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ON L-FUNCTIONS ASSOCIATED WITH THE VECTOR SPACE OF BINARY QUADRATIC FORMS

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Introduction

The purpose of this paper is to prove functional equations of L-functions associated with the vector space of binary quadratic forms and determine their poles and residues. For a commutative ring K, let $V(K)$ be the set of all symmetric matrices of degree 2 with coefficients in K. In $V(\mathbf{C})$, we consider the inner product

$$\langle x, y\rangle = \mathrm{tr}(xy^{\iota}) \quad (x, y \in V(\mathbf{C})),$$

where $y^{\iota} = \begin{pmatrix} y_3 & -y_2 \\ -y_2 & y_1 \end{pmatrix}$ for $y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}$. For $i = 1,2$, we set

$$V_i = \{x \in V(\mathbf{R}) \mid (-1)^i \det x > 0\},$$

and $V_i(\mathbf{Q}) = V(\mathbf{Q}) \cap V_i$. We define two subsets of $V_1(\mathbf{Q})$ by

$$V_1'(\mathbf{Q}) = \{x \in V_1(\mathbf{Q}) \mid \sqrt{-\det x} \notin \mathbf{Q}\},$$

$$V_1''(\mathbf{Q}) = \{x \in V_1(\mathbf{Q}) \mid \sqrt{-\det x} \in \mathbf{Q}\}.$$

Let $G = GL_2(\mathbf{C})$, $G(\mathbf{R}) = GL_2(\mathbf{R})$, $G(\mathbf{Q}) = GL_2(\mathbf{Q})$, and $G^+ = \{g \in G(\mathbf{R}) \mid \det g > 0\}$. Then G acts on $V(\mathbf{C})$ by $\rho(g)x = gx{}^tg$ for $x \in V(\mathbf{C})$ and $g \in G$, and the triple (G, ρ, V) is a prehomogeneous vector space with the singular set $S = \{x \in V \mid \det x = 0\}$. For $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, set $dg = (\det g)^{-2}\Pi_{1\le i,j\le 2}\, dg_{ij}$. Then dg defines a Haar measure on G^+. We consider the measures $dx = dx_1dx_2dx_3$ and $\omega(x) = |\det x|^{-3/2}\, dx$ $\left(x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}\right)$ on $V(\mathbf{R})$. The measure $\omega(x)$ is invariant under the action of G^+. Let Γ be a subgroup of $SL_2(\mathbf{Z})$ of finite index. We assume $\{\pm 1\} \subset \Gamma$. For $x \in V(\mathbf{R})$, let $G_x^+ = \{g \in G^+ \mid \rho(g)x = x\}$,

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and $\Gamma_x = G_x^+ \cap \Gamma$. Then there exists a measure $d\mu_x$ on G_x^+ which satisfies

$$\int_{G^+} F(g)\,dg = \int_{G^+/G_x^+} \omega(\rho(g)x) \int_{G_x^+} F(gh)\,d\mu_x(h) \quad \text{for} \quad F \in C_0(G^+).$$

It is known that the measure of G_x^+/Γ_x with respect to $d\mu_x$ is finite for $x \in V_1'(\mathbf{Q}) \cup V_2(\mathbf{Q})$, and we set

$$\mu(x) = \int_{G_x^+/\Gamma_x} d\mu_x.$$

For $x \in V_1''(\mathbf{Q})$, let $x = q\bar{x}$, $q \in \mathbf{Q}$. Here we assume $\bar{x}$ is primitive, i.e., $\bar{x}_1, 2\bar{x}_2, \bar{x}_3 \in \mathbf{Z}$ and $(\bar{x}_1, 2\bar{x}_2, \bar{x}_3) = 1$ for $\bar{x} = \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_2 & \bar{x}_3 \end{pmatrix}$. We define $\mu(x) = 2^{-3}\log(4\,|\det \bar{x}\,|)$ following Sato [5]. Then $\mu(x)$ satisfies $\mu(\rho(\gamma)x) = \mu(x)$ for $x \in V_1(\mathbf{Q}) \cup V_2(\mathbf{Q})$ and $\gamma \in \Gamma$.

Let p be a fixed prime. We consider a lattice L in $V(\mathbf{Q})$ which is closed under Γ and satisfies $L \otimes_{\mathbf{Z}} \mathbf{Z}_p = V(\mathbf{Z}_p)$. Let L^* be the dual lattice of L with respect to the inner product $\langle\ ,\ \rangle$. Then L^* is also closed under Γ and satisfies $L^* \otimes_{\mathbf{Z}} \mathbf{Z}_p = V(\mathbf{Z}_p)$ if $p \neq 2$, and $V(\mathbf{Z}_p)^*$ the set of all half-integral matrices in $V(\mathbf{Q}_p)$ if $p = 2$.

We define functions on L and L^*, which we call characters in this paper. First assume $p \neq 2$. Let ψ be a character of $(\mathbf{Z}/p\mathbf{Z})^\times$. For $x \in L \cup L^*$, we denote by $\tilde{x}$ the element of $V(\mathbf{Z}/p\mathbf{Z})$ obtained from x by reduction modulo p. For $x \in L \cup L^*$, we define

$$\psi^{(2)}(x) = \begin{cases} \psi(\det \tilde{x}) & \text{if rank } \tilde{x} = 2\,; \\ 0 & \text{otherwise.} \end{cases}$$

Let ψ_p and ψ_0 be the quadratic character and the trivial one of $(\mathbf{Z}/p\mathbf{Z})^\times$ respectively and let $\psi = \psi_p$ or ψ_0. Assume $\operatorname{rank}(\tilde{x}) = 1$ for $x \in L \cup L^*$. Then we can find $g \in GL_2(\mathbf{F}_p)$ such that ${}^t g\tilde{x}g = \begin{pmatrix} x_0 & 0 \\ 0 & 0 \end{pmatrix}$. We define

$$\psi^{(1)}(x) = \begin{cases} \psi(x_0) & \text{if rank } \tilde{x} = 1\,; \\ 0 & \text{otherwise.} \end{cases}$$

Then the value of this character is independent of the choice of g. For ψ with $\psi^2 = 1$, we set

$$\psi^{(0)}(x) = \begin{cases} 1 & \text{if rank } \tilde{x} = 0\,; \\ 0 & \text{otherwise.} \end{cases}$$

Let $p = 2$, and let ψ_0 be the trivial character of $(\mathbf{Z}/2\mathbf{Z})^\times$. For $x \in L$ and

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r, $0 \leq r \leq 2$, we define

$$\psi_0^{(r)}(x) = \begin{cases} 1 & \text{if rank } \tilde{x} = r\,; \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in L^*$, let $Q_x(v) = {}^t vxv$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, be the binary quadratic form associated with x, and let $\tilde{Q}_x(v)$ be the quadratic form over the prime field $\mathbf{F}_2$ of characteristic 2 obtained from $Q_x(v)$ by reduction modulo 2. Then $\tilde{Q}_x(v)$ is equivalent to one of v_1v_2, $v_1^2 + v_1v_2 + v_2^2$, v_1^2, or 0. We define $\psi_2^{(2)}$ and $\psi_2^{(0)}$ by

$$\psi_2^{(2)}(x) = \begin{cases} 1 & \text{if } \tilde{Q}_x \sim v_1v_2\,; \\ -1 & \text{if } \tilde{Q}_x \sim v_1^2 + v_1v_2 + v_2^2\,; \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_2^{(0)}(x) = \begin{cases} 1 & \text{if } \tilde{Q}_x \sim 0\,; \\ 0 & \text{otherwise.} \end{cases}$$

Let $L_i = L \cap V_i$, and $L_i^* = L^* \cap V_i$. We define for $i = 0{,}1$ and $\varphi = \psi^{(2)}$, $\psi_p^{(r)}(p \neq 2,\ r = 0,\ 1,\ 2)$, or $\psi_0^{(r)}(r = 0,\ 1)$

$$\zeta_1(s, L, \varphi) = \sum_{x \in \Gamma \backslash L_1} \varphi(x)\mu(x)\,|\det x|^{-s},$$

$$\zeta_2(s, L, \varphi) = \sum_{x \in \Gamma \backslash L_2,\ x_1 > 0} \varphi(x)\mu(x)\,|\det x|^{-s} \quad (s \in \mathbf{C}).$$

Here the summations on x are extended over all equivalence classes of L_1 with respect to Γ and over all equivalence classes of L_2 with respect to Γ satisfying $x_1 > 0$ for $x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$ respectively. We define $\zeta_i(s, L^*, \varphi)$ for $i = 1, 2$ and $\varphi = \psi^{(2)}$, $\psi_p^{(r)}(p \neq 2,\ r = 0,\ 1)$, or $\psi_2^{(r)}(r = 0,\ 2)$ by the same formulas taking summation over $\Gamma \backslash L_1^*$ and $\Gamma \backslash L_2^*$ with $x_1^* > 0$ for $x^* = \begin{pmatrix} x_1^* & x_2^* \\ x_2^* & x_3^* \end{pmatrix}$ respectively. Then these series converge absolutely for $\mathrm{Re}\, s > 3/2$. We set for $p \neq 2$

$$\zeta(s, M, \psi^{(2)}) = (\zeta_1(s, M, \psi^{(2)}), \zeta_2(s, M, \psi^{(2)})) \quad \text{for } M = L, L^*, \quad \text{and } \psi \neq \psi_p, \psi_0,$$

$$\zeta_o(s, M, \psi_p) = (\zeta_1(s, M, \psi_p^{(1)}), \zeta_2(s, M, \psi_p^{(1)})) \quad \text{for } M = L, L^*,$$

and for any prime p

$$\zeta_e(s, L^*, \psi_p) = \begin{pmatrix} \zeta_1(s, L^*, \psi_p^{(0)}) & \zeta_2(s, L^*, \psi_p^{(0)}) \\ \zeta_1(s, L^*, \psi_p^{(2)}) & \zeta_2(s, L^*, \psi_p^{(2)}) \end{pmatrix},$$

$$\zeta(s,L,\psi_0)=\begin{pmatrix}\zeta_1(s,L,\psi_0^{(0)}+\psi_0^{(1)}) & \zeta_2(s,L,\psi_0^{(0)}+\psi_0^{(1)})\\ \zeta_1(s,L,\psi_0^{(2)}) & \zeta_2(s,L,\psi_0^{(2)})\end{pmatrix}.$$

