1}{2}-\frac{\theta}{2}$	$\frac{\theta}{2}$	θ	$\frac{1}{6}$	α	β	γ	δ	$\alpha>1>\gamma>\beta>\delta$ (Lemma 4.1)
$1-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{1}{2}+\frac{3\theta}{2}$	θ	2θ	$\frac{1}{6}+\frac{\theta}{2}$	$\surd$ **Subcase** 5
$-1+\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{3}{2}+\frac{\theta}{2}$	θ	2θ	$-\frac{1}{6}+\frac{\theta}{2}$	$\delta>0\Rightarrow\theta>\frac{1}{3}$ but $\alpha+\beta+\gamma+\delta\le\frac{8}{3}\Rightarrow\theta\le\frac{1}{3}$
$2-\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{3}{2}+\frac{3\theta}{2}$	θ	2θ	$-\frac{1}{6}+\frac{\theta}{2}$	$\delta>0\Rightarrow\theta>\frac{1}{3}$ but $\alpha+\beta+\gamma+\delta\le\frac{8}{3}\Rightarrow\theta\le\frac{4}{15}$
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$\frac{3}{2}-\frac{3\theta}{2}$	θ	2θ	$\frac{1}{6}-\frac{\theta}{2}$	$\alpha>1\Rightarrow\theta<\frac{1}{3}$ but $\alpha+\beta+\gamma+\delta>2\Rightarrow\theta>\frac{1}{3}$
$-1+\alpha+\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$\frac{3}{2}-\frac{3\theta}{2}$	θ	2θ	$\frac{5}{6}-\frac{\theta}{2}$	$\alpha>1\Rightarrow\theta<\frac{1}{3}$ but $\gamma>\delta\Rightarrow\theta>\frac{1}{3}$
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta+\frac{\beta}{2}$	$\frac{3}{2}-\frac{\theta}{2}$	θ	2θ	$\frac{1}{6}-\frac{\theta}{2}$	$\surd$ **Subcase** 6
$2-\alpha-\frac{\gamma}{2}$	$\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1-\delta-\frac{\beta}{2}$	$\frac{3}{2}-\frac{\theta}{2}$	θ	2θ	$\frac{5}{6}-\frac{\theta}{2}$	$\beta<\delta$

5.1 Subcase $\alpha\beta\delta$ is a vertex

By Lemma 2.1, we get $\beta=\delta=\frac{1}{2},\gamma=\frac{4}{f}$. Then $\beta<\gamma$ implies $f=6$ and $\gamma=\frac{2}{3}$. So the $AVC=\{\alpha\beta\delta,\gamma^3\}$, and it gives a two-layer earth map tiling by Lemma 2.10. This is Case $(6,3,4,3)/6$ in Table 1.

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Table 7. All 29 solutions induced from Table 3

$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f	$(\alpha,\beta,\gamma,\delta)$	f
(35,16,18,11)/30	6	(55,16,18,7)/42	14	(79,16,18,13)/60	40
(35,16,18,3)/30	10	(49,16,30,1)/42	14	(43,6,8,5)/30	60
(33,16,22,1)/30	10	(43,6,16,1)/30	20	(39,8,10,5)/30	60
(19,7,9,1)/15	10	(43,4,18,1)/30	20	(35,8,18,1)/30	60
(41,10,16,3)/30	12	(83,16,18,13)/60	24	(49,4,6,3)/30	60
(17,5,9,4)/15	12	(71,16,42,1)/60	24	(39,6,16,1)/30	60
(19,3,11,2)/15	12	(23,3,5,1)/15	30	(47,4,10,1)/30	60
(67,12,50,11)/60	12	(41,8,10,5)/30	30	(77,10,36,1)/60	60
(71,8,54,7)/60	12	(37,8,18,1)/30	30	(59,6,20,1)/42	84
(41,8,18,3)/30	12	(67,16,42,1)/60	40		

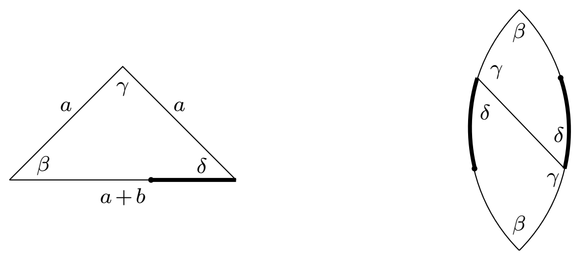

Figure 20.
Degenerate case $\alpha = 1$ and the subcase $\gamma+\delta = 1$.

5.2 Subcase $\alpha\gamma\delta$ is a vertex

By Lemma 2.1, we get $\beta = \delta = \frac{4}{f}, \gamma = 1-\frac{4}{f}$. Then $\beta < \gamma$ implies $f > 8$ and $\gamma > \frac{1}{2}$. By the angle values and Parity Lemma, we get the $AVC \subset \{\alpha\gamma\delta, \gamma^3, \beta^2\gamma^2, \alpha\beta^{\frac{f-4}{4}}\delta, \beta^{\frac{f+4}{4}}\gamma, \beta^{\frac{f}{2}}\}$.

When $f = 4k+2 (k \geq 2)$, we have the $AVC \subset \{\alpha\gamma\delta, \gamma^3, \beta^2\gamma^2, \beta^{\frac{f}{2}}\}$, and the only solution satisfying Balance Lemma 2.6 is $\{f\alpha\gamma\delta, 2\beta^{\frac{f}{2}}\}$ which gives a two-layer earth map tiling by Lemma 2.10$'$.

When $f = 4k$ $(k \geq 3)$, we have $\gamma = (k-1)\beta$, $\alpha+\delta = (k+1)\beta$ and the

$$AVC \subset \{\alpha\gamma\delta, \gamma^3, \beta^2\gamma^2, \alpha\beta^{k-1}\delta, \beta^{k+1}\gamma, \beta^{\frac{f}{2}}\}.$$

Trying out all possible β-vertices as the previous section, there are always four tilings as shown in Figure 10$'$, the second picture of Figure 21 (flip once), Figure 22 (flip twice with different spacing). Only when $f = 12$, we can apply the first flip modification in Figure 5 (after interchanging $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$) three times, as shown in the first picture of Figure 21. This is because $3(k-1) > 2k$ when $k \geq 4$.

All above tilings belong to the first infinite sequence in Table 2.

§6. Concave case $\beta > 1$

The quadrilateral with $\beta > 1$ is shown in Figure 23. We first prove some basic facts. Recall that Lemma 2.4$'$ implies $\gamma + 2\delta > 1$.

LEMMA 6.1. *In an a^3b-tiling with $\beta > 1$, we have $a < b$, $\alpha < \gamma, \delta$ and $\alpha < \frac{1}{2}$.*

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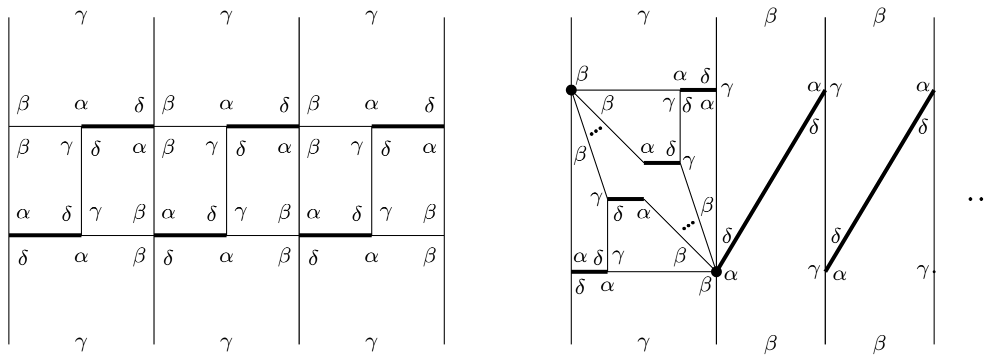

$T(6\alpha\gamma\delta, 2\gamma^3, 6\alpha\beta^2\delta)$ $T((4k-2)\alpha\gamma\delta, 2\alpha\beta^{k-1}\delta, 2\beta^{k+1}\gamma)$

Figure 21.
Two degenerate a^3b-tilings.

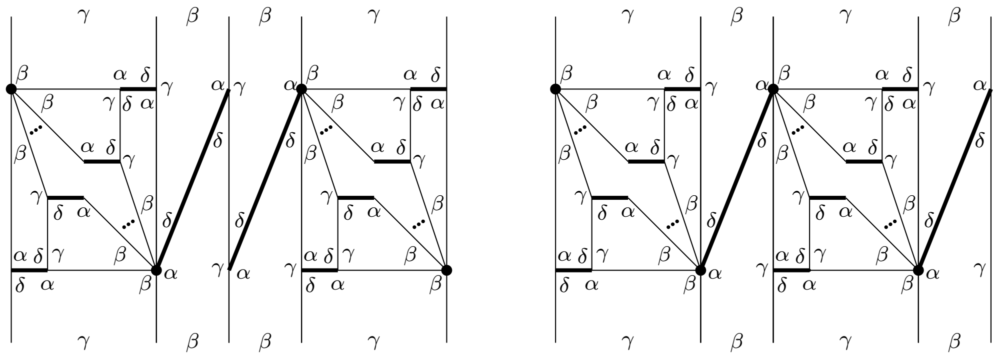

Figure 22.
Two tilings for $\{(4k-4)\alpha\gamma\delta, 2\beta^2\gamma^2, 4\alpha\beta^{k-1}\delta\}$.

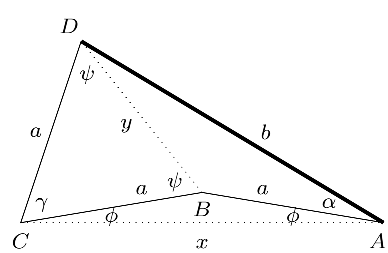

Figure 23.
a^3b-quadrilateral with $\beta > 1, \alpha, \gamma, \delta < 1$.

Proof. In Figure 23, by $\beta > \delta$, $\angle ABD = \beta - \psi > \delta - \psi = \angle ADB$. This implies $a < b$. Then $\angle CAD < \angle ACD$, that is, $\alpha + \phi < \gamma + \phi$. So $\alpha < \gamma$. By Lemma 2.2$'$, $\alpha < \delta$. If $\alpha \geq \frac{1}{2}$, then $\gamma, \delta > \frac{1}{2}$, and there is no $\beta\cdots$ vertex. So $\alpha < \frac{1}{2}$. ∎

LEMMA 6.2. *In an a^3b-tiling with $\beta > 1$, $\beta\delta\cdots$ is a vertex and $\beta + \delta < 2$.*

Proof. If $\beta\delta\cdots$ is not a vertex, by $\beta > 1$ and Parity Lemma, we get $\beta\cdots = \alpha^x\beta, \alpha^y\beta\gamma^z, \beta\gamma^w (x, y, w \geq 2, z \geq 1)$. Then $\#\alpha + \#\gamma \geq 2\#\beta = 2f$, and there is only one

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solution satisfying Balance Lemma 2.6: $\{\frac{f}{2}\alpha^2\beta, \frac{f}{2}\beta\gamma^2, \frac{f}{k}\delta^k\}$. But this implies $\alpha=\gamma$, contradicting Lemma 2.3′. Therefore, $\beta\delta\cdots$ is a vertex. ∎

To find rational a^3b-quadrilaterals by solving (2.5) or (2.6) via Proposition 2.14, we have to transform $\alpha-\frac{\gamma}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \delta-\frac{\beta}{2}, \alpha+\frac{\gamma}{2}, -\delta-\frac{\beta}{2}$ to the range $[0,\frac{1}{2}]$. For (2.5), by Lemma 6.1, we have $-\frac{1}{2}<\alpha-\frac{\gamma}{2}<\frac{1}{2}, \frac{1}{2}<\frac{\beta}{2}<1, 0<\frac{\gamma}{2}<\frac{1}{2}$ and $-1<\delta-\frac{\beta}{2}<\frac{1}{2}$. For (2.6), by Lemma 6.1, we have $0<\alpha+\frac{\gamma}{2}<1, \frac{1}{2}<\frac{\beta}{2}<1, 0<\gamma<\frac{1}{2}$, which implies $\sin(\delta+\frac{\beta}{2})<0$. By $\frac{1}{2}<\delta+\frac{\beta}{2}<2$, we get $1<\delta+\frac{\beta}{2}<2$. If $\frac{3}{2}\le\delta+\frac{\beta}{2}<2$, we get $\beta+\delta>2$, contradicting Lemma 6.2. So for the equation (2.6), we have $0<\alpha+\frac{\gamma}{2}<1, \frac{1}{2}<\frac{\beta}{2}<1, 0<\frac{\gamma}{2}<\frac{1}{2}$ and $1<\delta+\frac{\beta}{2}<\frac{3}{2}$. Thus, we have to consider the following five choices:

$$\{x_1,x_2,x_3,x_4\}=\begin{cases}\{\alpha-\frac{\gamma}{2}, 1-\frac{\beta}{2}, \frac{\gamma}{2}, \delta-\frac{\beta}{2}\},\\ \{-\alpha+\frac{\gamma}{2}, 1-\frac{\beta}{2}, \frac{\gamma}{2}, 1+\delta-\frac{\beta}{2}\},\\ \{-\alpha+\frac{\gamma}{2}, 1-\frac{\beta}{2}, \frac{\gamma}{2}, -\delta+\frac{\beta}{2}\},\\ \{\alpha+\frac{\gamma}{2}, 1-\frac{\beta}{2}, \frac{\gamma}{2}, -1+\delta+\frac{\beta}{2}\},\\ \{1-\alpha-\frac{\gamma}{2}, 1-\frac{\beta}{2}, \frac{\gamma}{2}, -1+\delta+\frac{\beta}{2}\}.\end{cases} \tag{6.1}$$

We will match these choices with four cases of solutions in Proposition 2.14 as follows.

6.1 Case 1: $x_1x_2=x_3x_4=0$

By $-1<\delta-\frac{\beta}{2}<\frac{1}{2}$, $-\frac{1}{2}<\alpha-\frac{\gamma}{2}<\frac{1}{2}$ and $1<\delta+\frac{\beta}{2}<\frac{3}{2}$, the only solution of $x_1x_2=x_3x_4=0$ for (6.1) comes from $\alpha=\frac{\gamma}{2}, \delta=\frac{\beta}{2}$. Then we get $3\alpha+\beta+\delta>2$. By $R(\beta\delta\cdots)<3\alpha,\beta,\delta$, Parity Lemma and Lemma 6.2, we deduce that $\alpha\beta\delta$ is a vertex. This implies $\alpha=\frac{2}{f}, \beta=\frac{4}{3}-\frac{4}{3f}, \gamma=\frac{4}{f}, \delta=\frac{2}{3}-\frac{2}{3f}$. By the angle values and Parity Lemma, we get the $AVC\subset\{\alpha\beta\delta, \alpha\delta^3, \alpha^x\beta\gamma^{\frac{f-3x+2}{6}}, \alpha^y\gamma^{\frac{f-3y+2}{6}}\delta^2, \alpha^z\gamma^{\frac{2f-3z+1}{6}}\delta, \alpha^w\gamma^{\frac{f-w}{2}}\}$. Then we know there is no $\beta^2\cdots$ vertex, which further implies that (by AAD) the

$$AVC\subset\{\alpha\beta\delta, \alpha\delta^3, \alpha^2\beta\gamma^{\frac{f-4}{6}}, \alpha^2\gamma^{\frac{f-4}{6}}\delta^2, \beta\gamma^{\frac{f+2}{6}}, \gamma^{\frac{f+2}{6}}\delta^2, \alpha\gamma^{\frac{f-1}{3}}\delta, \gamma^{\frac{f}{2}}\}.$$

When $f=6k$ or $6k+2$ ($k\ge1$), we have the $AVC\subset\{\alpha\beta\delta, \alpha\delta^3, \gamma^{\frac{f}{2}}\}$, and the only solution satisfying Balance Lemma 2.6 is $\{f\alpha\beta\delta, 2\gamma^{\frac{f}{2}}\}$ which gives a two-layer earth map tiling by Lemma 2.10.

When $f=6k+4$ ($k\ge1$), we have $\beta=(2k+1)\gamma$, $\alpha+\delta=(k+1)\gamma$ and the

$$AVC\subset\{\alpha\beta\delta, \alpha\delta^3, \alpha^2\beta\gamma^k, \alpha^2\gamma^k\delta^2, \beta\gamma^{k+1}, \gamma^{k+1}\delta^2, \alpha\gamma^{2k+1}\delta, \gamma^{\frac{f}{2}}\}.$$

We will discuss all possible γ-vertices in any tiling as follows. Whenever $\gamma^{\frac{f}{2}}$ is a vertex, the tiling must be a two-layer earth map tiling by Lemma 2.10. If $\gamma^{\frac{f}{2}}$ never appears, we have the following subcases.

6.1.1. Subcase $\alpha^2\beta\gamma^k$ appears ($\gamma^{\frac{f}{2}}$ never appears)

By the AVC, $\beta^2\cdots$ is never a vertex. Then $\alpha^2\beta\gamma^k$ has only two possible AAD. In Figure 24, $\alpha^2\beta\gamma^k=\vert^{\beta}\gamma^{\delta}\vert^{\beta}\alpha^{\delta}\pmb{\vert}^{\delta}\alpha^{\beta}\vert^{\delta}\gamma^{\beta}\vert\cdots$ determines T_1,T_2,T_3,T_4. Then $\beta_2\delta_1\cdots=\alpha_5\beta_2\delta_1$ determines T_5. So $\vert\beta_5\vert\gamma_2\vert\cdots=\vert\beta_5\vert\gamma_2\vert\alpha\pmb{\vert}\cdots$, $\vert\beta_5\vert\gamma_2\vert^{\beta}\gamma^{\delta}\vert\cdots$ or $\vert\beta_5\vert\gamma_2\vert^{\delta}\gamma^{\beta}\vert\cdots$. If $\vert\beta_5\vert\gamma_2\vert\cdots=\vert\beta_5\vert\gamma_2\vert\alpha\pmb{\vert}\cdots$ or $\vert\beta_5\vert\gamma_2\vert^{\beta}\gamma^{\delta}\vert\cdots$, we get $\delta_2\delta_3\cdots=\beta\delta_2\delta_3\cdots$, contradicting the AVC. So we have $\vert\beta_5\vert\gamma_2\vert\cdots=\vert\beta_5\vert\gamma_2\vert^{\delta}\gamma^{\beta}\vert\cdots$ which determines T_6. Similarly, we can determine T_7,T_8. Then we get $\delta_2\delta_3\delta_6\delta_7\cdots$, contradicting the AVC. Therefore, $\alpha^2\beta\gamma^k=\vert^{\beta}\gamma^{\delta}\vert^{\beta}\alpha^{\delta}\pmb{\vert}^{\delta}\alpha^{\beta}\vert\beta\vert\cdots$.

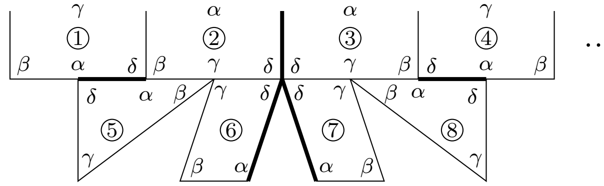

Figure 24.
One possible AAD of $\alpha^2\beta\gamma^k = |^\beta\gamma^\delta|^\beta\alpha^\delta\Vert^\delta\alpha^\beta|^\delta\gamma^\beta|\cdots$.

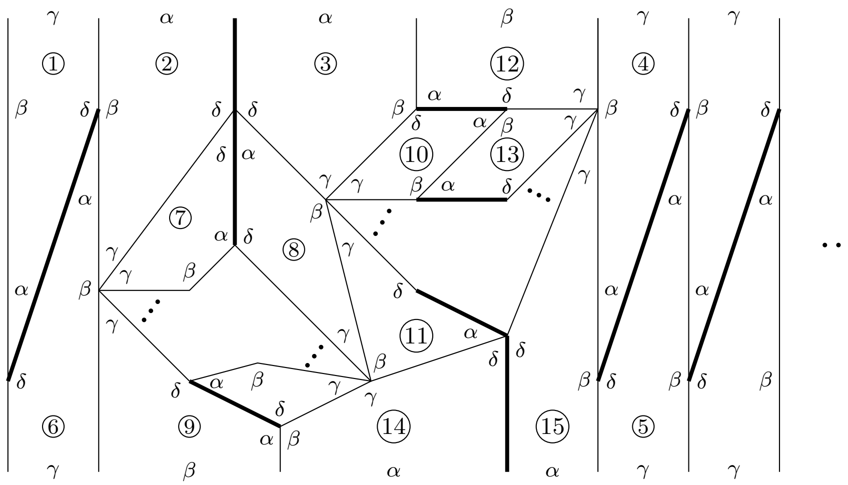

Figure 25.
One special tiling for $\{(6k-2)\alpha\beta\delta, 2\alpha\delta^3, 2\alpha^2\beta\gamma^k, 4\beta\gamma^{k+1}\}$.

The AAD $\alpha^2\beta\gamma^{\frac{f-4}{6}} = |^\beta\gamma_1^\delta|^\beta\alpha_2^\delta\Vert^\delta\alpha_3^\beta|\beta|\cdots$ determines T_1, T_2, T_3 in Figure 25. Then $R(|\alpha\Vert\alpha|\beta|\cdots) = \gamma^k$ and this γ^k determines k time zones ($2k$ or $\frac{f-4}{3}$ tiles). We have $\beta_6\gamma_2\cdots = |\beta_6|\gamma_2|^\beta\alpha^\delta\Vert\cdots$, $|\beta_6|\gamma_2|^\beta\gamma^\delta|\cdots$ or $|\beta_6|\gamma_2|^\delta\gamma^\beta|\cdots$. If $\beta_6\gamma_2\cdots = |\beta_6|\gamma_2|^\beta\alpha^\delta\Vert\cdots$ or $|\beta_6|\gamma_2|^\beta\gamma^\delta|\cdots$, then we get $\beta\delta_2\delta_3\cdots$, contradicting the *AVC*. Therefore, $\beta_6\gamma_2\cdots = |\beta_6|\gamma_2|^\delta\gamma^\beta|\cdots$. This determines T_7. Then $\delta_2\delta_3\delta_7\cdots = \alpha_8\delta_2\delta_3\delta_7$ determines T_8. We have $|\beta_6|\gamma_2|\gamma_7|\cdots = \alpha^2\beta\gamma^k$ or $\beta\gamma^{k+1}$. If $|\beta_6|\gamma_2|\gamma_7|\cdots = \alpha^2\beta\gamma^k = |^\beta\alpha\Vert\alpha|\beta_6|\gamma_2|^\delta\gamma_7^\beta|\cdots$, this gives a vertex $\beta^2\cdots$, contradicting the *AVC*. Therefore, $|\beta_6|\gamma_2|\gamma_7|\cdots = \beta\gamma^{k+1} = |\beta_6|\gamma_2|^\delta\gamma_7^\beta|^\delta\gamma^\beta|\cdots$ determines T_9. Then $R(\beta_6\gamma_2\cdots) = \gamma^k$ and this γ^k determines k time zones ($2k$ or $\frac{f-4}{3}$ tiles). Similarly, we get $\beta_8\gamma_3\cdots = |\beta_8|^\delta\gamma_3^\beta|^\delta\gamma_{10}^\beta|\cdots = \beta\gamma^{k+1}$ which determines T_{10}. Then $R(\beta_8\gamma_3\gamma_{10}\cdots) = \gamma^{k-1}$ and this γ^{k-1} determines $k-1$ time zones ($2k-2$ or $\frac{f-10}{3}$ tiles). Then $\beta_3\delta_{10}\cdots = \alpha_{12}\beta_3\delta_{10}$ determines T_{12}. So only two tiles are undetermined. By checking all possibilities, it turns out there are 3 different ways to arrange these last two tiles, and Figure 25 shows one way with $\beta_{11}\gamma^{\frac{f-4}{6}}\cdots = \beta\gamma^{\frac{f+2}{6}}$. Then $\alpha_9\delta\cdots = \alpha_9\beta_{14}\delta$ determines T_{14}; $\alpha_{14}\beta_9\gamma^{\frac{f-4}{6}}\cdots = \alpha_{14}\alpha_{15}\beta_9\gamma^{\frac{f-4}{6}}, \alpha_{11}\delta_{14}\delta\cdots = \alpha_{11}\delta_{14}\delta_{15}\delta, \beta_4\gamma^{\frac{f-4}{6}}\cdots = \beta_4\gamma^{\frac{f-4}{6}}\gamma_{15}, \alpha_4\delta_5\cdots = \alpha_4\beta_{15}\delta_5$determine T_{15}. Centering T_{14}, T_{15} in Figure 26, it becomes clear that they form a hexagon with three-fold symmetry, and the other two ways are obtained by rotating the b-edge $120°$ and $240°$, respectively.