Here $\zeta_i(s,L,\psi_0^{(0)}+\psi_0^{(1)})=\zeta_i(s,L,\psi_0^{(0)})+\zeta_i(s,L,\psi_0^{(1)})$. Let $M_1'=M_1\cap V_1'(\mathbf{Q})$, $M_1''=M_1\cap V_1''(\mathbf{Q})$ for $M=L,L^*$. We set

$$\eta(s,L,\varphi)=\sum_{x\in\Gamma\backslash L_1''}\varphi(x)\,|\det x|^{-s},$$

and we define $\eta(s,L^*,\varphi)$ by the same formula taking the summation over $\Gamma\backslash L_1^{*\prime\prime}$. Then we can show

THEOREM 1. *Let φ be one of the characters defined above. The series $\zeta_i(s,L,\varphi)$ and $\zeta_i(s,L^*,\varphi)$ can be extended to the whole s-plane as meromorphic functions and satisfy the functional equations*:

(1) *If $\varphi\neq\psi_p,\psi_0$, then*

$$\begin{aligned}&\zeta(3/2-s,L^*,(\bar\psi\psi_p)^{(2)})\\&=v(L)2^{2-2s}\pi^{1/2-2s}p^{2s-3}\Gamma(s-1/2)\Gamma(s)W_2^2(\bar\psi\psi_p)\\&\quad\times\left\{\zeta(s,L,\psi^{(2)})\begin{pmatrix}\sin\pi s & 0\\ 1 & \cos\pi s\end{pmatrix}\right.\\&\quad\left.+2^{-3}\eta(s,L,\psi^{(2)})\left(\left(\frac{\Gamma'(s)}{\Gamma(s)}-\frac{\Gamma'(s-1/2)}{\Gamma(s-1/2)}\right)\sin\pi s,\ -\pi\right)\right\}.\end{aligned}$$

(2) *Let $\varphi=\psi_p$ and $r=1$.*

(i) *If $\psi_p(-1)=1$, then*

$$\begin{aligned}\zeta_o(3/2-s,L^*,\psi_p)&=v(L)2^{2-2s}\pi^{1/2-2s}p^{2s-3}\Gamma(s-1/2)\Gamma(s)\\&\quad\times W_o^2(\psi_p)\zeta_o(s,L,\psi_p)\begin{pmatrix}\sin\pi s & 0\\ 1 & \cos\pi s\end{pmatrix}.\end{aligned}$$

(ii) *If $\psi_p(-1)=-1$, $\zeta_1(s,M,\psi_p^{(1)})=0$ for $M=L,L^*$, and*

$$\begin{aligned}&\zeta_2(3/2-s,L^*,\psi_p^{(1)})\\&=\sqrt{-1}^{-1}v(L)2^{2-2s}\pi^{1/2-2s}p^{2s-3}\Gamma(s-1/2)\Gamma(s)\sin(\pi s)W_o^2(\psi_p)\zeta_2(s,L,\psi_p^{(1)}).\end{aligned}$$

(3) *If $\varphi=\psi_p^{(r)}$, $r=0,2$, or $\psi_0^{(r)}$, $r=0,1,2$, then*

$$\begin{aligned}&\zeta_e(3/2-s,L^*,\psi_p)\\&=v(L)2^{2-2s}\pi^{1/2-2s}p^{2s}\Gamma(s-1/2)\Gamma(s)W_e^2(\psi_p)^{-1}\end{aligned}$$

$$\times \left\{\zeta(s, L, \phi_0)\begin{pmatrix} \sin \pi s & 0 \\ 1 & \cos \pi s \end{pmatrix} + 2^{-3}\begin{pmatrix} \eta(s, L, \phi_0^{(0)} + \phi_0^{(1)}) & 0 \\ \eta(s, L, \phi_0^{(2)}) & 0 \end{pmatrix}\right.$$
$$\left.\times \begin{pmatrix} (\Gamma'(s)/\Gamma(s) - \Gamma'(s-1/2)/\Gamma(s-1/2)) \sin \pi s & -\pi \\ 0 & 0 \end{pmatrix}\right\}.$$

Here $v(L)$ is the volume of a fundamental paralelleogram of L with respect to dx, and $\eta(s, L, \phi_0^{(0)} + \phi_0^{(1)}) = \eta(s, L, \phi_0^{(0)}) + \eta(s, L, \phi_0^{(1)})$.

In the above theorem, $W_2^2(\psi)$, $W_o^2(\psi_p)$ and $W_e^2(\psi_p)$ are the Gauss sums introduced in [4], the definition and explicit values of which will be given in Section 1. In Theorem 5 in Section 1, we will determine the poles of $\zeta_i(s, M, \phi^{(r)})$, and give formulas for the principal parts at their poles. The proof of Theorem 1 and Theorem 5 is a modification of Shintani [8] and Sato [5]. Namely, we consider the prehomogeneous vector space $(\tilde{G}, \tilde{\rho}, \tilde{V})$, where

$$\tilde{G} = GL_2(\mathbf{C}) \times GL_1(\mathbf{C}),\ \tilde{V} = V \oplus \mathbf{C}^2 \quad \text{(considered as column vectors)},$$

and

$$\tilde{\rho}(g, t)(x, y) = (gx{}^tg, {}^tg^{-1}yt) \quad (g \in GL_2(\mathbf{C}),\ t \in GL_1(\mathbf{C}),\ x \in V,\ y \in \mathbf{C}^2).$$

Then our L-functions can be obtained as residues of L-functions of $(\tilde{G}, \tilde{\rho}, \tilde{V})$ and the functional equations can be derived from those of L-functions associated with $(\tilde{G}, \tilde{\rho}, \tilde{V})$. The L-function $\zeta_2(s, L, \phi_p^{(1)})$ was introduced by Hashimoto. Arakawa [1] proposed interesting conjectures concerning its special values at $s = 0$. The L-functions associated with the space of quadratic forms of degree $n \geq 3$ with the characters of the above type were treated in [4]. We refer to Sato [6] for a general treatment of L-functions associated with prehomogeneous vector spaces.

§1. Preliminaries

We retain the notation introduced in Introduction. In this section, we will give the definition of Gauss sums and state the result on the poles and residues of the L-functions defined in Introduction. Let p be a prime number. For $a \in \mathbf{C}$ let $e(a) = \exp(2\pi\sqrt{-1}\, a)$, and $e_p(a) = \exp(2\pi\sqrt{-1}\, a/p)$. For a character ψ of $\mathbf{F}_p^\times$, we denote by $G(\psi)$ the ordinary Gauss sum $\sum_{a \in \mathbf{F}_p^\times} \psi(a) e_p(a)$. First assume p is odd. For $x \in V(\mathbf{F}_p)$, a character ψ of $\mathbf{F}_p^\times$, and an integer r, $0 \leq r \leq 2$, we define

$$W_r^2(x, \psi) = \sum_{y \in V(\mathbf{F}_p)} \psi^{(r)}(y) e_p(\mathrm{tr}(xy)).$$

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Here we assume $r = 2$ if $\psi^2 \neq \psi_0$. Then the following result was proved in [4], §1.

PROPOSITION 2. *Assume $p \neq 2$.*

(1) *Let $\psi^2 \neq \psi_0$. Then one has*

$$W_2^2(x, \psi) = (\bar{\psi}\psi_p)^{(2)}(x) G(\psi) G(\psi_p) G(\psi\psi_p).$$

(2) *Let $\psi = \psi_p$. Then one has*

$$W_1^2(x, \psi_p) = \psi_p^{(1)}(x) p G(\psi_p).$$

(3) *Let $\psi = \psi_0$. If* rank $x = 1$, $x \in V(\mathbf{F}_p)$, *then*

$$W_0^2(x, \psi_0) + W_1^2(x, \psi_0) = W_2^2(x, \psi_0) = 0,$$

and let rank $x_0 = 0$, rank $x_2 = 2$, *for* $x_0, x_2 \in V(\mathbf{F}_p)$. *Then*

$$\begin{pmatrix} W_0^2(x_0, \psi_0) + W_1^2(x_0, \psi_0) & W_0^2(x_2, \psi_0) + W_1^2(x_2, \psi_0) \\ W_2^2(x_0, \psi_0) & W_2^2(x_0, \psi_0) \end{pmatrix}$$
$$= \begin{pmatrix} p^2 & \psi_p(-1)p \\ p^3 - p^2 & -\psi_p(-1)p \end{pmatrix} \begin{pmatrix} \psi_p^{(0)}(x_0) & 0 \\ 0 & \psi_p^{(2)}(x_2) \end{pmatrix}.$$

We set

$$W_2^2(\psi) = G(\psi) G(\psi_p) G(\psi\psi_p) \quad \text{for} \quad \psi \quad \text{with} \quad \psi^2 \neq \psi_0,$$
$$W_o^2(\psi_p) = pG(\psi_p),$$
$$W_e^2(\psi_p) = \begin{pmatrix} p^2 & \psi_p(-1)p \\ p^3 - p^2 & -\psi_p(-1)p \end{pmatrix},$$

and set for simplicity

$$W_e^2(\psi_p) = \begin{pmatrix} w(0,0) & w(0,2) \\ w(2,0) & w(2,2) \end{pmatrix}.$$

We note $W_2^2(\psi) W_2^2(\bar{\psi}\psi_p) = p^3$ for $\psi^2 \neq \psi_0$, and $W_o^2(\psi_p)^2 = \psi_p(-1)p^3$. We set

$$\tilde{W}_e^2(\psi_p) = \begin{pmatrix} p^2 & \psi_p(-1)p \\ \psi_p(-1)p(p^3 - p^2) & -p^2 \end{pmatrix} G(\psi_p).$$

Then we have (cf. Proposition 1.17, [4])

$$\tilde{W}_e^2(\psi_p)^2 = p^6 \psi_p(-1) E_2.$$

Here E_2 is the identity matrix of degree 2.