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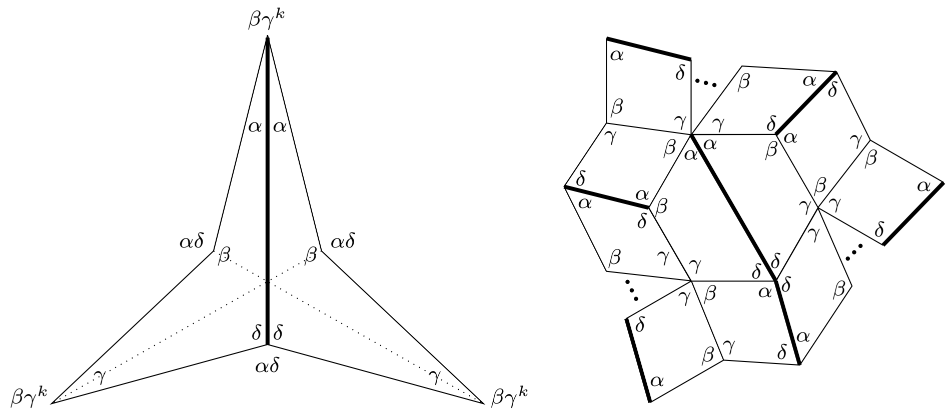

Figure 26.
Three special tilings for $\{(6k-2)\alpha\beta\delta, 2\alpha\delta^3, 2\alpha^2\beta\gamma^k, 4\beta\gamma^{k+1}\}$.

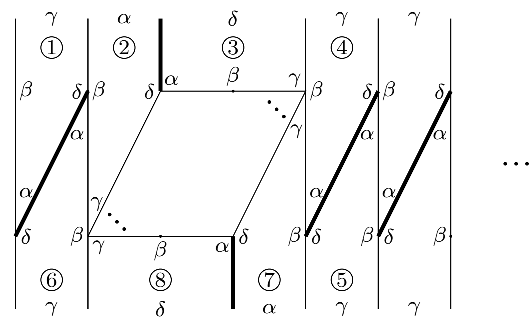

Figure 27.
$T((6k+2)\alpha\beta\delta, 2\beta\gamma^{k+1}, 2\alpha\gamma^{2k+1}\delta)$.

6.1.2. Subcase $\alpha\gamma^{2k+1}\delta$ appears ($\alpha^2\beta\gamma^k, \gamma^{\frac{f}{2}}$ never appear)

If $\alpha^2\beta\gamma^k, \gamma^{\frac{f}{2}}$ never appear, Balance Lemma 2.6 implies the

$$AVC \subset \{\alpha\beta\delta, \alpha^2\gamma^k\delta^2, \beta\gamma^{k+1}, \alpha\gamma^{2k+1}\delta\}.$$

In Figure 27, we have the unique AAD $\alpha\gamma^{2k+1}\delta = |^\beta\gamma_1^\delta|^\beta\alpha_2^\delta\Vert^\alpha\delta_3^\gamma|^\beta\gamma_4^\delta|\cdots$ which determines T_2, T_3. Then $R(\alpha_2\delta_3\cdots) = \gamma^{2k+1}$ and this γ^{2k+1} determines $2k+1$ time zones ($4k+2$ or $\frac{2f-2}{3}$ tiles). Then $|\beta_4|^\delta\gamma_3^\beta|\cdots = |\beta_4|^\delta\gamma_3^\beta|^\delta\gamma^\beta|\cdots = \beta\gamma^{k+1}$. Then $R(\beta_4\gamma_3\cdots) = \gamma^k$ and this γ^k determines k time zones ($2k$ or $\frac{f-4}{3}$ tiles). This tiling is exactly the second flip modification in Figure 5.

6.1.3. Subcase $\alpha^2\gamma^k\delta^2$ appears ($\alpha^2\beta\gamma^k, \alpha\gamma^{2k+1}\delta, \gamma^{\frac{f}{2}}$ never appear)

By the AVC, $\beta^2\cdots$ is never a vertex. Then $\alpha^2\gamma^k\delta^2$ has only two possible AAD. In Figure 28, $\alpha^2\gamma^k\delta^2 = |^\beta\alpha_1^\delta\Vert^\delta\alpha_2^\beta|^\delta\gamma_3^\beta|\cdots$ determines T_1, T_2, T_3. Then $\beta_2\delta_3\cdots = \alpha_4\beta_2\delta_3$ determines T_4; $\beta_4\gamma_2\cdots = \beta_4\gamma_2\gamma_5\cdots = \beta\gamma^{k+1}$. By γ_5, we get $\beta\delta_1\delta_2\cdots$ or $\delta_1\delta_2\delta\cdots$, contradicting the AVC. Therefore, $\alpha^2\gamma^k\delta^2 = |\alpha\Vert\delta|\cdots|\alpha\Vert\delta|\cdots$.

In Figure 29, $\alpha^2\gamma^k\delta^2 = |^\beta\alpha_1^\delta\Vert^\alpha\delta_2^\gamma|\cdots|^\beta\gamma^\delta|\cdots|^\beta\alpha_3^\delta\Vert^\alpha\delta_4^\gamma|\cdots|^\beta\gamma^\delta|\cdots$ determines T_1, T_2, T_3, T_4. Then $R(\alpha^2\delta^2\cdots) = \gamma^k$ determines k time zones ($2k$ or $\frac{f-4}{3}$ tiles). Then $R(\beta\gamma_2\cdots) =$

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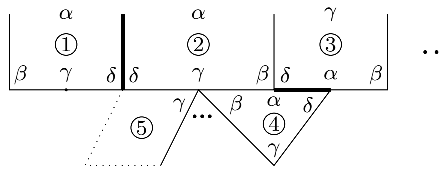

Figure 28.
One possible AAD of $\alpha^2\gamma^k\delta^2 = |\alpha\|\alpha|\cdots$.

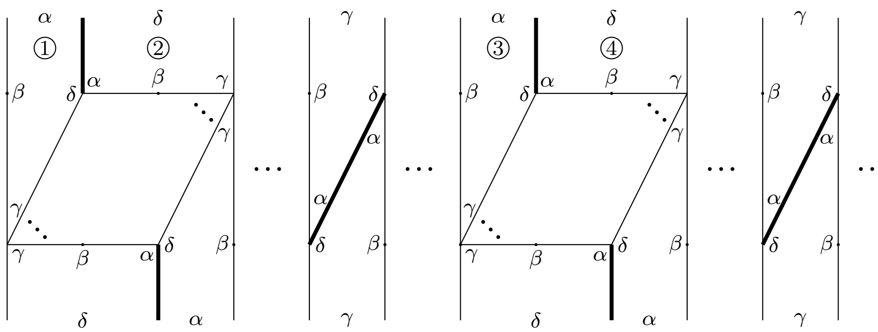

Figure 29.
Many different tilings for $\{6k\alpha\beta\delta, 2\alpha^2\gamma^k\delta^2, 4\beta\gamma^{k+1}\}$.

Table 8. All 10 subcases of $\{x_1,x_2\}=\{x_3,x_4\}$

$x_1(=x_3=\frac{\gamma}{2})$	$x_4(=x_2=1-\frac{\beta}{2})$	
$\alpha-\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\delta=1$
$-\alpha+\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$\delta=0$
$-\alpha+\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\alpha=0$
$\alpha+\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\alpha=0$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$f=4$

$x_1=x_4$ ($\&1-\frac{\beta}{2}=\frac{\gamma}{2}$)	
$\alpha-\frac{\gamma}{2}=\delta-\frac{\beta}{2}$	$f<4$
$-\alpha+\frac{\gamma}{2}=1+\delta-\frac{\beta}{2}$	$\alpha+\delta=0$
$-\alpha+\frac{\gamma}{2}=-\delta+\frac{\beta}{2}$	
$\alpha+\frac{\gamma}{2}=-1+\delta+\frac{\beta}{2}$	$\beta+\delta>2$
$1-\alpha-\frac{\gamma}{2}=-1+\delta+\frac{\beta}{2}$	$f=4$

$R(\beta\gamma_4\cdots)=\gamma^k$ and each of these two γ^k determines k time zones ($2k$ or $\frac{f-4}{3}$ tiles). This tiling can also be obtained by applying the second flip modification in Figure 5 two times.

If the $AVC \subset \{\alpha\beta\delta, \beta\gamma^{k+1}\}$, there is no solution satisfying Balance Lemma 2.6.

In fact, one special tiling in Figure 29, as shown in the first picture of Figure 30, is related to Figure 25 by a special flip modification along L_3 in Figure 30.

All of the above tilings belong to the second infinite sequence in Table 2.

6.2 Case 2: $\{x_1,x_2\}=\{x_3,x_4\}$

We can fix $x_2=1-\frac{\beta}{2}$ and $x_3=\frac{\gamma}{2}$ in (6.1). Then either $x_1=x_3$, $x_4=x_2$ as listed in the left of Table 8, or $x_1=x_4, x_2=x_3$ as listed in the right. All solutions are ruled out by the fact listed in the other column of Table 8 except one solution $-\alpha+\frac{\gamma}{2}=-\delta+\frac{\beta}{2}$, $1-\frac{\beta}{2}=\frac{\gamma}{2}$. By $\beta+\gamma=2$, we have $\beta\cdots=\alpha^x\beta\delta^y$. By $\#\alpha+\#\delta\geq 2\#\beta=2f$ and $\alpha\neq\delta$, there is only one solution satisfying Balance Lemma 2.6: $\{f\alpha\beta\delta, 2\gamma^{\frac{f}{2}}\}$. Then $\alpha=-\frac{1}{2}+\frac{4}{f}, \beta=2-\frac{4}{f}, \gamma=\frac{4}{f}, \delta=\frac{1}{2}$. By $\alpha>0$, we get $f<8$ which forces $f=6$. This solution admits only a two-layer earth map tiling $T(6\alpha\beta\delta, 2\gamma^3)$, and is Case $(1,8,4,3)/6$ in Table 1.