Now assume $p = 2$. For a quadratic form x on $\mathbf{F}_2$, choose a half integral symmetic matrix $\bar{x} \in V(\mathbf{Z})^*$ such that $\tilde{Q}_{\bar{x}} = x$. For y, take $\bar{y} \in V(\mathbf{Z})$ such that $\bar{y} \bmod 2 = y$. Here we identify $\mathbf{F}_2$ with $\mathbf{Z}/2\mathbf{Z}$. Let $\mathrm{tr}(xy) = (\bar{x}\bar{y}) \bmod 2$. Then $\mathrm{tr}(xy)$ is independent of the choice of $\bar{x}$ and $\bar{y}$. We define

$$W_r^2(x, \phi_0) = \sum_{y \in V(\mathbf{F}_2)} \phi_0^{(r)}(y) e_2(\mathrm{tr}(xy)).$$

We see the definition is independent of the choice of $\bar{x}$. The following result was proved in [3], §2.

Proposition 3. *Assume $p = 2$ and let the notation be as above. For a quadratic form $x \sim v_1^2$, one has $W_2^2(x, \phi_0) = W_0^2(x, \phi_0) + W_1^2(x, \phi_0) = 0$. Let $\phi_2^{(2)}(x_2) \neq 0$ and $\phi_2^{(0)}(x_0) \neq 0$ for quadratic forms x_0, x_2. Then*

$$\begin{pmatrix} W_0^2(x_0, \phi_0) + W_1^2(x_0, \phi_0) & W_0^2(x_2, \phi_0) + W_1^2(x_2, \phi_0) \\ W_2^2(x_0, \phi_0) & W_2^2(x_2, \phi_0) \end{pmatrix} = \begin{pmatrix} 2^2 & 2 \\ 2^3 - 2^2 & -2 \end{pmatrix} \begin{pmatrix} \phi_2^{(0)}(x_0) & 0 \\ 0 & \phi_2^{(2)}(x_2) \end{pmatrix}.$$

We set

$$W_e^2(\phi_2) = \begin{pmatrix} 2^2 & 2 \\ 2^3 - 2^2 & -2 \end{pmatrix} = \begin{pmatrix} w(0,0) & w(0,2) \\ w(2,0) & w(2,2) \end{pmatrix},$$

as before. We note that if we set $p = 2$ and $\phi_p(-1) = 1$ in $W_e^2(\phi_p)$, we obtain $W_e^2(\phi_2)$.

Let M be a lattice in $V(\mathbf{Q})$. Let $x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in V(\mathbf{R})$, and set

$$\lambda(M) = \min\{|x_1| \mid x \in M, x_1 \neq 0\},$$

$$\mu(M) = \min\{|x_2| \mid x \in M, x_1 = 0, x_2 \neq 0\},$$

$$\lambda'(M) = \min\{|x_3| \mid x \in M, x_1 = x_2 = 0, x_3 \neq 0\}.$$

Then we see easily

Lemma 4. *Let M^* be the dual lattice of M. Then one has:*

(1) $\lambda(M)\lambda'(M^*) = 1$, $\lambda(M^*)\lambda'(M) = 1$.

(2) $v(M) = \lambda(M)\mu(M)\lambda'(M)$, $v(M^*) = \lambda(M^*)\mu(M^*)\lambda'(M^*)$.
(3) $\mu(M)\mu(M^*) = 1/2$, $v(M)v(M^*) = 1/2$.

Let $\mathfrak{H}$ be the complex upper half plane. Then G^+ acts on $\mathfrak{H}$ by the linear fractional transformation $gz = (az+b)(cz+d)^{-1}$, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+$, $z \in \mathfrak{H}$. Let Γ and L be as in Introduction, and let $\kappa_1, \kappa_2, \ldots, \kappa_\nu$ ($\kappa_i \in \mathbf{Q} \cup \infty$) be a complete set of representatives of cusps with respect to Γ. For each i ($1 \leq i \leq \nu$), take $\sigma_i \in SL_2(\mathbf{Z})$ such that $\kappa_i = \sigma_i 0$ and set $y^{(i)} = {}^t\sigma_i^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let $\Gamma_\infty^{(i)} = \{\gamma \in \Gamma \mid {}^t\gamma^{-1} y^{(i)} = y^{(i)}\}$, and set $\Gamma_\infty(m) = \left\{\begin{pmatrix} 1 & 0 \\ mn & 1 \end{pmatrix} \mid n \in \mathbf{Z}\right\}$. Then there exists a positive integer δ_i such that $\Gamma_\infty^{(i)} = \sigma_i \Gamma_\infty(\delta_i)\sigma_i^{-1}$. We set $L^{(i)} = \sigma_i^{-1} L\, {}^t\sigma_i^{-1}$ and $\lambda_i = \lambda(L^{(i)})$, $\mu_i = \mu(L^{(i)})$, $\lambda_i' = \lambda'(L^{(i)})$, $\lambda_i^* = \lambda(L^{(i)*})$, $\mu_i^* = \mu(L^{(i)*})$, $\lambda_i^{*\prime} = \lambda'(L^{(i)*})$. Let $v(\Gamma\backslash\mathfrak{H})$ be the volume of the fundamental domain of $\mathfrak{H}$ by Γ with respect to the invariant measure $y^{-2}dxdy$ for $z = x + \sqrt{-1}\,y \in \mathfrak{H}$. In these notations, we can state our result on the poles and residues as follows.

THEOREM 5. *Let the notation be as above.*

(1) *If* $\psi^2 \neq \psi_0$, *the L-functions* $\zeta_i(s, L, \psi^{(2)})$ *are entire for* $i = 1, 2$.

(2) *Let* p *be an odd prime,* $\psi = \psi_p$, *and* $r = 1$.

(i) *If* $\psi_p(-1) = 1$, $\zeta_i(s, L, \psi_p^{(1)})$ *is holomorphic except at* $s = 1$ *for* $i = 1,2$, *and the principal part of* $\zeta_o(s, L, \psi^{(1)})$ *at* $s = 1$ *is given by*

$$2^{-2}v(L)^{-1}pW_o^2(\psi_p)^{-1}L(1, \psi_p)\left(\sum_{i=1}^{\nu} \psi_p(\lambda_i)\delta_i\lambda_i\right)(s-1)^{-1}(1,0).$$

(ii) *If* $\psi_p(-1) = -1$, $\zeta_2(s, L, \psi_p^{(1)})$ *is holomorphic except at* $s = 1$, *and the principal part of* $\zeta_2(s, L, \psi_p^{(1)})$ *at* $s = 1$ *is given by*

$$2^{-2}\sqrt{-1}\,v(L)^{-1}pW_o^2(\psi_p)^{-1}L(1, \psi_p)\left(\sum_{i=1}^{\nu} \psi_p(\lambda_i)\delta_i\lambda_i\right)(s-1)^{-1}.$$

(3) *Let* p *be a prime. If* $\psi = \psi_p^{(r)}$ ($r = 0,2$), *or* $\psi = \psi_0^{(r)}$ ($r = 0,1,2$), $\zeta_i(s, L, \psi_0^{(0)} + \psi_1^{(0)})$, $\zeta_i(s, L^*, \psi_0^{(2)})$, $\zeta_i(s, L^*, \psi_p^{(0)})$, $\zeta_i(s, L^*, \psi_0^{(2)})$ *are holomorphic except at* $s = 3/2$ *or* $s = 1$, *and the principal parts of* $\zeta(s, L, \psi_0)$ *at* $s = 3/2$ *and* $s = 1$ *are given respectively by*

$$2^{-2}v(L)^{-1}p^{-3}v(\Gamma\backslash\mathfrak{H})\pi(s-3/2)^{-1}\begin{pmatrix} w(0,0) & w(0,0) \\ w(2,0) & w(2,0) \end{pmatrix},$$

$$-2^{-3}v(L)^{-1}p^{-3}\Big\{(\sum_{i=1}^{\nu}\delta_i\lambda_i)(s-1)^{-2}\begin{pmatrix} w(0,0) & 0 \\ w(2,0) & 0\end{pmatrix}$$

$$-(s-1)^{-1}\begin{pmatrix} 2w(0,0)(\sum_{i=1}^{\nu}\delta_i\lambda_i\log(\lambda_i/4\pi)) & -w(0,0)(\sum_{i=1}^{\nu}\delta_i\lambda_i)\pi \\ 2w(2,0)(\sum_{i=1}^{\nu}\delta_i\lambda_i\log(\lambda_i/4\pi)) & -w(2,0)(\sum_{i=1}^{\nu}\delta_i\lambda_i)\pi\end{pmatrix}\Big\},$$

and those of $W_e^2(\psi_p)\zeta_e(s, L^*, \psi_p)$ *at* $s = 3/2$ *and* $s = 1$ *are given respectively by*

$$2^{-1}v(L)v(\Gamma\backslash\mathfrak{H})\pi(s-3/2)^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 0\end{pmatrix},$$

$$-2^{-3}v(L)p\Big\{(\sum_{i=1}^{\nu}\delta_i\lambda_i^*)(s-1)^{-2}\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}$$

$$-(s-1)^{-1}\begin{pmatrix} 2(\sum_{i=1}^{\nu}\delta_i\lambda_i^*\log(p\lambda_i^*/4\pi)) & -\pi(\sum_{i=1}^{\nu}\delta_i\lambda_i^*) \\ 0 & 0\end{pmatrix}\Big\}.$$

Finally we give a formula for $\eta(s, M, \varphi)$ in terms of λ_i, μ_i, λ_i^* and δ_i.

LEMMA 6. *Let* $M = L$ *or* L^*. *The function* $\eta(s, M, \varphi)$ *is given by*

$$\begin{cases} (\sum_{i=1}^{\nu}2\psi(-\mu_i^2)\delta_i\lambda_i^*\mu_i^{-(2s-1)})L(2s-1,\psi^2) & \text{if } \varphi = \psi^{(2)} \text{ with } \psi^2 \neq \psi_0 \text{ and } M = L, \\ (\text{resp. } (\sum_{i=1}^{\nu}2\psi(-\mu_i^{*2})\delta_i\lambda_i\mu_i^{*-(2s-1)})L(2s-1,\psi^2)) & (\text{resp. } M = L^*) \\ 0 & \text{if } \varphi = \psi_p^{(1)}, \\ (\sum_{i=1}^{\nu}2\delta_i\lambda_i^*\mu_i^{-(2s-1)})p^{-(2s-1)}\zeta(2s-1) & \text{if } \varphi = \psi_0^{(0)} + \psi_0^{(1)} \text{ and } M = L, \\ (\sum_{i=1}^{\nu}2\delta_i\lambda_i^*\mu_i^{-(2s-1)})(1-p^{-(2s-1)})\zeta(2s-1) & \text{if } \varphi = \psi_0^{(2)} \text{ and } M = L, \\ (\sum_{i=1}^{\nu}2\delta_i\lambda_i\mu_i^{*-(2s-1)})p^{-2s}\zeta(2s-1) & \text{if } \varphi = \psi_0^{(2)} \text{ and } M = L^*, \\ (\sum_{i=1}^{\nu}2\psi_p(-1)\delta_i\lambda_i\mu_i^{*-(2s-1)})(1-p^{-(2s-1)})\zeta(2s-1) & \text{if } \varphi = \psi_p^{(2)} \text{ and } M = L^*. \end{cases}$$

Here we set $\psi_2(-1) = 1$.

Proof. We give a proof of the case $\varphi = \psi^{(2)}$. The other cases can be treated in the same way. We see easily that $L_1'' \subset \cup_{1\le i\le\nu}\{\rho(\gamma\sigma_i)\begin{pmatrix} 0 & \alpha \\ \alpha & \beta\end{pmatrix} \mid \gamma\in\Gamma, \alpha, \beta \in \mathbf{Q}, \alpha > 0\}$. If $\rho(\gamma)\begin{pmatrix} 0 & \alpha \\ \alpha & \beta\end{pmatrix} = \begin{pmatrix} 0 & \alpha' \\ \alpha' & \beta'\end{pmatrix}$ for $\gamma \in SL_2(\mathbf{Z})$, then $\alpha = \alpha'$ and $\gamma \in$

$\Gamma_\infty(1)\{\pm 1\}$. Hence the above union is disjoint. The condition $\rho(\gamma\sigma_i)\begin{pmatrix}0 & \alpha\\ \alpha & \beta\end{pmatrix} \in L$ implies $\begin{pmatrix}0 & \alpha\\ \alpha & \beta\end{pmatrix} \in \rho(\sigma_i^{-1})L$. We note by the definition of λ_i, μ_i and λ'_i, the lattice $p(\sigma_i^{-1})L$ has a $\mathbf{Z}$-basis consisting of elements of the form $\begin{pmatrix}\lambda_i & *\\ * & *\end{pmatrix}$, $\begin{pmatrix}0 & \mu_i\\ \mu_i & *\end{pmatrix}$, and $\begin{pmatrix}0 & 0\\ 0 & \lambda'_i\end{pmatrix}$. Hence the above condition implies in particular $\alpha = \mu_i n$ for some positive integer n. On the other hand, $\rho(\sigma_i)\begin{pmatrix}0 & \mu_i n\\ \mu_i n & \beta\end{pmatrix} = \rho(\gamma\sigma_i)\begin{pmatrix}0 & \mu_i n\\ \mu_i n & \beta'\end{pmatrix}$ for $\gamma \in \Gamma$ if and only if $\beta - \beta' \in 2n\mu_i\delta_i\mathbf{Z}$ by the definition of δ_i. If $\varphi = \psi^{(2)}$, $\varphi\left(\rho(\sigma_i)\begin{pmatrix}0 & \mu_i n\\ \mu_i n & \beta\end{pmatrix}\right) = \psi(-\mu_i^2 n^2)$. From this we see

$$\eta(s, L, \psi^{(2)}) = \sum_{i=1}^{\nu}\sum_{n=1}^{\infty} 2\psi(-\mu_i^2 n^2) n\mu_i\delta_i\lambda_i'^{-1}(\mu_i^2 n^2)^{-s}$$
$$= \Big(\sum_{i=1}^{\nu} 2\psi(-\mu_i^2)\lambda_i^*\delta_i\mu_i^{-(2s-1)}\Big)L((2s-1, \psi^2).$$

This completes the proof.

§2. L-functions associated with $(\tilde{G}, \tilde{\rho}, \tilde{V})$

Let $(\tilde{G}, \tilde{\rho}, \tilde{V})$ be as in Introduction, and let

$$P_1(x, y) = {}^t yxy, \quad P_2(x, y) = P_2(x) = \det x \ (x \in V,\ y \in \mathbf{C}^2).$$

Then the triple $(\tilde{G}, \tilde{\rho}, \tilde{V})$ is a prehomogeneous vector space with the singular set $\tilde{S} = \{(x, y) \in \tilde{V} \mid P_1(x, y)P_2(x, y) = 0\}$, and the polynomials P_1 and P_2 are its irreducible relative invariants corresponding to the characters $\chi_1(g, t) = \chi_1(t) = t^2$ and $\chi_2(g, t) = \chi_2(g) = \det g^2((g, t) \in GL_2(\mathbf{C}) \times GL_1(\mathbf{C}))$. We consider the standard $\mathbf{Q}$-structure on $(\tilde{G}, \tilde{\rho}, \tilde{V})$:

$$\tilde{G}(\mathbf{Q}) = GL_2(\mathbf{Q}) \times GL_1(\mathbf{Q}), \quad \tilde{V}(\mathbf{Q}) = V(\mathbf{Q}) \oplus \mathbf{Q}^2.$$

Let $\tilde{G}^+ = G^+ \times \mathbf{R}_+$, $\mathbf{R}_+ = \{a \in \mathbf{R} \mid a > 0\}$, and $\tilde{V}_i = (V_i \oplus \mathbf{R}^2) \cap (\tilde{V}(\mathbf{R}) - \tilde{S}(\mathbf{R}))$, where $\tilde{S}(\mathbf{R}) = \tilde{S} \cap \tilde{V}(\mathbf{R})$. For L, L^*, we set $\tilde{L} = L \oplus \mathbf{Z}^2$, $\tilde{L}^* = L^* \oplus \mathbf{Z}^2$. Let $\tilde{\Gamma} = \Gamma \times \{1\} \subset \tilde{G}$. Then $\tilde{L}$ and $\tilde{L}^*$ are $\tilde{\Gamma}$-invariant lattices of $\tilde{V}(\mathbf{R})$. We define an L-function in two variables s_1, s_2 associated with $(\tilde{G}, \tilde{\rho}, \tilde{V})$, L and a character φ defined on L by

$$\xi_i(s_1, s_2, \tilde{L}, \varphi) = \sum_{(x,y) \in \tilde{\Gamma}\backslash\tilde{L}_i,\ P_1(x,y)>0} \varphi(x) \mid P_1(x, y) \mid^{-s_1} \mid P_2(x, y) \mid^{-s_2}$$
$$(i = 1,2,\ s_1, s_2 \in \mathbf{C}),$$

and for a character φ defined on L^* we define $\xi_i(s_1, s_2, \tilde{L}^*, \varphi)$ by the same formula taking summation over $(x, y) \in \tilde{\Gamma}\backslash \tilde{L}_i^*$ with $P_1(x, y) > 0$. Here $\tilde{L}_i = \tilde{L} \cap \tilde{V}_i$, $\tilde{L}_i^* = \tilde{L}^* \cap \tilde{V}_i$. These series converge absolutely for Re s_1, Re $s_2 > 1$. We set

$$\xi(s_1, s_2, \tilde{M}, \psi^{(2)}) = (\xi_1(s_1, s_2, \tilde{M}, \psi^{(2)}), \xi_2(s_1, s_2, \tilde{M}, \psi^{(2)})) \quad \text{for } \tilde{M} = \tilde{L}, \tilde{L}^* \text{ and } \psi \neq \psi_p, \psi_0,$$

$$\xi_o(s_1, s_2, \tilde{M}, \psi_p) = (\xi_1(s_1, s_2, \tilde{M}, \psi_p^{(1)}), \xi_2(s_1, s_2, \tilde{M}, \psi_p^{(1)})) \text{ for } \tilde{M} = \tilde{L}, \tilde{L}^*,$$

$$\xi_e(s_1, s_2, \tilde{L}^*, \psi_p) = \begin{pmatrix} \xi_1(s_1, s_2, \tilde{L}^*, \psi_p^{(0)}) & \xi_2(s_1, s_2, \tilde{L}^*, \psi_p^{(0)}) \\ \xi_1(s_1, s_2, \tilde{L}^*, \psi_p^{(2)}) & \xi_2(s_1, s_2, \tilde{L}^*, \psi_p^{(2)}) \end{pmatrix},$$

$$\xi(s_1, s_2, \tilde{L}, \psi_0) = \begin{pmatrix} \xi_1(s_1, s_2, \tilde{L}, \psi_0^{(0)} + \psi_0^{(1)}) & \xi_2(s_1, s_2, \tilde{L}, \psi_0^{(0)} + \psi_0^{(1)}) \\ \xi_1(s_1, s_2, \tilde{L}, \psi_0^{(2)}) & \xi_2(s_1, s_2, \tilde{L}, \psi_0^{(2)}) \end{pmatrix}.$$

To describe the properties of these functions, we set

$$A(s_1, s_2) = 2^{2-s_1-2s_2}\pi^{1/2-s_1-2s_1}\Gamma(s_2)\Gamma(s_1 + s_2 - 1/2),$$

$$B_1(s_1) = 2\pi^{-2s_1}\Gamma(s_1)^2 \sin \pi s_1/2, \; B_2(s_1) = 2\pi^{-2s_1}\Gamma(s_1)^2 \cos \pi s_1/2,$$

$$C(s_1) = 2^{1-s_1}\pi^{1/2-s_1}\Gamma(s_1 - 1/2)\zeta(2s_1 - 1),$$

$$\mathfrak{U}_1(s_1, s_2) = \begin{pmatrix} \cos \pi(s_1 + 2s_2)/2 & \sin \pi s_1/2 \\ \cos \pi s_1/2 & \sin \pi(s_1 + 2s_2)/2 \end{pmatrix},$$

$$\mathfrak{U}_2(s_1, s_2) = \begin{pmatrix} \sin \pi(s_1 + 2s_2)/2 & \cos \pi s_1/2 \\ \sin \pi s_1/2 & \cos \pi(s_1 + 2s_2)/2 \end{pmatrix},$$

$$\mathfrak{B}_1(s_1) = \begin{pmatrix} -\sin \pi s_1/2 & 0 \\ 0 & \cos \pi s_1/2 \end{pmatrix}, \; \mathfrak{B}_2(s_1) = \begin{pmatrix} \cos \pi s_1/2 & 0 \\ 0 & \sin \pi s_1/2 \end{pmatrix},$$

and let C be the Euler constant. Then we can prove

PROPOSITION 7. *The functions $\xi_i(s_1, s_2, \tilde{L}, \varphi)$ and $\xi_i(s_1, s_2, \tilde{L}^*, \varphi)$ can be continued to meromorphic functions of (s_1, s_2) on $\mathbf{C}^2$ such that the functions*

$$(s_1 - 1)^2(s_2 - 1)(s_1 + s_2 - 3/2)\xi_i(s_1, s_2, \tilde{L}, \varphi),$$

$$(s_1 - 1)^2(s_2 - 1)(s_1 + s_2 - 3/2)\xi_i(s_1, s_2, \tilde{L}^*, \varphi)$$

are entire. Moreover these functions satisfy the following properties.