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6.3 Case 3: $\{x_1, x_2\} = \{\frac{1}{6}, \theta\}$ and $\{x_3, x_4\} = \{\frac{\theta}{2}, \frac{1}{2} - \frac{\theta}{2}\}$, or $\{x_3, x_4\} = \{\frac{1}{6}, \theta\}$ and $\{x_1, x_2\} = \{\frac{\theta}{2}, \frac{1}{2} - \frac{\theta}{2}\}$, for some $0 < \theta \le \frac{1}{2}$

In Table 9, we list each of these $5 \times 8 = 40$ options. It turns out 36 of these options are dismissed by Lemma 2.1, Lemma 2.4 ′, and Lemma 6.2. We list the corresponding details in the right-hand column. The remaining four options are summarized as follows:

1. $\alpha = -\frac{1}{6} + \frac{\gamma}{2}$, $\beta = 2\gamma$, $\delta = -\frac{1}{2} + \frac{3\gamma}{2}$, $\frac{8}{15} < \gamma \le \frac{2}{3}$;
2. $\beta = \frac{2}{3} + 2\alpha$, $\gamma = \frac{1}{3}$, $\delta = 3\alpha$, $\frac{1}{6} < \alpha \le \frac{5}{18}$;
3. $\alpha = \frac{3\gamma}{4}$, $\beta = 1 + \frac{\gamma}{2}$, $\delta = \frac{2}{3} + \frac{\gamma}{4}$, $\frac{2}{15} < \gamma \le \frac{2}{5}$;
4. $\alpha = \frac{\gamma}{4}$, $\beta = 1 + \frac{\gamma}{2}$, $\delta = \frac{1}{3} + \frac{\gamma}{4}$, $\frac{1}{3} < \gamma \le \frac{2}{3}$.

6.3.1. Subcase $\alpha = -\frac{1}{6} + \frac{\gamma}{2}, \beta = 2\gamma, \delta = -\frac{1}{2} + \frac{3\gamma}{2}, \frac{8}{15} < \gamma \le \frac{2}{3}$

By the angle values and Parity Lemma, only $\alpha\beta\delta$, $\beta\delta^2$ and γ^3 can be degree 3 vertices. If $\beta\delta^2$ is a vertex, we have $\alpha = \frac{2}{15}$, $\beta = \frac{6}{5}$, $\gamma = \frac{3}{5}$, $\delta = \frac{2}{5}$. Then $\beta\cdots = \beta\delta^2$, $\alpha^3\beta\delta$ or $\alpha^6\beta$. So $\#\alpha + \#\delta > 2\#\beta = 2f$, contradicting Balance Lemma 2.6. So $\alpha\beta\delta$ or γ^3 is a vertex. Both cases imply $\alpha = \frac{1}{6}$, $\beta = \frac{4}{3}$, $\gamma = \frac{2}{3}$, $\delta = \frac{1}{2}$, and $f = 6$. This implies all vertices have degree 3. There is only one solution satisfying Balance Lemma 2.6: $\{6\alpha\beta\delta, 2\gamma^3\}$, and it gives a two-layer earth map tiling by Lemma 2.10. This also gives Case $(1,8,4,3)/6$ in Table 1 (see Remark 2.16).

6.3.2. Subcase $\beta = \frac{2}{3} + 2\alpha, \gamma = \frac{1}{3}, \delta = 3\alpha, \frac{1}{6} < \alpha \le \frac{5}{18}$

By $R(\beta\delta\cdots) < 3\alpha, \beta, \delta$, Parity Lemma and Lemma 6.2, we get $\alpha\beta\delta$ is a vertex. This implies $\alpha = \frac{2}{9}, \beta = \frac{10}{9}, \gamma = \frac{1}{3}, \delta = \frac{2}{3}$. Then $\beta\cdots = \alpha\beta\delta, \alpha^4\beta$. By $\#\beta = \#\alpha$, we have $\beta\cdots = \alpha\beta\delta$. There is only one solution satisfying Balance Lemma 2.6: $\{12\alpha\beta\delta, 2\gamma^6\}$, and it gives a two-layer earth map tiling by Lemma 2.10. This is Case $(2,10,3,6)/9$ in Table 1.

6.3.3. Subcase $\alpha = \frac{3\gamma}{4}, \beta = 1 + \frac{\gamma}{2}, \delta = \frac{2}{3} + \frac{\gamma}{4}, \frac{2}{15} < \gamma \le \frac{2}{5}$

By $R(\beta\delta\cdots) < 3\alpha, \beta, \delta$, Parity Lemma and Lemma 6.2, we get $\alpha\beta\delta$ is a vertex. This implies $\alpha = \frac{1}{6}, \beta = \frac{10}{9}, \gamma = \frac{2}{9}, \delta = \frac{13}{18}$. Then $\alpha + \delta = 4\gamma$ and $\beta = 5\gamma$. Then we get $f = 18$. By the angle values and Parity Lemma, we get the

$$AVC \subset \{\alpha\beta\delta, \alpha^2\gamma\delta^2, \beta\gamma^4, \alpha^4\beta\gamma, \alpha\gamma^5\delta, \alpha^5\gamma^2\delta, \gamma^9, \alpha^4\gamma^6, \alpha^8\gamma^3, \alpha^{12}\}.$$

By $\#\delta = \#\alpha$, we have $\alpha\cdots = \delta\cdots = \alpha\beta\delta, \alpha^2\gamma\delta^2$ or $\alpha\gamma^5\delta$. Therefore, the

$$AVC \subset \{\alpha\beta\delta, \alpha^2\gamma\delta^2, \beta\gamma^4, \alpha\gamma^5\delta, \gamma^9\}.$$

We will discuss all possible vertices containing γ in any tiling as follows.

If γ^9 appears, the tiling is a two-layer earth map tiling by Lemma 2.10. This is Case $(3,20,4,13)/18$ in Table 1.

If $\alpha\gamma^5\delta$ appears (γ^9 never appears), then $\alpha\gamma^5\delta = |^{\beta}\alpha_1^{\delta}\Vert^{\alpha}\delta_2^{\gamma}|^{\beta}\gamma_3^{\delta}|\cdots$ determines $T_1, T_2, \ldots, T_7$ in Figure 31. Then $\beta_4\delta_3\cdots = \alpha_8\beta_4\delta_3$ determines T_8. Similarly, we can determine $T_9, T_{10}, T_{11}, T_{12}$. Then $\beta_3\gamma_2\cdots = \beta_3\gamma_2\gamma_{13}\gamma_{14}\gamma_{15}$. By γ_{13}, we get $\beta_2\cdots = \beta_2\delta_{13}\cdots$ which determines T_{13}. Similarly, we can determine T_{14}, T_{15}. Then $\beta_2\delta_{13}\cdots = \alpha_{16}\beta_2\delta_{13}$ and $\alpha_2\delta_1\cdots = \alpha_2\beta_{16}\delta_1$ determine T_{16}. Similarly, we can determine T_{17}, T_{18}. This tiling is exactly the second flip modification in Figure 5.

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Table 9. Case $\{x_1,x_2\}=\{\frac{1}{6},\theta\}$ *and* $\{x_3,x_4\}=\{\frac{\theta}{2},\frac{1}{2}-\frac{\theta}{2}\}$ *or* $\{x_3,x_4\}=\{\frac{1}{6},\theta\}$ *and* $\{x_1,x_2\}=\{\frac{\theta}{2},\frac{1}{2}-\frac{\theta}{2}\}$*, for some* $0<\theta\le\frac{1}{2}$

θ	$\frac{1}{6}$	$\frac{\theta}{2}$	$\frac{1}{2}-\frac{\theta}{2}$	α	β	γ	δ	$\gamma+2\delta>1,\beta+\delta<2$ (Lemmas 2.4′ and 6.2)
$\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{3\theta}{2}$	$\frac{5}{3}$	θ	$\frac{4}{3}-\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$-\frac{\theta}{2}$	$\frac{5}{3}$	θ	$\frac{1}{3}-\frac{\theta}{2}$	$\alpha<0$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$-\frac{\theta}{2}$	$\frac{5}{3}$	θ	$\frac{1}{3}+\frac{\theta}{2}$	$\alpha<0$
$\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{\theta}{2}$	$\frac{5}{3}$	θ	$\frac{2}{3}-\frac{\theta}{2}$	$\beta+\delta>2$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$1-\frac{3\theta}{2}$	$\frac{5}{3}$	θ	$\frac{2}{3}-\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
θ	$\frac{1}{6}$	$\frac{1}{2}-\frac{\theta}{2}$	$\frac{\theta}{2}$	α	β	γ	δ	$\gamma+2\delta>1,\beta+\delta<2$ (Lemmas 2.4′ and 6.2)
$\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{1}{2}+\frac{\theta}{2}$	$\frac{5}{3}$	$1-\theta$	$\frac{5}{6}+\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta=4>\frac{8}{3}$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$\frac{1}{2}-\frac{3\theta}{2}$	$\frac{5}{3}$	$1-\theta$	$-\frac{1}{6}+\frac{\theta}{2}$	$\gamma+2\delta=\frac{2}{3}$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{1}{2}-\frac{3\theta}{2}$	$\frac{5}{3}$	$1-\theta$	$\frac{5}{6}-\frac{\theta}{2}$	$\beta+\delta>2$
$\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$-\frac{1}{2}+\frac{3\theta}{2}$	$\frac{5}{3}$	$1-\theta$	$\frac{1}{6}+\frac{\theta}{2}$	$\alpha>0\Rightarrow\theta>\frac{1}{3}$ but $\beta+\delta<2\Rightarrow\theta<\frac{1}{3}$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{1}{2}-\frac{\theta}{2}$	$\frac{5}{3}$	$1-\theta$	$\frac{1}{6}+\frac{\theta}{2}$	$\alpha<\delta\Rightarrow\theta>\frac{1}{3}$ but $\alpha+\beta+\gamma+\delta>2\Rightarrow\theta<\frac{1}{3}$
$\frac{1}{6}$	θ	$\frac{\theta}{2}$	$\frac{1}{2}-\frac{\theta}{2}$	α	β	γ	δ	$\gamma+2\delta>1,\beta+\delta<2$ (Lemmas 2.4′ and 6.2)
$\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{1}{6}+\frac{\theta}{2}$	$2-2\theta$	θ	$\frac{3}{2}-\frac{3\theta}{2}$	$\beta>1\Rightarrow\theta<\frac{1}{2}$ but $\alpha+\beta+\gamma+\delta\le\frac{8}{3}\Rightarrow\theta\ge\frac{1}{2}$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$-\frac{1}{6}+\frac{\theta}{2}$	$2-2\theta$	θ	$\frac{1}{2}-\frac{3\theta}{2}$	$\gamma+2\delta<1$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$-\frac{1}{6}+\frac{\theta}{2}$	$2-2\theta$	θ	$\frac{1}{2}-\frac{\theta}{2}$	$\gamma+2\delta=1$
$\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{1}{6}-\frac{\theta}{2}$	$2-2\theta$	θ	$\frac{1}{2}+\frac{\theta}{2}$	$\alpha>0\Rightarrow\theta<\frac{1}{3}$ but $\beta+\delta<2\Rightarrow\theta>\frac{1}{3}$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{5}{6}-\frac{\theta}{2}$	$2-2\theta$	θ	$\frac{1}{2}+\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
$\frac{1}{6}$	θ	$\frac{1}{2}-\frac{\theta}{2}$	$\frac{\theta}{2}$	α	β	γ	δ	$\gamma+2\delta>1,\beta+\delta<2$ (Lemmas 2.4′ and 6.2)
$\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{2}{3}-\frac{\theta}{2}$	$2-2\theta$	$1-\theta$	$1-\frac{\theta}{2}$	$\beta>1\Rightarrow\theta<\frac{1}{2}$ but $\alpha+\beta+\gamma+\delta\le\frac{8}{3}\Rightarrow\theta\ge\frac{1}{2}$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$\frac{1}{3}-\frac{\theta}{2}$	$2-2\theta$	$1-\theta$	$-\frac{\theta}{2}$	$\delta<0$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{1}{3}-\frac{\theta}{2}$	$2-2\theta$	$1-\theta$	$1-\frac{3\theta}{2}$	$\surd$ **Subcase**1
$\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$-\frac{1}{3}+\frac{\theta}{2}$	$2-2\theta$	$1-\theta$	$\frac{3\theta}{2}$	$\alpha<0$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{1}{3}+\frac{\theta}{2}$	$2-2\theta$	$1-\theta$	$\frac{3\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
$\frac{\theta}{2}$	$\frac{1}{2}-\frac{\theta}{2}$	$\frac{1}{6}$	θ	α	β	γ	δ	$\gamma+2\delta>1,\beta+\delta<2$ (Lemmas 2.4′ and 6.2)
$\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{1}{6}+\frac{\theta}{2}$	$1+\theta$	$\frac{1}{3}$	$\frac{1}{2}+\frac{3\theta}{2}$	$\surd$ **Subcase**2
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$\frac{1}{6}-\frac{\theta}{2}$	$1+\theta$	$\frac{1}{3}$	$-\frac{1}{2}+\frac{3\theta}{2}$	$\alpha>0\Rightarrow\theta<\frac{1}{3}$ but $\delta>0\Rightarrow\theta>\frac{1}{3}$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{1}{6}-\frac{\theta}{2}$	$1+\theta$	$\frac{1}{3}$	$\frac{1}{2}-\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta=2$
$\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$-\frac{1}{6}+\frac{\theta}{2}$	$1+\theta$	$\frac{1}{3}$	$\frac{1}{2}+\frac{\theta}{2}$	$\alpha>0\Rightarrow\theta>\frac{1}{3}$ but $\beta+\delta<2\Rightarrow\theta<\frac{1}{3}$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{5}{6}-\frac{\theta}{2}$	$1+\theta$	$\frac{1}{3}$	$\frac{1}{2}+\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{3}{8}$
$\frac{\theta}{2}$	$\frac{1}{2}-\frac{\theta}{2}$	θ	$\frac{1}{6}$	α	β	γ	δ	$\gamma+2\delta>1,\beta+\delta<2$ (Lemmas 2.4′ and 6.2)
$\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{3\theta}{2}$	$1+\theta$	2θ	$\frac{2}{3}+\frac{\theta}{2}$	$\surd$ **Subcase**3
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$\frac{\theta}{2}$	$1+\theta$	2θ	$-\frac{1}{3}+\frac{\theta}{2}$	$\delta<0$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$\frac{\theta}{2}$	$1+\theta$	2θ	$\frac{1}{3}+\frac{\theta}{2}$	$\surd$ **Subcase**4
$\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$-\frac{\theta}{2}$	$1+\theta$	2θ	$\frac{2}{3}-\frac{\theta}{2}$	$\alpha<0$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$1-\frac{3\theta}{2}$	$1+\theta$	2θ	$\frac{2}{3}-\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
$\frac{1}{2}-\frac{\theta}{2}$	$\frac{\theta}{2}$	$\frac{1}{6}$	θ	α	β	γ	δ	$\gamma+2\delta>1,\beta+\delta<2$ (Lemmas 2.4′ and 6.2)
$\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{2}{3}-\frac{\theta}{2}$	$2-\theta$	$\frac{1}{3}$	$1+\frac{\theta}{2}$	$\delta>1$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$-\frac{1}{3}+\frac{\theta}{2}$	$2-\theta$	$\frac{1}{3}$	$\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta=2$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$-\frac{1}{3}+\frac{\theta}{2}$	$2-\theta$	$\frac{1}{3}$	$1-\frac{3\theta}{2}$	$\alpha<0$
$\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{1}{3}-\frac{\theta}{2}$	$2-\theta$	$\frac{1}{3}$	$\frac{3\theta}{2}$	$\beta+\delta>2$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{1}{3}+\frac{\theta}{2}$	$2-\theta$	$\frac{1}{3}$	$\frac{3\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$