(1) *Let $\psi \neq \psi_p, \psi_0$. Then one has*

(a) $\xi(s_1, 3/2 - s_1 - s_2, \tilde{L}^*, (\bar{\psi}\psi_p)^{(2)})$

$$= v(L)p^{s_1+2s_2-3}A(s_1, s_2)W_2^2(\bar{\psi}\psi_p)\xi(s_1, s_2, \tilde{L}, \psi^{(2)})\mathfrak{U}_2(s_1, s_2),$$

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(b) $\xi(1-s_1,\ s_1+s_2-1/2,\ \tilde{M},\ \psi^{(2)})=B_2(s_1)\xi(s_1,\ s_2,\ M,\ \psi^{(2)})\mathfrak{B}_2(s_1)$,

(c) $\lim_{s_2\to 1}(s_1-1)\xi(s_1,\ s_2,\ \tilde{L},\ \psi^{(2)})=0$,

(d) $\lim_{s_2\to -s_1+3/2}(s_1+s_2-3/2)\xi(s_1,\ s_2,\ \tilde{L},\ \psi^{(2)})=0$,

(e) $\lim_{s_1\to 1}(s_1-1)^2\xi(s_1,\ s-s_1/2,\ \tilde{M},\ \psi^{(2)})=\frac{1}{2}\eta(s,\ M,\ \psi^{(2)})(1{,}0)$,

(f) $\lim_{s_1\to 1}\frac{\partial}{\partial s_1}\{(s_1-1)^2\xi(s_1,\ s-s_1/2,\ \tilde{M},\ \psi^{(2)})\}$

$$=2\zeta(s,\ M,\ \psi^{(2)})+\frac{2C-\log 2}{2}\eta(s,\ M,\ \psi^{(2)})(1{,}0),$$

for $M=L,\ L^*$.

(2) *Let* $\varphi=\psi_p^{(1)}$, *and let* $\psi_p(-1)=(-1)^{\varepsilon}$ *with* $\varepsilon=1$ *or* 2. *Then one has*

(a) $\xi_o(s_1,\ 3/2-s_1-s_2,\ \tilde{L}^*,\ \psi_p)$
$$=v(L)\sqrt{-1}^{\varepsilon-2}p^{s_1+2s_2-3}A(s_1,\ s_2)W_o^2(\psi_p)\xi_o(s_1,\ s_2,\ \tilde{L},\ \psi_p)\mathfrak{U}_{\varepsilon}(s_1,\ s_2),$$

(b) $\xi_o(1-s_1,\ s_1+s_2-1/2,\ \tilde{M},\ \psi_p)=B_{\varepsilon}(s_1)\xi_o(s_1,\ s_2,\ \tilde{M},\ \psi_p)\mathfrak{B}_{\varepsilon}(s_1)$,

(c) $\lim_{s_2\to 1}(s_1-1)\xi_o(s_1,\ s_2,\ \tilde{L},\ \psi_p)$

$$=v(L)^{-1}W_o^2(\psi_p)^{-1}G(\psi_p)L(s_1,\ \psi_p)\zeta(2s_1)\Big(\sum_{i=1}^{\nu}\psi_p(\lambda_i)\delta_i\lambda_i^{1-s_1}\Big)(1{,}1),$$

(d) $\lim_{s_2\to -s_1+3/2}(s_1+s_2-3/2)\xi_o(s_1,\ s_2,\ \tilde{L},\ \psi_p)$

$$=v(L)^{-1}\sqrt{-1}^{2-\varepsilon}\psi_p(-1)p^{s_1}C(s_1)W_o^2(\psi_p)^{-1}L(s_1,\ \psi_p)\Big(\sum_{i=1}^{\nu}\psi_p(\lambda_i)\delta_i\lambda_i^{s_1}\Big)$$

$$\times\begin{cases}(\sin\pi s_1/2,\ \cos\pi s_1/2) & \text{if}\quad \varepsilon=2\\ (-\cos\pi s_1/2,\ \sin\pi s_1/2) & \text{if}\quad \varepsilon=1,\end{cases}$$

(e) $\lim_{s_1\to 1}(s_1-1)^2\xi_o(s_1,\ s-s_1/2,\ \tilde{M},\ \psi_p)=0$,

(f) $\lim_{s_1\to 1}\frac{\partial}{\partial s_1}\{(s_1-1)^2\xi_o(s_1,\ s-s_1/2,\ \tilde{M},\ \psi_p)\}=2\zeta_o(s_1,\ M,\ \varphi_p)$,

for $M=L,\ L^*$.

(3) *Let* $\varphi=\psi_p^{(r)}$, $r=0{,}2$, *or* $\psi_0^{(r)}$, $r=0{,}1{,}2$. *Then one has*

(a) $\xi_e(s_1,\ 3/2-s_1-s_2,\ \tilde{L}^*,\ \psi_p)$

$$=v(L)p^{s_1+2s_2}A(s_1,\ s_2)W_e^2(\psi_p)^{-1}\xi(s_1,\ s_2,\ \tilde{L},\ \psi_0)\mathfrak{U}_2(s_1,\ s_2),$$

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(b) $\xi_e(1-s_1, s_1+s_2-1/2, \tilde{L}^*, \psi_p) = B_2(s_1)\xi_e(s_1, s_2, \tilde{L}^*, \psi_p)\mathfrak{B}_2(s_1)$,

(b)′ $\xi(1-s_1, s_1+s_2-1/2, \tilde{L}, \psi_0) = B_2(s_1)\xi(s_1, s_2, \tilde{L}, \psi_0)\mathfrak{B}_2(s_1)$,

(c) $$\lim_{s_2\to 1}(s_2-1)v(L)^{-1}p^{-3}W_e^2(\psi_p)\xi_e(s_1, s_2, \tilde{L}^*, \psi_p)$$
$$= 2p^{-s_1-2}\zeta(s_1)\zeta(2s_1)\left(\sum_{i=1}^{\nu}\delta_i(\lambda_i^*)^{1-s_1}\right)\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix},$$

(c)′ $$\lim_{s_2\to 1}(s_2-1)\xi(s_1, s_2, \tilde{L}, \psi_0)$$
$$= v(L)^{-1}p^{-3}\zeta(s_1)\zeta(2s_1)\left(\sum_{i=1}^{\nu}\delta_i\lambda_i^{1-s_1}\right)\begin{pmatrix}w(0,0) & w(0,0)\\ w(2,0) & w(2,0)\end{pmatrix},$$

(d) $$\lim_{s_2\to -s_1+\frac{3}{2}}(s_1+s_2-3/2)v(L)^{-1}p^{-3}W_e^2(\psi_p)\xi_e(s_1, s_2, \tilde{L}^*, \psi_p)$$
$$= 2p^{-3}p^{s_1}C(s_1)\zeta(s_1)\left(\sum_{i=1}^{\nu}\delta_i(\lambda_i^*)^{s_1}\right)\begin{pmatrix}\sin\pi s_1/2 & \cos\pi s_1/2\\ 0 & 0\end{pmatrix},$$

(d)′ $$\lim_{s_2\to -s_1+\frac{3}{2}}(s_1+s_2-3/2)\xi(s_1, s_2, \tilde{L}, \psi_0)$$
$$= v(L)^{-1}p^{-3}C(s_1)\zeta(s_1)\left(\sum_{i=1}^{\nu}\delta_i\lambda_i^{s_1}\right)\begin{pmatrix}w(0,0)\sin\pi s_1/2 & w(0,0)\cos\pi s_1/2\\ w(2,0)\sin\pi s_1/2 & w(2,0)\cos\pi s_1/2\end{pmatrix},$$

(e) $$\lim_{s_1\to 1}(s_1-1)^2\xi_e(s_1, s-s_1/2, \tilde{L}^*, \psi_p) = \frac{1}{2}\begin{pmatrix}\eta(s, L^*, \psi_p^{(0)}) & 0\\ \eta(s, L^*, \psi_p^{(2)}) & 0\end{pmatrix},$$

(e)′ $$\lim_{s_1\to 1}(s_1-1)^2\xi(s_1, s-s_1/2, \tilde{L}, \psi_0) = \frac{1}{2}\begin{pmatrix}\eta(s, L, \psi_0^{(0)}+\psi_0^{(1)}) & 0\\ \eta(s, L, \psi_0^{(2)}) & 0\end{pmatrix},$$

(f) $$\lim_{s_1\to 1}\frac{\partial}{\partial s_1}\{(s_1-1)^2\xi_e(s_1, s-s_1/2, \tilde{L}^*, \psi_p)\}$$
$$= 2\zeta_e(s, L^*, \psi_p) + \frac{2C-\log 2}{2}\begin{pmatrix}\eta(s, L^*, \psi_p^{(0)}) & 0\\ \eta(s, L^*, \psi_p^{(2)}) & 0\end{pmatrix},$$

(f)′ $$\lim_{s_1\to 1}\frac{\partial}{\partial s_1}\{(s_1-1)^2\xi(s_1, s-s_1/2, \tilde{L}, \psi_0)\}$$
$$= 2\zeta(s, L, \psi_0) + \frac{2C-\log 2}{2}\begin{pmatrix}\eta(s, L, \psi_0^{(0)}+\psi_0^{(1)}) & 0\\ \eta(s, L, \psi_0^{(2)}) & 0\end{pmatrix}.$$

We can easily deduce our Theorem 1 and Theorem 5 from this proposition as in the proof of Theorem 1, [5], and we will give an outline of the proof. Namely, for $M = L, L^*$, the function $(s-1)^2(s-3/2)\zeta_i(s, M, \varphi)$ is entire as a function in s on $\mathbf{C}$ because $(s_1-1)^2(s-s_1/2-1)(s+s_1/2-3/2)\xi_i(s_1, s-s_1/2, \tilde{M}, \varphi)$ is entire as a function in s_1 and s on $\mathbf{C}^2$. The functional equations can be deduced from (a) by setting $s_2 = s - s_1/2$ and comparing the coefficients of the expansions at $s_1 = 1$ of the both sides. The result on the residue at $s = 3/2$ follows from (c), (c)′, (e), (e)′, (f), and (f)′. Finally the formulas (d), (d)′, (e), (e)′, (f), and (f)′ give the result on the principal part at $s = 1$.