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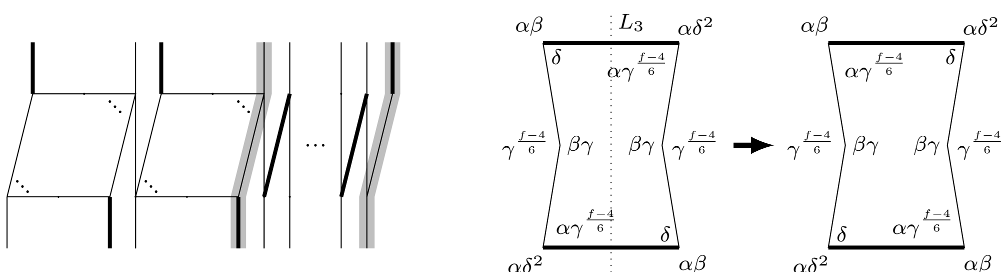
Table 9. (Continued)

$\frac{1}{2}-\frac{\theta}{2}$	$\frac{\theta}{2}$	θ	$\frac{1}{6}$	α	β	γ	δ	$\gamma+2\delta>1, \beta+\delta<2$ (Lemmas 2.4′ and 6.2)
$\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$\delta-\frac{\beta}{2}$	$\frac{1}{2}+\frac{\theta}{2}$	$2-\theta$	2θ	$\frac{7}{6}-\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$1+\delta-\frac{\beta}{2}$	$-\frac{1}{2}+\frac{3\theta}{2}$	$2-\theta$	2θ	$\frac{1}{6}-\frac{\theta}{2}$	$\gamma+2\delta<1$
$-\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-\delta+\frac{\beta}{2}$	$-\frac{1}{2}+\frac{3\theta}{2}$	$2-\theta$	2θ	$\frac{5}{6}-\frac{\theta}{2}$	$\alpha>0\Rightarrow\theta>\frac{1}{3}$ but $\alpha+\cdots+\delta\le\frac{8}{3}\Rightarrow\theta\le\frac{1}{6}$
$\alpha+\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{1}{2}-\frac{3\theta}{2}$	$2-\theta$	2θ	$\frac{1}{6}+\frac{\theta}{2}$	$\alpha>0\Rightarrow\theta<\frac{1}{3}$ but $\beta+\delta<2\Rightarrow\theta>\frac{1}{3}$
$1-\alpha-\frac{\gamma}{2}$	$1-\frac{\beta}{2}$	$\frac{\gamma}{2}$	$-1+\delta+\frac{\beta}{2}$	$\frac{1}{2}-\frac{\theta}{2}$	$2-\theta$	2θ	$\frac{1}{6}+\frac{\theta}{2}$	$\alpha+\beta+\gamma+\delta>\frac{8}{3}$



$\{(6k-2)\alpha\beta\delta, 2\alpha^2\gamma^k\delta^2, 4\beta\gamma^{k+1}\}$

Figure 30.
A special flip modification ($\frac{f+2}{3}$ tiles flipped).

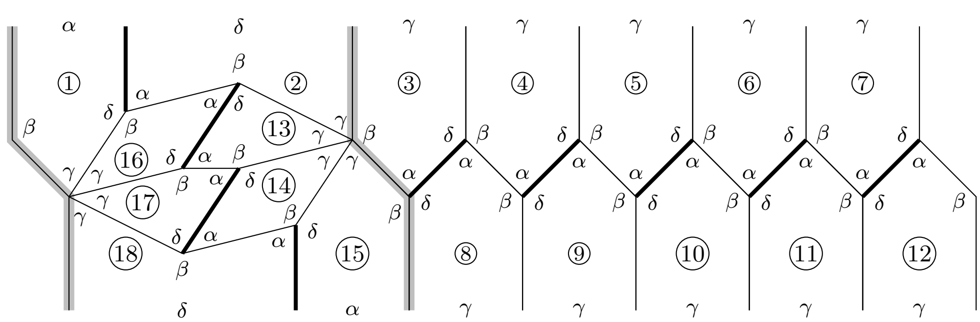


Figure 31.
$T(16\alpha\beta\delta, 2\beta\gamma^4, 2\alpha\gamma^5\delta)$.

If $\alpha^2\gamma\delta^2$ appears ($\alpha\gamma^5\delta$, γ^9 never appear), it has only two possible AAD since there is no vertex $\beta^2\cdots$ by $\beta>1$. In the first picture of Figure 32, $\alpha^2\gamma\delta^2 = |^{\beta}\alpha_1^{\delta}\Vert^{\delta}\alpha_2^{\beta}|^{\gamma}\delta_3^{\alpha}\Vert^{\alpha}\delta_4^{\gamma}|^{\beta}\gamma_5^{\delta}|$ determines T_1, T_2, T_3, T_4, T_5. Then $\beta_1\delta_5\cdots = \alpha_6\beta_1\delta_5$ determines T_6. We have $\beta_6\gamma_1\cdots = \beta_6\gamma_1\gamma_7\gamma^2$. By γ_7, we have $\beta_7\delta_1\delta_2\cdots$ or $\delta_1\delta_2\delta_7\cdots$, contradicting the *AVC*. In the second picture of Figure 32, $\alpha^2\gamma\delta^2 = |^{\beta}\alpha_1^{\delta}\Vert^{\alpha}\delta_2^{\gamma}|^{\beta}\alpha_3^{\delta}\Vert^{\alpha}\delta_4^{\gamma}|^{\beta}\gamma_5^{\delta}|$ determines T_1, T_2, T_3, T_4, T_5. Then $\beta_1\delta_5\cdots = \alpha_6\beta_1\delta_5$ determines T_6. Then $\beta_5\gamma_4\cdots = \beta_5\gamma_4\gamma_7\gamma_8\gamma_9$. By γ_7, we get $\beta_4\cdots = \beta_4\delta_7\cdots$ which determines T_7. Similarly, we can determine T_8, T_9. Then $\beta_4\delta_7\cdots = \alpha_{10}\beta_4\delta_7, \alpha_4\delta_3\cdots = \alpha_4\beta_{10}\delta_3$ determine T_{10}. Similarly, we can determine T_{11}, T_{12}. After repeating the process

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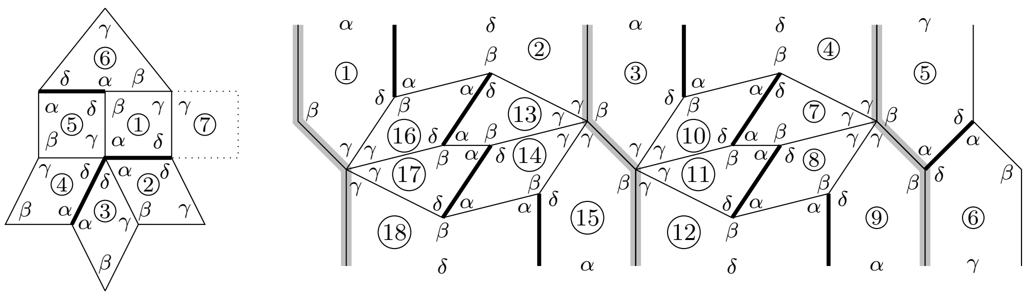

Figure 32.
$T(14\alpha\beta\delta, 2\alpha^2\gamma\delta^2, 4\beta\gamma^4)$.

Table 10. All 10 *solutions induced from Table 3*

$(\alpha,\beta,\gamma,\delta)$	f	
(5,32,18,11)/30	20	No degree 3 vertex
(5,32,6,23)/30	20	
(1,42,4,17)/30	30	
(1,17,9,4)/15	60	No degree 3 vertex
(1,21,5,8)/15	12	

$(\alpha,\beta,\gamma,\delta)$	f	
(13,66,32,29)/60	12	No degree 3 vertex
(5,32,14,13)/30	30	$\beta\cdots=\beta\gamma^2$ or $\alpha^3\beta\delta$
(3,32,22,7)/30	30	No degree 3 vertex
(1,19,3,8)/15	60	No degree 3 vertex
(7,66,8,49)/60	24	No degree 3 vertex

one more time, we can determine T_{13}, T_{14}, ..., T_{18}. This tiling can also be obtained by applying the second flip modification in Figure 5 two times.

If the $AVC \subset \{\alpha\beta\delta, \beta\gamma^4\}$, there is no solution satisfying Balance Lemma 2.6.