§3. Proof of Proposition 7

In this section, we give a proof of Proposition 7, and complete the proof of Theorem 1 and Theorem 5. We will treat mainly the cases of $\phi^{(2)}$ and $\phi_p^{(1)}$. The formulas of Lemma 15 in the case of $\phi_p^{(r)}$ $(r = 0,2)$ and $\phi_0^{(r)}$ $(1 \leq r \leq 2)$ are lengthy and will be given in Appendix.

Let $d^{\times}t$ be the measure $t^{-1}dt$ on $\mathbf{R}_+$. For a $\tilde{\rho}(\tilde{\Gamma})$-stable subset A of $\tilde{L}$ or $\tilde{L}^*$, $F \in \mathcal{S}(\tilde{V}(\mathbf{R}))$, and a character φ defined on A, we set

$$Z(F, A, \varphi, s_1, s_2) = \int_{\tilde{G}^+/\tilde{\Gamma}} \chi_1(t)^{s_1}\chi_2(g)^{s_2} \sum_{(x,y)\in A-\tilde{S}} \varphi(x)F(\tilde{\rho}(g, t)(x, y))dgd^{\times}t.$$

For $\varepsilon = 1, 2$, let

$$\tilde{V}_i^{\varepsilon} = \{(x, y) \in \tilde{V}_i \mid P_1(x, y)(-1)^{\varepsilon} > 0\}.$$

Then $\tilde{V}_i = \tilde{V}_i^1 \cup \tilde{V}_i^2$ (disjoint) and the set $V(\mathbf{R}) - S(\mathbf{R})$ is a disjoint union of the four sets $\tilde{V}_i^{\varepsilon}$ $(i = 1, 2, \varepsilon = 1, 2)$. Let dy be the standard Euclidean measure on $\mathbf{R}^2$. For F, we define

$$\Psi_i^{\varepsilon}(F, s_1, s_2) = \int_{\tilde{V}_i} (\mathrm{sgn}(P_1(x, y)))^{\varepsilon} \mid P_1(x, y) \mid^{s_1} \mid P_2(x, y) \mid^{s_2} F(x, y)dxdy.$$

Then $\Psi_i^{\varepsilon}(F, s_1, s_2)$ converges for $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 > 0$, and can be continued to a meromorphic function on $\mathbf{C}^2$ (cf. [2], [7]). We see easily

Lemma 8. *Let $M = L$ or L^*. Let φ be a character defined on M. The integrals $Z(F, M, \varphi, s_1, s_2)$ converges absolutely for* $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 > 1$ *and one has*

$$Z(F, \tilde{M}, \varphi, s_1, s_2) = 4^{-1} \sum_{i=1}^{2} \xi_i(s_1, s_2, \tilde{M}, \varphi)\Psi_i^{\varepsilon}(F, s_1 - 1, s_2 - 1).$$

Here $\varepsilon = 1$ and $\varphi = \phi_p^{(1)}$, and $\phi_p^{(1)}(-1) = -1$, and $\varepsilon = 2$ otherwise.

Let $Z_+^{(1)}(F, A, \varphi, s_1, s_2)$ and $\hat{Z}_+^{(1)}(F, A, \varphi, s_1, s_2)$ and $Z_+^{(2)}(F, A, \varphi, s_1, s_2)$ be the integrals obtained from $Z(F, A, \varphi, s_1, s_2)$ by restricting the domains of integration to $\{(g, t) \in \tilde{G}_+/\tilde{\Gamma} \mid \chi_1(t) \geq 1\}$, $\{(g, t) \in \tilde{G}_+/\tilde{\Gamma} \mid \chi_1(t) \geq \chi_2(g)\}$ and $\{(g, t) \in \tilde{G}^+/\tilde{\Gamma} \mid \chi_2(g) \geq 1\}$, respectively. Then by Lemma 8 we see

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LEMMA 9. *Let A and F be as above.*
(1) $Z_+^{(1)}(F, A, \varphi, s_1, s_2)$ *converges absolutely if* $\operatorname{Re} s_2 > 1$.
(2) $\hat{Z}_+^{(1)}(F, A, \varphi, s_1, s_2)$ *converges absolutely if* $\operatorname{Re} s_1 + \operatorname{Re} s_2 > 2$ *and* $\operatorname{Re} s_2 > 1$.
(3) $Z_+^{(2)}(F, A, \varphi, s_1, s_2)$ *converges absolutely if* $\operatorname{Re} s_1 > 1$.

Let B be the subgroup of G consisting of lower triangular matrices, and let $S = \{x \in V \mid x_1 P_2(x) = 0\}$. We denote also by ρ the restriction of ρ to B. Then the triple (B, ρ, V) is a prehomogeneous vector space with the singular set S with the irreducible relative invariants x_1, $P_2(x) = \det x$. Let ρ^* be the contragredient representation of ρ, which is given by $\rho^*(g) = \chi_2^{-1}(g)\rho(g)$ for $g \in G$. For $f \in \mathcal{S}(V(\mathbf{R}))$, $i = 1, 2$, $\varepsilon = 1, 2$, we set

$$\Phi_i^\varepsilon(f, s_1, s_2) = \int_{V_i} (\operatorname{sgn} x_1)^\varepsilon |x_1|^{s_1} |P_2(x)|^{s_2} f(x)\, dx,$$

$$\Sigma^\varepsilon(f, s) = \int_{\mathbf{R}^2} (\operatorname{sgn} t)^\varepsilon |t|^{s-1} f\left(\begin{pmatrix} t & u \\ u & u^2/t \end{pmatrix}\right) dt du.$$

The integral $\Phi_i^\varepsilon(f, s_1, s_2)$ (resp. $\Sigma^\varepsilon(f, s)$) converges absolutely for $(s_1, s_2) \in \mathbf{C}^2$ with $\operatorname{Re} s_1 > 0$, $\operatorname{Re} s_2 > 0$ (resp. $s \in \mathbf{C}$ with $\operatorname{Re} s > 1$). These integrals are slight generalizations of those introduced by Shintani [8]. For $f \in \mathcal{S}(V(\mathbf{R}))$, we define the Fourier transform f^* of f by

$$f^*(x) = \int_{V(\mathbf{R})} f(y) e(\langle x, y \rangle)\, dy.$$

Then we can prove

LEMMA 10. *The function $\Phi_i^\varepsilon(f, s_1, s_2)$ has an analytic continuation to meromorphic function of (s_1, s_2) in $\mathbf{C}^2$ and satisfies the functional equation*:

$$\begin{pmatrix} \Phi_1^\varepsilon(f, s_1 - 1, s_2 - 1) \\ \Phi_2^\varepsilon(f, s_1 - 1, s_2 - 1) \end{pmatrix} = \sqrt{-1}^{\varepsilon - 2} A(s_1, s_2) \mathfrak{U}_\varepsilon(s_1, s_2) \begin{pmatrix} \Phi_1^\varepsilon(f^*, s_1 - 1, 1/2 - s_1 - s_2) \\ \Phi_2^\varepsilon(f^*, s_1 - 1, 1/2 - s_1 - s_2) \end{pmatrix}.$$

Moreover if f is $SO(2)$ invariant, then

$$2^{s_1 - 2s_2} \sin(\pi s_1/2) \Gamma(s_2)^{-1} \Gamma(s_1 + s_2 - 1/2)^{-1} \Phi_1^2(f, s_1 - 1, s_2 - 1),$$

$$2^{s_1 - 2s_2} \cos(\pi s_1/2) \Gamma(s_2)^{-1} \Gamma(s_1 + s_2 - 1/2)^{-1} \Phi_1^1(f, s_1 - 1, s_2 - 1),$$

and

$$2^{s_1-2s_2}\Gamma(s_2)^{-1}\Gamma(s_1+s_2-1/2)^{-1}\Phi_2^{\varepsilon}(f,\ s_1-1,\ s_2-1),\quad \varepsilon=1,\ 2,$$

are entire functions of (s_1, s_2) *which are invariant under the substitution*

$$(s_1,\ s_2)\mapsto(1-s_1,\ s_1-s_2-1/2).$$

Proof. The assertion for $\varepsilon = 2$ is nothing but Lemma 1 of [8]. When $\varepsilon = 1$, we may assume the function f is odd. Then in the same way as in the case of $\varepsilon = 2$, we can show that Φ_i can be continued meromorphically to $\mathbf{C}^2$ and that for $f^* \in C_0^\infty(V(\mathbf{R}) - S(\mathbf{R}))$, it holds

$$\Phi_i^1(f,\ s_1-1,\ s_2-1)=\int_{V(\mathbf{R})}f^*(x)J(x)\,dx,$$

with

$$J(x)=(-1)^{\varepsilon-2}A(s_1,\ s_2)\,|\,x_1\,|^{s_1-1}\,|\,P_2(x)\,|^{1/2-s_1-s_2}$$
$$\times\, e((\operatorname{sgn}x_1)\{(-1)^i s_2/4+(\operatorname{sgn}P_2(x))(1/2-s_1-s_2)/4+1/8\}).$$

The functional equation follows from this and the fact that f is odd. The functional equation for general $f \in \mathcal{S}(V(\boldsymbol{R}))$ can be shown as in the proof of Theorem 1 of Shintani [9]. For $(\theta,\ u,\ t) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}_+$, set

$$x^i(\theta,\ u,\ t)=\rho(k_\theta)\begin{pmatrix} u & \sqrt{u^2-(-1)^i t} \\ \sqrt{u^2-(-1)^i t} & u\end{pmatrix}\in V(\mathbf{R}),$$

where $k_\theta = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$. Assume f is odd and $SO(2)$-invariant. We see

$$\Phi_1^1(f,\ s_1-1,\ s_2-1)=\int_{\mathbf{R}}du\int_{\mathbf{R}_+}t^{s_2-1}dt\int_0^\pi d\theta\, f(x^1(0,\ u,\ t))$$
$$\times\operatorname{sgn}(u+\sqrt{u^2+t}\sin 2\theta)\,|\,u+\sqrt{u^2+t}\sin 2\theta\,|^{s_1-1},$$

$$\Phi_2^1(f,\ s_1-1,\ s_2-1)=2\int_{\sqrt{t}}^{\infty}du\int_{\mathbf{R}_+}t^{s_2-1}dt\int_0^\pi d\theta\, f(x^1(0,\ u,\ t))$$
$$\times\,(u+\sqrt{u^2-t}\sin 2\theta)^{s_1-1}.$$

Since

$$\int_0^\pi \operatorname{sgn}(u+\sqrt{u^2+t}\sin\theta)\,|\,u+\sqrt{u^2+t}\sin 2\theta\,|^s\,d\theta$$
$$=t^{s/2}\pi(2\sqrt{-1}\sin\pi s/2)^{-1}(P_s(\sqrt{-1}\,ut^{-1/2})-P_s(-\sqrt{-1}\,ut^{-1/2}))\ \ (t>0),$$

$$\int_0^{\pi} (u+\sqrt{u^2-t}\sin 2\theta)^s d\theta = t^{s/2}\pi P_s(ut^{-1/2})\ \ (u > t^{1/2} > 0),$$

with the Legendre function $P_s(z)$, our assertion can be proved in the same way as in the case of $\varepsilon = 2$.

The following lemma for $\varepsilon = 2$ is Lemma 2 in [8], and the case of $\varepsilon = 1$ can be proved in the same way by taking an odd function f as above.

LEMMA 11. *The function $\Sigma^{\varepsilon}(f, s)$ has an analytic continuation to a meromorphic function on* $\mathbf{C}$ *and if* $f \in C_0^{\infty}(V(\mathbf{R}) - S(\mathbf{R}))$, *it satisfies*

$$\Sigma^{\varepsilon}(f^*, s-1) = \begin{cases} 2^{1-s}\pi^{1/2-s}\Gamma(s-1/2)\{\sin(\pi s/2)\Phi_1^2(f, s-1, 1/2-s) \\ \quad + \cos(\pi s/2)\Phi_2^2(f, s-1, 1/2-s)\} & \text{if } \varepsilon = 2, \\ \sqrt{-1}\,2^{1-s}\pi^{1/2-s}\Gamma(s-1/2)\{\cos(\pi/2)\Phi_1^1(f, s-1, 1/2-s) \\ \quad + \sin(\pi s/2)\Phi_2^1(f, s-1, 1/2-s)\} & \text{if } \varepsilon = 1. \end{cases}$$

For $F \in \mathcal{S}(\tilde{V}(\mathbf{R}))$, we set

$$F^*(x^*, y) = \int_{V(\mathbf{R})} F(x, y)e(\langle x, x^*\rangle)dx,$$

$$\hat{F}(x, y^*) = \int_{\mathbf{R}^2} F(x, y)e(\langle y, y^*\rangle)dy.$$

Here $\langle y, y^*\rangle = y_1y_2^* - y_2y_1^*$. We note

$$\Psi_i^{\varepsilon}(F, s_1, s_2) = \int_0^{\infty} t^{2s_1+2}\Phi_i^{\varepsilon}(F_0(*, \begin{pmatrix} t \\ 0 \end{pmatrix}), s_1, s_2)d^{\times}t, \tag{3.1}$$

where $F_0(x, y) = \int_0^{2\pi} F(\tilde{\rho}(k_{\theta}, 1)(x, y))d\theta$. The first functional equation of the following lemma follows from Lemma 9 and (3.1).

LEMMA 12. *Let* $F \in \mathcal{S}(\tilde{V}(\mathbf{R}))$. *For* $i = 1, 2$, $\varepsilon = 1, 2$, *the function* $\Psi_i^{\varepsilon}(F, s_1, s_2)$ *satisfies the following functional equations*:

$$\begin{pmatrix} \Psi_1^{\varepsilon}(F, s_1-1, s_2-1) \\ \Psi_2^{\varepsilon}(F, s_1-1, s_2-1) \end{pmatrix} = \sqrt{-1}^{\varepsilon-2}A(s_1, s_2)\mathfrak{U}_{\varepsilon}(s_1, s_2) \times \begin{pmatrix} \Psi_1^{\varepsilon}(F^*, s_1-1, 1/2-s_1-s_2) \\ \Psi_2^{\varepsilon}(F^*, s_1-1, 1/2-s_1-s_2) \end{pmatrix},$$

$$\begin{pmatrix} \Psi_1^\varepsilon(F, s_1 - 1, s_2 - 1) \\ \Psi_2^\varepsilon(F, s_1 - 1, s_2 - 1) \end{pmatrix} = B_\varepsilon(s_1)\mathfrak{U}_\varepsilon(s_1) \begin{pmatrix} \Psi_1^\varepsilon(\hat{F}, -s_1, s_1 + s_2 - 1/2) \\ \Psi_2^\varepsilon(\hat{F}, -s_1, s_1 + s_2 - 1/2) \end{pmatrix}.$$

Proof. The second equation for $\varepsilon = 2$ is (2.11) of Lemma 2.9 of [5]. The formula in the case of $\varepsilon = 1$ follows from the fact that the Fourier transform of $\operatorname{sgn}(x^2 - y^2)\,|x^2 - y^2|^s$ is $-2(2\pi)^{-s-1}\cos^2(\pi s/2)\operatorname{sgn}(x^2 - y^2)\,|x^2 - y^2|^{-s-1}$ (cf. [3], Ch. III, 2.6).

Remark. For any $(s_1, s_2) \in \mathbf{C}^2$, there exists a $SO(2)$-invariant $f \in C_0^\infty(V(\mathbf{R}) - S(\mathbf{R}))$ satisfying $\Gamma(1 + s_1/2)\Phi_1^2(f, s_1, s_2,) \neq 0$, $\Gamma(s_1/2)\Phi_1^1(f, s_1, s_2) \neq 0$ and $\Phi_2^\varepsilon(f, s_1, s_2) \neq 0$ for $\varepsilon = 1, 2$, and there exists $F \in C_0^\infty(\tilde{V}(\mathbf{R}) - \tilde{S}(\mathbf{R}))$ satisfying $\Gamma(1 + s_1/2)\Psi_1^2(F, s_1, s_2) \neq 0$, $\Gamma(s_1/2)\Psi_1^1(F, s_1, s_2) \neq 0$, and $\Psi_2^\varepsilon(F, s_1, s_2) \neq 0$ for $\varepsilon = 1, 2$.

We note $x \in V(\mathbf{R})$ of rank 1 can be written uniquely as $x = k_\theta \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} {}^t k_\theta$ ($v \in \mathbf{R}_+$, $0 \leq \theta < \pi$). We set

$$\Omega = \{(x, y) \in \tilde{V}(\mathbf{R}) \mid \operatorname{rank} x = 1,\ P_1(x, y) \neq 0\},$$

$$\tilde{\Sigma}^\varepsilon(F, s) = \int_\Omega (\operatorname{sgn} P_1(x, y))^\varepsilon\,|P_1(x, y)|^s\,F(x, y)\,dv\,d\theta\,dy.$$

Then the integral $\tilde{\Sigma}^\varepsilon(F, s)$ converges absolutely for $\operatorname{Re} s > 0$, and has an analytic continuation to a meromorphic function on $\mathbf{C}$. We note

$$\tilde{\Sigma}^\varepsilon(F, s) = \int_0^\infty t^{2s+2}\,\Sigma^\varepsilon\left(F_0\left(*, \begin{pmatrix} t \\ 0 \end{pmatrix}\right), s\right) d^\times t. \tag{3.2}$$

Let B^+ be the connected component of the $\mathbf{R}$-valued points of B, and let $db = t_1^{-2}t_2^{-1}dt_1dt_2du$ for $b = \begin{pmatrix} t_1 & 0 \\ u & t_2 \end{pmatrix}$. Then db gives a Haar measure on B^+. For a positive integer δ, let M be a $\rho(\Gamma_\infty(\delta))$ invariant lattice in $V(\mathbf{Q})$. We assume a character φ is defined on M. We set

$$I(f, M, \varphi, s_1, s_2) = \int_{B^+/\Gamma_\infty(\delta)} t_1^{2(s_1+s_2)}t_2^{2s_2} \sum_{x \in M - S} \varphi(x) f(\rho(b)x)\,db,$$

$$I_+(f, M, \varphi, s_1, s_2) = \int_{B^+/\Gamma_\infty(\delta),\, t_1t_2 \geq 1} t_1^{2(s_1+s_2)}t_2^{2s_2} \sum_{x \in M - S} \varphi(x) f(\rho(b)x)\,db.$$

Then the integral $I(f, M, \varphi, s_1, s_2)$ converges absolutely for $\operatorname{Re} s_1, \operatorname{Re} s_2 > 1$ and the integral $I_+(f, M, \varphi, s_1, s_2)$ converges absolutely in the domain

$\{(s_2, s_2) \in \mathbf{C}^2 \mid \mathrm{Re}\, s_1 > 1\}$ (cf. [9], Lemma 3). Let $M_0 = \{x \in M \mid x_1 \neq 0, P_2(x) = 0\}$. Then M_0 is $\rho(\Gamma_\infty(\delta))$-stable. We set for $\varphi = \psi_p^{(1)}$

$$\zeta_0(s, M, \varphi_p^{(1)}) = 2 \sum_{x \in \Gamma_\infty(\delta)\backslash M_0,\, x_1 > 0} \varphi_p^{(1)}(x) \,|\, x_1 \,|^{-s},$$

and for the trivial character

$$\zeta_0(s, M) = \sum_{x \in \Gamma_\infty(\delta)\backslash M_0} |\, x_1 \,|^{-s}.$$