6.3.4. Subcase $\alpha=\frac{\gamma}{4}, \beta=1+\frac{\gamma}{2}, \delta=\frac{1}{3}+\frac{\gamma}{4}, \frac{1}{3}<\gamma\le\frac{2}{3}$

By $R(\beta\delta\cdots)<5\alpha,\beta,\delta$, Parity Lemma and Lemma 6.2, we deduce that $\alpha\beta\delta$ or $\alpha^3\beta\delta$ is a vertex. If $\alpha\beta\delta$ is a vertex, then $\alpha=\frac{1}{6}, \beta=\frac{4}{3}, \gamma=\frac{2}{3}, \delta=\frac{1}{2}$. There is only one solution satisfying Balance Lemma 2.6: $\{6\alpha\beta\delta, 2\gamma^3\}$, and it gives a two-layer earth map tiling by Lemma 2.10. This also gives Case (1,8,4,3)/6 in Table 1. If $\alpha^3\gamma\delta$ is a vertex, then $\alpha=\frac{1}{9}, \beta=\frac{11}{9}, \gamma=\frac{4}{9}, \delta=\frac{4}{9}$, which does not admit any any degree 3 vertex, a contradiction.

6.4 Case 4: $\{x_1,x_2,x_3,x_4\}$ are in Table 3.

There are $8\times5\times15=600$ subcases to consider, but most are ruled out by violating $0<\alpha,\gamma,\delta<1, 1<\beta<2$, f being even integer or $\beta+\delta<2$. Only 10 subcases are left in Table 10. But six of them are ruled out by not admitting any degree 3 vertex. Four remaining subcases are boxed.

6.4.1. Subcase $(5,32,6,23)/30$

By the angle values and Parity Lemma, we get $\beta\cdots=\alpha\beta\delta$ or $\alpha^2\beta\gamma^3$. By $\#\beta=\#\alpha$, we get $\beta\cdots=\alpha\beta\delta$, which determines a two-layer earth map tiling $T(20\alpha\beta\delta, 2\gamma^{10})$ in Table 1 by Lemma 2.10.

6.4.2. Subcase $(1,42,4,17)/30$

By the angle values and Parity Lemma, we get $\beta\cdots=\alpha\beta\delta$, $\alpha^2\beta\gamma^4$, $\alpha^6\beta\gamma^3$, $\alpha^{10}\beta\gamma^2$, $\alpha^{14}\beta\gamma$ or $\alpha^{18}\beta$. By $\#\beta=\#\alpha$, we get $\beta\cdots=\alpha\beta\delta$, which determines a two-layer earth map tiling $T(30\alpha\beta\delta, 2\gamma^{15})$ in Table 1 by Lemma 2.10.

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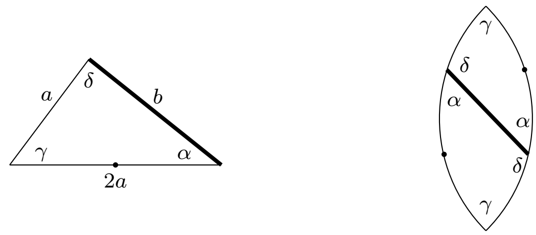
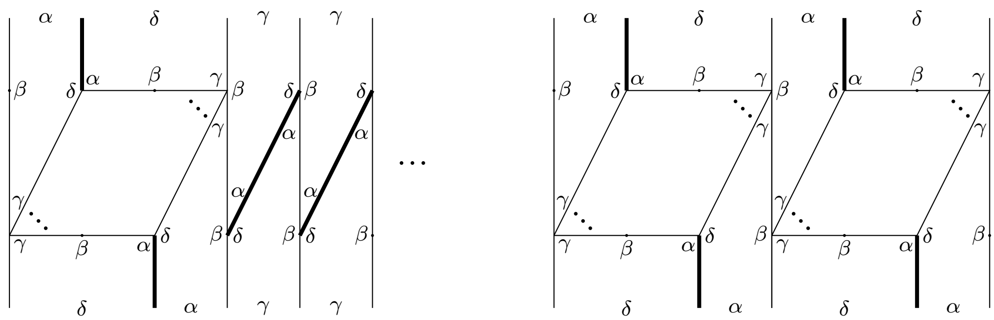


Figure 33.
$\beta=1$ and the subcase $\alpha+\delta=1$.



$T((4k-2)\alpha\beta\delta, 2\alpha\gamma^k\delta, 2\beta\gamma^k)$ $T((4k-4)\alpha\beta\delta, 2\alpha^2\delta^2, 4\beta\gamma^k)$

Figure 34.
Two flips of $T(4k\alpha\beta\delta, 2\gamma^{2k})$ if $\beta=1$ and $\alpha+\delta=1$.

6.4.3. Subcase $(1,21,5,8)/15$

By the angle values and Parity Lemma, we get $\beta\cdots=\alpha\beta\delta$ or $\alpha^4\beta\gamma$. By $\#\beta=\#\alpha$, we get $\beta\cdots=\alpha\beta\delta$, which determines a two-layer earth map tiling $T(12\alpha\beta\delta, 2\gamma^6)$ in Table 1 by Lemma 2.10.

6.4.4. Subcase $(5,32,14,13)/30$

By the angle values and Parity Lemma, we get $\beta\cdots=\beta\gamma^2$ or $\alpha^3\beta\delta$. There is no solution satisfying Balance Lemma 2.6.

§7. Degenerate case $\beta=1$

If $\beta=1$, the quadrilateral degenerates to an isosceles triangle in Figure 33.

By $\beta=1$, we have $\alpha+\gamma+\delta=(1+\frac{4}{f})$. By Lemmas 2.2 ′ and 2.4 ′, we get $\delta>\alpha$ and $\gamma+2\delta>1$. By $a+b>2a$, we get $b>a$. This implies $\gamma>\alpha$. If $\alpha\geq\frac{1}{2}$, then $R(\beta\cdots)=1\leq 2\alpha<2\gamma, 2\delta$. So $\beta\cdots=\alpha^2\beta$, contradicting Balance Lemma. We conclude that $\alpha<\frac{1}{2}$ and $\alpha^2\beta$ is never a vertex.

If $\alpha\beta\delta$ is a vertex, we have $\alpha+\delta=1, \beta=1, \gamma=\frac{4}{f}$, as shown in the second picture of Figure 33. So $a=\frac{1}{3}$, and we get $\alpha=\arctan(2\tan\frac{2\pi}{f})$ by the cosine law. This is equivalent to $\cos(\frac{\pi}{2}-\alpha-\frac{2\pi}{f})-3\cos(\frac{\pi}{2}-\alpha+\frac{2\pi}{f})=0$ by the product to sum formula. Then Theorem 6 of Conway–Jones [3] implies that α is irrational for any even integer $f\geq 6$. Thus, this belongs to the irrational angle case in [11]. Such quadrilaterals always admit two-layer earth map tilings for any even integer $f\geq 6$, together with their flip modifications when $f=4k$ as shown in Figure 34.

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If $\alpha\beta\delta$ is not a vertex, then we will find all tilings by discussing all possible β-vertices. If $\alpha^x\beta(x\ge 4)$ is a vertex, then its unique AAD $\Vert\alpha^\beta|^\beta\alpha\Vert\cdots$ at $\alpha^x\beta(x\ge 4)$ gives a vertex $\beta^2\cdots$, contradicting $\beta=1$. Similarly, $\alpha^y\beta\delta(y\ge 5)$, $\alpha^w\beta\gamma(w\ge 4)$, $\alpha^p\gamma^q(p\ge 2)$ and $\alpha^z\delta^2(z\ge 4)$ are not vertices.

7.0.1 Subcase $\gamma>\delta$

Then we have $\beta>\gamma>\delta>\alpha$. By $\gamma+2\delta>1$, we get $\gamma>\frac{1}{3}$. By the angle values and Parity Lemma, we get $\beta\cdots=\beta\gamma^2,\alpha^2\beta\gamma,\beta\delta^x(x\ge 2)$ or $\alpha^p\beta\delta^q(p\ge 1,p+q\ge 4)$.

If $\beta\gamma^2$ is a vertex, then $\gamma=\frac{1}{2},\alpha+\delta>\frac{1}{2},\frac{1}{4}<\delta<\frac{1}{2}$. So we have $\beta\cdots=\beta\gamma^2$, $\alpha^2\beta\gamma$ or $\alpha^3\beta\delta$. They all satisfy $\#\alpha+\#\gamma\ge 2\#\beta$. If $\alpha^2\beta\gamma$ or $\alpha^3\beta\delta$ is a vertex, then $\#\alpha+\#\gamma>2\#\beta$, contradicting Balance Lemma 2.6. If $\beta\cdots=\beta\gamma^2$, then $\#\gamma>\#\beta$, again a contradiction.

Therefore, we have $\beta\cdots=\alpha^2\beta\gamma,\beta\delta^x$ or $\alpha^p\beta\delta^q$. They all satisfy $\#\alpha+\#\delta\ge 2\#\beta=2f$. There is only one solution satisfying Balance Lemma 2.6: $\{\frac{f}{2}\beta\delta^2,\frac{f}{2}\alpha^2\beta\gamma,2\gamma^{\frac{f}{4}}\}$. This implies $\alpha=\frac{1}{2}-\frac{4}{f}$, $\gamma=\frac{8}{f}$, $\delta=\frac{1}{2}$. By $1>\gamma>\delta$, we get $8<f<16$, which do not satisfy (2.5) in Lemma 2.13. We conclude that there is no tiling in this case.

7.0.2 Subcase $\gamma<\delta$

Then we have $\beta>\delta>\gamma>\alpha$. By $\gamma+2\delta>1$, we get $\delta>\frac{1}{3}$. By the angle values and Parity Lemma, we get $\beta\cdots=\beta\delta^2,\alpha^3\beta\delta,\beta\gamma^x(x\ge 2)$ or $\alpha^p\beta\gamma^q(p\ge 2,q\ge 1)$.

If $\beta\delta^2$ is a vertex, then $\gamma<\delta=\frac{1}{2}$, $\alpha+\gamma>\frac{1}{2}$. So $\beta\cdots=\beta\delta^2,\alpha^2\beta\gamma,\alpha^3\beta\delta$ or $\beta\gamma^y(y\ge 3)$. If $\alpha^3\beta\delta$ is a vertex, then $\beta\cdots=\beta\delta^2$ or $\alpha^3\beta\delta$. So $\#\alpha+\#\delta>2\#\beta$, contradicting Balance Lemma 2.6. Similarly, $\beta\gamma^y$ is not a vertex. So $\beta\cdots=\beta\delta^2$ or $\alpha^2\beta\gamma$. There is only one solution satisfying Balance Lemma 2.6: $\{\frac{f}{2}\beta\delta^2,\frac{f}{2}\alpha^2\beta\gamma,2\gamma^{\frac{f}{4}}\}$. We get $\alpha=\frac{1}{2}-\frac{4}{f},\beta=1,\gamma=\frac{8}{f},\delta=\frac{1}{2}$. By $\gamma<\delta$, we get $f>16$. By (2.5) in Lemma 2.13, we get $f=16$, a contradiction.

If $\beta\delta^2$ is not a vertex, we have $\beta\cdots=\alpha^3\beta\delta,\beta\gamma^x$ or $\alpha^p\beta\gamma^q$. They all satisfy $\#\alpha+\#\gamma\ge 2\#\beta$. If $\alpha^3\beta\delta$ or $\alpha^p\beta\gamma^q$ is a vertex, then $\#\alpha+\#\gamma>2\#\beta$, contradicting Balance Lemma 2.6. If $\beta\cdots=\beta\gamma^x$, then $\#\gamma>\#\beta$, again a contradiction.

7.0.3 Subcase $\gamma=\delta$

By $\gamma+2\delta>1$, we get $\gamma=\delta>\frac{1}{3}$. By the angle values and Parity Lemma, we get $\beta\cdots=\beta\gamma^2,\beta\delta^2,\alpha^2\beta\gamma$ or $\alpha^3\beta\delta$. If $\beta\gamma^2$ and $\beta\delta^2$ are not vertices, we have $\beta\cdots=\alpha^2\beta\gamma$ or $\alpha^3\beta\delta$, contradicting Balance Lemma 2.6. Therefore, $\beta\gamma^2$ or $\beta\delta^2$ is a vertex. So we get $\gamma=\delta=\frac{1}{2}$. Then we get $\alpha=\frac{4}{f}$. By $\gamma=\delta=\frac{1}{2}$, we get $b=2a=\frac{1}{2}$. By the sine law $\frac{\sin\alpha}{\sin a}=\frac{\sin\gamma}{\sin 2a}$, we have $\alpha=\frac{1}{4}$. This implies $f=16$. By the angle values and Parity Lemma, we get the

$$AVC\subset\{\beta\gamma^2,\beta\delta^2,\alpha^2\beta\gamma,\gamma^4,\gamma^2\delta^2,\delta^4,\alpha^2\gamma\delta^2\}.$$

If $\alpha^2\gamma\delta^2$ is a vertex, it has only two possible AAD. In the left of Figure 35, $\alpha^2\gamma\delta^2=|^\beta\alpha_1^\delta\Vert^\delta\alpha_2^\beta|^\gamma\delta_3^\alpha\Vert^\alpha\delta_4^\gamma|^\beta\gamma_5^\delta|$ determines T_1,T_2, T_3,T_4,T_5. We have $\beta_2\gamma_3\cdots=\beta\gamma^2$ or $\alpha^2\beta\gamma$. If $\beta_2\gamma_3\cdots=\alpha^2\beta\gamma$, we get the AAD $\Vert\alpha^\beta|^\beta\gamma|\cdots$ at $\alpha^2\beta\gamma$. This gives a vertex $\beta^2\cdots$, contradicting the AVC. Therefore, $\beta_2\gamma_3\cdots=|^\gamma\beta_2^\alpha|^\delta\gamma_3^\beta|^\delta\gamma_6^\beta|$ determines T_6. Then $\beta_3\delta_6\cdots=\beta_3\delta_6\delta_7$ determines T_7. Similarly, we can determine T_8,T_9. Then we get $\alpha_3\alpha_4\gamma_7\gamma_8\cdots$, contradicting the AVC.

In the right of Figure 35, $\alpha^2\gamma\delta^2=|^\beta\alpha_1^\delta\Vert^\alpha\delta_2^\gamma|^\beta\alpha_3^\delta\Vert\cdots$ determines T_1,T_2,T_3. We have $\beta_3\gamma_2\cdots=\beta\gamma^2$ or $\alpha^2\beta\gamma$. If $\beta_3\gamma_2\cdots=\alpha^2\beta\gamma$, we get the AAD $\Vert\alpha^\beta|^\beta\gamma|\cdots$ at $\alpha^2\beta\gamma$. This gives a vertex $\beta^2\cdots$, contradicting the AVC. Therefore, $\beta_3\gamma_2\cdots=|^\gamma\beta_3^\alpha|^\delta\gamma_2^\beta|^\delta\gamma_4^\beta|$ determines T_4. Then $\beta_2\delta_4\cdots=\beta_2\delta_4\delta_5$ determines T_5; $|\gamma_5|\alpha_2\Vert\delta_1|\cdots=|\gamma_5|\alpha_2\Vert\delta_1|\alpha_6\Vert\delta_7|$ determines T_6,T_7.

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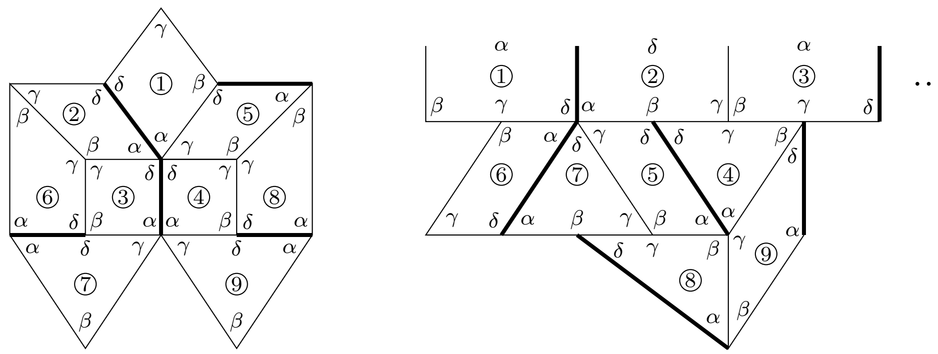

Figure 35.
Two possible $\alpha^2\gamma\delta^2$ and their AAD.

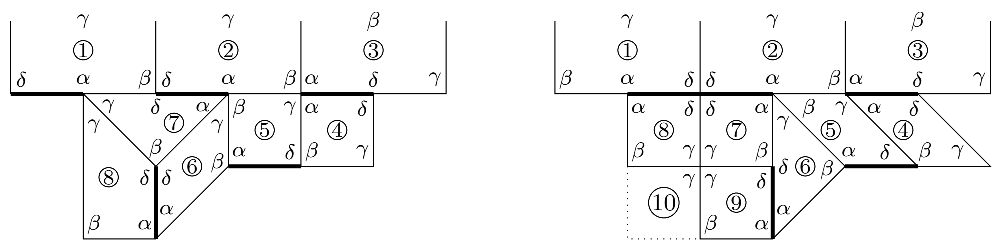

Figure 36.
Two possible AAD for $\beta\gamma^2$.

Similarly, we can determine T_8. Then $\alpha_4\alpha_5\beta_8\cdots = |^{\beta}\alpha_4^{\delta}\Vert^{\delta}\alpha_5^{\beta}|^{\gamma}\beta_8^{\alpha}|^{\beta}\gamma_9^{\delta}|$ determines T_9. We get $\beta_4\gamma_3\delta_9\cdots$, contradicting the *AVC*.

Therefore, $\alpha^2\gamma\delta^2$ is not a vertex. This implies $\alpha\cdots = \alpha^2\beta\gamma$.

If $\beta\gamma^2$ is a vertex, it has only two possible AAD. In the first picture of Figure 36, $\beta\gamma^2 = |^{\delta}\gamma_1^{\beta}|^{\delta}\gamma_2^{\beta}|^{\alpha}\beta_3^{\gamma}|$ determines T_1, T_2, T_3. Then $\Vert\alpha_3|\beta_2|\cdots = \Vert^{\delta}\alpha_3^{\beta}|^{\gamma}\beta_2^{\alpha}|^{\beta}\gamma_5^{\delta}|^{\beta}\alpha_4^{\delta}\Vert$ determines T_4, T_5; $\Vert\alpha_2|\beta_5|\cdots = \Vert^{\delta}\alpha_2^{\beta}|^{\gamma}\beta_5^{\alpha}|^{\beta}\gamma_6^{\delta}|^{\beta}\alpha_7^{\delta}\Vert$, $\beta_1\delta_2\delta_7\cdots = \beta_1\delta_2\delta_7$ determines T_6, T_7; $\beta_7\delta_6\cdots = \beta_7\delta_6\delta_8$ determines T_8. We get $\alpha_1\gamma_7\gamma_8\cdots$, contradicting the *AVC*.

In the second picture of Figure 36, $\beta\gamma^2 = |^{\beta}\gamma_1^{\delta}|^{\delta}\gamma_2^{\beta}|^{\alpha}\beta_3^{\gamma}|$ determines T_1, T_2, T_3. Then $\Vert\alpha_3|\beta_2|\cdots = |^{\delta}\alpha_3^{\beta}|^{\gamma}\beta_2^{\alpha}|^{\beta}\gamma_5^{\delta}|^{\beta}\alpha_4^{\delta}\Vert$ determines T_4, T_5; $\Vert\alpha_2|\beta_5|\cdots = \Vert^{\delta}\alpha_2^{\beta}|^{\gamma}\beta_5^{\alpha}|^{\beta}\gamma_6^{\delta}|^{\beta}\alpha_7^{\delta}\Vert$ determines T_6, T_7; $\delta_1\delta_2\delta_7\cdots = \delta_1\delta_2\delta_7\delta_8$ determines T_8; $\beta_7\delta_6\cdots = \beta_7\delta_6\delta_9$ determines T_9; $\gamma_7\gamma_8\gamma_9\cdots = \gamma_7\gamma_8\gamma_9\gamma_{10}$. We get $\beta_8\beta_{10}\cdots$ or $\beta_9\beta_{10}\cdots$, contradicting the *AVC*.

Therefore, $\beta\gamma^2$ is not a vertex.

This implies the $AVC \subset \{\beta\delta^2, \alpha^2\beta\gamma, \gamma^4, \gamma^2\delta^2, \delta^4\}$. There is only one solution satisfying Balance Lemma 2.6: $\{8\beta\delta^2, 8\alpha^2\beta\gamma, 2\gamma^4\}$. We have the AAD $\gamma^4 = |^{\beta}\gamma_1^{\delta}|^{\beta}\gamma_2^{\delta}|^{\beta}\gamma_3^{\delta}|^{\beta}\gamma_4^{\delta}|$ which determines T_1, T_2, T_3, T_4. Then $\beta_2\delta_1\cdots = \beta_2\delta_1\delta_5$ determines T_5. Similarly, we can determine T_6, T_7, T_8. Then $\alpha_2\alpha_6\gamma_5\cdots = |^{\beta}\alpha_2^{\delta}\Vert^{\delta}\alpha_6^{\beta}|^{\gamma}\beta_9^{\alpha}|^{\beta}\gamma_5^{\delta}|$. So $\beta_5\cdots = \alpha_9\beta_5\cdots$ or $\beta_5\gamma_9\cdots$, shown in two pictures of Figure 37, respectively.

In the left of Figure 37, $\beta_5\cdots = \alpha_9\beta_5\cdots$ determines T_9. Then $\alpha_3\alpha_7\gamma_6\cdots = \alpha_3\alpha_7\beta_{10}\gamma_6, \beta_6\gamma_9\cdots = \alpha_{10}\beta_6\gamma_9\cdots$ determine T_{10}. Similarly, we can determine T_{11}, T_{12}. Then $\alpha_9\beta_5\gamma_{12}\cdots = \alpha_9\alpha_{13}\beta_5\gamma_{12}$ determines T_{13}. Similarly, we can determine T_{14}, T_{15}, T_{16}.

In the right of Figure 37, $\beta_5\cdots = \beta_5\gamma_9\cdots$ determines T_9. Then we get a different tiling by similar deductions. The 3D pictures for these two tilings are shown in Figure 3. Their authentic pictures of the stereo-graphic projection are shown in Figure 38. This is Case $(1,4,2,2)/4$ in Table 1.

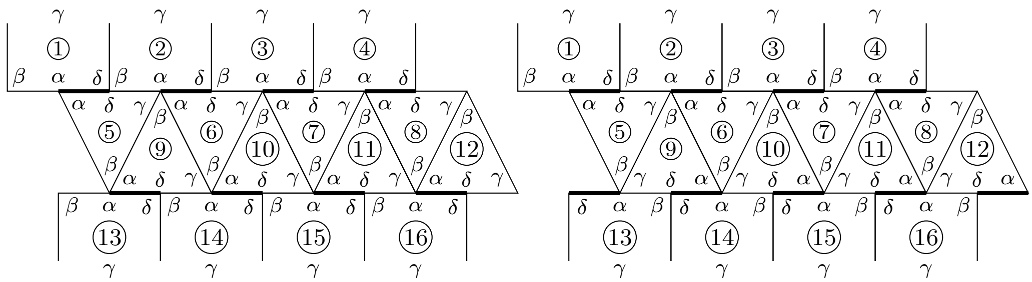

Figure 37.
Two tilings for $\{8\beta\delta^2, 8\alpha^2\beta\gamma, 2\gamma^4\}$.

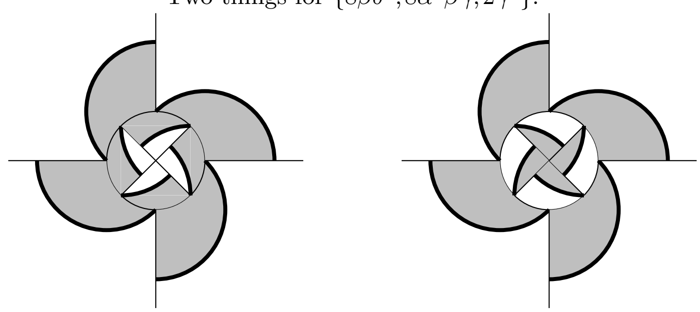

Figure 38.
Stereo-graphic projection for two tilings of $\{8\beta\delta^2, 8\alpha^2\beta\gamma, 2\gamma^4\}$.

Appendix: Exact and numerical geometric data

Angles $(\alpha,\beta,\gamma,\delta)$	Edges
$(6,3,4,3)/6$	$a=1/2, b=1/6$
$(1,8,4,3)/6$	$a=\arccos(1/3)\approx 0.3918, b=1$
$(12,4,6,2)/9$	$a=1-\arcsin\left(\frac{\sqrt{2}}{\sqrt{\cos\left(\frac{2\pi}{9}\right)\left(2-2\cos\left(\frac{4\pi}{9}\right)\right)}}\right)\approx 0.5673$
	$b=\arccos\left(\frac{\sqrt{3}\cot\left(\frac{2\pi}{9}\right)-\cot\left(\frac{2\pi}{9}\right)\sin\left(\frac{\pi}{9}\right)}{1+\cos\left(\frac{\pi}{9}\right)}\right)\approx 0.1741$
$(2,10,3,6)/9$	$a=\arccos\left(\frac{4\sqrt{3}\sin\left(\frac{2\pi}{9}\right)}{3}-1\right)\approx 0.3390$
	$b=\arccos\left(\frac{8\cos\left(\frac{\pi}{9}\right)-4\sqrt{3}\sin\left(\frac{4\pi}{9}\right)-1}{3}\right)\approx 0.5324$
$(1,21,5,8)/15$	$a=\arccos\left(\frac{2\sin\left(\frac{\pi}{15}\right)-\sqrt{3}\cos\left(\frac{7\pi}{15}\right)}{\sin\left(\frac{7\pi}{15}\right)}\right)\approx 0.4241$
	$b=\arccos\left(\frac{51-90\sqrt{3}\sin\left(\frac{2\pi}{5}\right)-96\sqrt{3}\sin\left(\frac{7\pi}{15}\right)+88\cos\left(\frac{2\pi}{15}\right)+184\cos\left(\frac{\pi}{15}\right)}{1+6\cos\left(\frac{7\pi}{15}\right)-2\cos\left(\frac{2\pi}{15}\right)+6\cos\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{\pi}{5}\right)}\right)\approx 0.7413$
$(4,9,5,17)/15$	$a=\arccos\left(\frac{2\sin\left(\frac{\pi}{15}\right)-\sqrt{3}\cos\left(\frac{7\pi}{15}\right)}{\sin\left(\frac{7\pi}{15}\right)}\right)\approx 0.4241$
	$b=\arccos\left(\frac{-3+9\sqrt{5}-5\sqrt{3}\sqrt{10-2\sqrt{5}}}{-9-9\sqrt{5}+\sqrt{3}\left(\sqrt{5}+4\right)\sqrt{10-2\sqrt{5}}}\right)\approx 0.1654$
$(9,28,10,23)/30$	$a=\arccos\left(\frac{\cot\left(\frac{\pi}{10}\right)-2\cot\left(\frac{\pi}{10}\right)\cos\left(\frac{7\pi}{30}\right)\sin\left(\frac{\pi}{5}\right)}{2\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{7\pi}{30}\right)}\right)\approx 0.3353$
	$b=\arccos\left(\frac{30+2\sqrt{5}-\sqrt{3}\left(5+\sqrt{5}\right)\sqrt{10-2\sqrt{5}}}{2-10\sqrt{5}+3\sqrt{3}\left(\sqrt{5}+1\right)\sqrt{10-2\sqrt{5}}}\right)\approx 0.4159$

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Angles $(\alpha,\beta,\gamma,\delta)$	Edges
$(3,16,10,41)/30$	$a=\arccos\left(\frac{\sqrt{3}\cos(\frac{11\pi}{30})-2\sin(\frac{\pi}{10})}{\sin(\frac{11\pi}{30})}\right)\approx 0.4698$
	$b=\arccos\left(\frac{-28+60\sqrt{3}\sin(\frac{7\pi}{15})+61\sqrt{3}\sin(\frac{4\pi}{15})+61\sqrt{3}\sin(\frac{\pi}{15})-61\cos(\frac{2\pi}{15})-120\cos(\frac{\pi}{15})}{\cos(\frac{2\pi}{5})+3\cos(\frac{2\pi}{15})}\right)$ ≈ 0.1461
$(5,32,6,23)/30$	$a=\arccos\left(\frac{\cot(\frac{\pi}{10})-2\cot(\frac{\pi}{10})\cos(\frac{7\pi}{30})\sin(\frac{\pi}{5})}{2\sin(\frac{\pi}{5})\sin(\frac{7\pi}{30})}\right)\approx 0.3353$
	$b=\arccos\left(\frac{30+2\sqrt{5}-\sqrt{3}\,(5+\sqrt{5})\sqrt{10-2\sqrt{5}}}{2-10\sqrt{5}+3\sqrt{3}\,(\sqrt{5}+1)\sqrt{10-2\sqrt{5}}}\right)\approx 0.4159$
$(1,16,6,43)/30$	$a=\arccos\left(\frac{\sqrt{3}\cos(\frac{11\pi}{30})-2\sin(\frac{\pi}{10})}{\sin(\frac{11\pi}{30})}\right)\approx 0.4698$
	$b=\arccos\left(\frac{-7\sqrt{3}+22\sqrt{3}\cos(\frac{\pi}{15})-24\sqrt{3}\cos(\frac{2\pi}{15})+32\sin(\frac{7\pi}{15})-18\sin(\frac{2\pi}{5})}{21\sqrt{3}-66\sqrt{3}\cos(\frac{\pi}{15})+80\sqrt{3}\cos(\frac{2\pi}{15})-104\sin(\frac{7\pi}{15})+58\sin(\frac{2\pi}{5})}\right)\approx 0.2730$
$(1,42,4,17)/30$	$a=\arccos\left(\frac{2\sin(\frac{\pi}{15})-\sqrt{3}\cos(\frac{7\pi}{15})}{\sin(\frac{7\pi}{15})}\right)\approx 0.4241$
	$b=\arccos\left(\frac{\sqrt{3}\,(9\sqrt{5}+29)\sqrt{10-2\sqrt{5}}-58\sqrt{5}-70}{(15\sqrt{5}+27)\sqrt{3}\sqrt{10-2\sqrt{5}}-46\sqrt{5}-146}\right)\approx 0.5493$
$(3,20,4,13)/18$	$a=\arccos\left(\frac{4\sqrt{3}\sin(\frac{2\pi}{9})}{3}-1\right)\approx 0.3390$
	$b=\arccos\left(\frac{\cos(\frac{\pi}{9})-1}{2\sqrt{3}\sin(\frac{4\pi}{9})-3\cos(\frac{\pi}{9})-1}\right)\approx 0.4527$
$(1,4,2,2)/4$	$a=1/4, b=1/2$
$(5,4,7,3)/9$	$a=\arccos\left(\frac{\sqrt{3}\cot(\frac{2\pi}{9})-\cot(\frac{2\pi}{9})\sin(\frac{\pi}{9})}{1+\cos(\frac{\pi}{9})}\right)\approx 0.1741$
	$b=\arccos\left(\frac{68\sqrt{3}+47\sqrt{3}\cos(\frac{\pi}{9})+162\sin(\frac{2\pi}{9})+162\sin(\frac{\pi}{9})}{99\sqrt{3}+69\sqrt{3}\cos(\frac{\pi}{9})+234\sin(\frac{2\pi}{9})+234\sin(\frac{\pi}{9})}\right)\approx 0.2584$
$(15,6,10,7)/18$	$a=\arccos\left(4\cos\left(\frac{\pi}{9}\right)-3\right)\approx 0.2258$
	$b=\arccos\left(28\sqrt{3}\sin\left(\frac{4\pi}{9}\right)-36\cos\left(\frac{\pi}{9}\right)-13\right)\approx 0.1183$
$(\frac{4}{f},1-\frac{4}{f},\frac{4}{f},1)$	$a=\arccos\left(\frac{\cos(\frac{4\pi}{f})(1-\cos(\frac{4\pi}{f}))}{\sin^2(\frac{4\pi}{f})}\right), b=1-2a$
	$f=10, a\approx 0.4241, b\approx 0.1517;\ \lim_{f\to\infty} a=\lim_{f\to\infty} b=1/3$
$(\frac{2}{f},\frac{4f-4}{3f},\frac{4}{f},\frac{2f-2}{3f})$	$a=\arccos\left(\frac{\sqrt{3}\sin(\frac{8\pi}{3f})-\sqrt{3}\sin(\frac{4\pi}{3f})-\cos(\frac{4\pi}{3f})-\cos(\frac{8\pi}{3f})+2}{\sqrt{3}\sin(\frac{8\pi}{3f})+\sqrt{3}\sin(\frac{4\pi}{3f})+\cos(\frac{4\pi}{3f})-\cos(\frac{8\pi}{3f})}\right)$
	$b=\arccos\left(\frac{\sqrt{3}\sin(\frac{2\pi}{3f})+4\cos(\frac{2\pi}{f})-\cos(\frac{2\pi}{3f})}{\sqrt{3}\sin(\frac{2\pi}{3f})+3\cos(\frac{2\pi}{3f})}\right)$
	$+\ \arccos\left(\frac{\sqrt{3}\,(\cos(\frac{2\pi}{f})-\cos(\frac{2\pi}{3f})+\sqrt{3}\sin(\frac{2\pi}{3f}))}{3\sin(\frac{2\pi}{f})}\right)$
	$f=6, a\approx 0.3390, b\approx 0.8065;\ \lim_{f\to\infty} a=\lim_{f\to\infty} b=\arccos(1/3)$
$(\frac{2}{f},\frac{2f-4}{3f},\frac{4}{f},\frac{4f-2}{3f})$	$a=\arccos\left(\frac{\sqrt{3}\sin(\frac{2\pi}{3f})\cos(\frac{2\pi}{f})+\cos(\frac{2\pi}{3f})\cos(\frac{2\pi}{f})-1}{\sin(\frac{2\pi}{f})(\sqrt{3}\cos(\frac{2\pi}{3f})-\sin(\frac{2\pi}{3f}))}\right)$
	$\phi=\arccos\left(\frac{\sin(\frac{2\pi}{f})-\sin(\frac{(f+4)\pi}{6f})\sin(\frac{4\pi}{f})}{\sqrt{-2\sin(\frac{4\pi}{f})\sin(\frac{2\pi}{f})\sin(\frac{(f+4)\pi}{6f})-\cos(\frac{2\pi}{f})^2-\cos(\frac{4\pi}{f})^2+2}}\right)$
	$b=\arcsin\left(\frac{\sin a\sin(-\frac{2\pi}{3}+\frac{4\pi}{3f}+\phi)}{\sin(-\frac{4\pi}{3}+\frac{2\pi}{3f}+\phi)}\right)$
	$f=10, a\approx 0.4698, b\approx 0.0898;\ \lim_{f\to\infty} a=\lim_{f\to\infty} b=\arccos(1/3)$

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