Let M_0^p be the set of primitive elements in M_0 such that $x_1 > 0$. Then

$$\zeta_0(s, M, \varphi^{(1)}) = 2L(s, \psi_p) \sum_{x \in \Gamma_\infty(\delta)\backslash M_0^p} \varphi_p^{(1)}(x) \,|\, x_1 \,|^{-s},$$

$$\zeta_0(s, M) = 2\zeta(s) \sum_{x \in \Gamma_\infty(\delta)\backslash M_0^p} |\, x_1 \,|^{-s}.$$

We set

$$\zeta_0^p(s, M, \psi_p^{(1)}) = \sum_{x \in \Gamma_\infty(\delta)\backslash M_0^p} \psi_p^{(1)}(x) \,|\, x_1 \,|^{-s},\ \zeta_0^p(s, M) = \sum_{x \in \Gamma_\infty(\delta)\backslash M_0^p} |\, x_1 \,|^{-s}.$$

Then we can show easily that $\zeta_0^p(s, M, \psi_p^{(1)})$ and $\zeta_0^p(s, M)$ coincide with $\zeta(2s-1)/\zeta(2s)$ up to elementary factors. In these notations, we can prove

LEMMA 13. *Let M be a lattice in $V(\mathbf{Q})$, and assume a character φ is defined on M. For $f \in \mathcal{S}(V(\mathbf{R}))$, let $f_p^*(y) = f^*(y/p)$.*

(1) *Let $\varphi = \psi^{(2)}$ with $\psi^2 \neq \psi_0$ and set $C(M, \psi) = v(M)^{-1} p^{-3} W_2^2(\psi)$. Then for* $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 > 1$, *one has*

$$I(f, M, \psi^{(2)}, s_1, s_2) = I_+(f, M, \psi^{(2)}, s_1, s_2) + C(M, \psi^{(2)}) I_+(f_p^*, M^*, (\bar{\psi}\psi_p)^{(2)}, s_1, 3/2 - s_1 - s_2).$$

(2) *Let $\varphi = \psi^{(1)}$, $\psi_p(-1) = (-1)^\varepsilon$ and set $C(M, \psi_p) = v(M)^{-1} p^{-3} \psi_p(-1) W_o^2(\psi_p)$. Then for* $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 > 1$, *one has*

$$\begin{aligned} I(f, M, \psi_p^{(2)}, s_1, s_2) &= I_+(f, M, \psi_p^{(2)}, s_1, s_2) \\ &\quad + C(M, \psi_p) I_+(f_p^*, M^*, \psi_p^{(1)}, s_1, 3/2 - s_1 - s_2) \\ &\quad - \frac{1}{8s_2} \zeta_0(s_1, M, \psi_p^{(1)}) \Sigma^\varepsilon(f, s_1 - 1) \\ &\quad + \frac{1}{8(s_1 + s_2 - 3/2)} C(M, \psi_p) \zeta_0(s_1, M^*, \psi_p^{(1)}) \\ &\qquad \times \Sigma^\varepsilon(f_p^*, s_1 - 1) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2s_1 + 2s_2 - 1} \delta\phi_p(-\lambda')\lambda^{*1-s_1}p^{s_1-1}G(\phi_p)L(s_1, \phi_p) \\
&\quad \times \{\Phi_1^{\varepsilon}(f^*, s_1 - 1, 0) + \Phi_2^{\varepsilon}(f^*, s_1 - 1, 0)\} \\
&+ \frac{1}{4(s_2 - 1)} \delta\phi_p(\lambda)\lambda^{1-s_1}C(M, \phi_p)G(\phi_p)L(s_1, \phi_p) \\
&\quad \times \{\Phi_1^{\varepsilon}(f, s_1 - 1, 0) + \Phi_1^{\varepsilon}(f, s_1 - 1, 0)\},
\end{aligned}
$$

where $\lambda = \lambda(M)$, $\lambda' = \lambda'(M)$, *and* $\lambda^* = \lambda(M^*)$.

Proof. For a lattice N in $V(\mathbf{Q})$, we set $R_1(N) = \{x \in N \mid x_1 \neq 0, P_2(x) = 0\}$, $R_2(N) = \{x \in N \mid x_1 = 0, P_2(x) \neq 0\}$, and $R_3(N) = \{x \in N \mid x_1 = 0, P_2(x) = 0\}$. Then $N \cap S = R_1(N) \cup R_2(N) \cup R_3(N)$ (disjoint). Let $\phi^2 \neq \phi_0$. If $x \in R_1(N) \cup R_3(N)$ for $N = M, M^*$, then $\phi^{(2)}(x) = 0$. By the Poisson summation formula and Proposition 2, we have

$$
\sum_{x \in M} \phi^{(2)}(x)\, f(\rho(b)x) = C(M, \phi)\chi_2(b)^{-3/2}\{\sum_{x \in M^*} (\bar{\phi}\phi_p)^{(2)}(y) f_p^*(\rho^*(b)y)\}.
$$

From this we see

$$
\begin{aligned}
(3.3)\quad & I(f, M, \phi^{(2)}, s_1, s_2) \\
&= I_+(f, M, \phi^{(2)}, s_1, s_2) + C(M, \phi)I_+(f_p^*, M^*, s_1, 3/2 - s_1 - s_2) \\
&\quad + \int_{-\infty}^{\infty}\int_{\mathbf{R}_+ \times \mathbf{R}_+,\, t_1t_2 \leq 1} (S_2^*(b) - S_2(b))t_1^{-2}t_2^{-1}\, dt_1dt_2du,
\end{aligned}
$$

where

$$
\begin{aligned}
S_2(b) &= t_1^{2s_1+2s_2}t_2^{2s_2} \sum_{x \in \Gamma_\infty(\delta)\backslash R_2(M)} \phi^{(2)}(x)f(\rho(b)x), \\
S_2^*(b) &= C(M, \phi)t_1^{2s_1+2s_2-3}t_1^{2s_2-3} \sum_{x \in \Gamma_\infty(\delta)\backslash R_2(M^*)} (\bar{\phi}\phi_0)^{(2)}(y)f^*(\rho^*(b)y/p).
\end{aligned}
$$

Let $f_1(x) = \int_{\mathbf{R}^2} f(x_1, y_2, y_3)e(-x_2y_2 - x_3y_3)dy_2dy_3$ and let $\mu = \mu(M)$. Then, by the Poisson summation formula, we see

$$
\begin{aligned}
\int_{-\infty}^{\infty} S_2(b)du &= \delta\lambda'^{-1}t_1^{2s_1+2s_2-2}t_2^{2s_2-2}\phi(-\mu^2) \\
&\quad \times \sum_{m=-\infty}^{\infty} \phi(m)^2 \int_{-\infty}^{\infty} f\left(\begin{pmatrix} 0 & \mu m t_1 t_2 \\ \mu m t_1 t_2 & u \end{pmatrix}\right) du \\
&= \delta(p\lambda'\mu)^{-1}\phi(-\mu^2)G(\phi^2)t_1^{2t_1+t_2-3}t_2^{2s_2-3}
\end{aligned}
$$

$$\times \sum_{m=-\infty}^{\infty} \bar{\psi}(m)^2 f_1\left(\left(\begin{array}{cc} 0 & (p\mu t_1 t_2)^{-1} m \\ (p\mu t_1 t_2)^{-1} m & 0 \end{array}\right)\right).$$

In the same way, by means of (3) of Lemma 4 we obtain

$$\int_{-\infty}^{\infty} S_2^*(b)\,du = C(M, \psi) p \delta \lambda^{*\prime -1} \psi_p(-1) \bar{\psi}(-\mu^{*2}) t_1^{2s_1+2s_2-3} t_2^{2s_2-3}$$
$$\times \sum_{m=-\infty}^{\infty} \bar{\psi}(m^2) f_1\left(\left(\begin{array}{cc} 0 & (p\mu t_1 t_2)^{-1} m \\ (p\mu t_1 t_2)^{-1} m & 0 \end{array}\right)\right),$$

with $\mu^* = \mu(M^*)$ and $\lambda^{*\prime} = \lambda'(M^*)$. Hence

$$\int_{-\infty}^{\infty} (S_2^*(b) - S_2(b))\,du$$
$$= -\delta (p\lambda'\mu)^{-1} \psi(-\mu^2)(G(\psi^2) - p^{-1}\psi(4)\psi_p(-1) W_2^2(\psi))$$
$$\times t_1^{2s_1+2s_2-3} t_2^{2s_2-3} \sum_{m=-\infty}^{\infty} f_1\left(\left(\begin{array}{cc} 0 & (p\mu t_1 t_2)^{-1} m \\ (p\mu t_1 t_2)^{-1} m & 0 \end{array}\right)\right).$$

But we see the integral

$$\int_{\mathbf{R}_+ \times \mathbf{R}_+,\, t_1 t_2 < 1} t_1^{2s_1+2s_2-3} t^{2s_2-3} \left(\sum_{m \in \mathbf{Z}} \bar{\psi}(m^2) f_1\left(\left(\begin{array}{cc} 0 & (p\mu t_1 t_2)^{-1} m \\ (p\mu t_1 t_2)^{-1} m & 0 \end{array}\right)\right)\right) dt_1 dt_2$$

does not converge. Hence we obtain

$$G(\psi^2) = p^{-1}\psi(4)\psi_p(-1) W_2^2(\psi)$$

and the last term of (3.3) vanishes. This completes the proof of (1).

Let us consider the case (2). By the Poisson summation formula, we obtain

$$\sum_{x \in M} \psi_p^{(1)}(x) f(\rho(b)x) = C(M, \psi_p) \chi_2(b)^{-3/2} \sum_{y \in M^*} \psi_p^{(1)}(y) f^*(\rho^*(b) y/p).$$

We note $\psi_p^{(1)}(x) = 0$ for $x \in R_2(M) \cup R_2(M^*)$. Hence we have

$$I(f, M, \psi^{(1)}, s_1, s_2) = I_+(f, M, \psi_p^{(1)}, s_1, s_2)$$
$$+ C(M, \psi_p) I_+(f_p^*, M^*, \psi_p^{(1)}, s_1, 3/2 - s_1 - s_2)$$
$$+ \int_{B^+/\Gamma_\infty(\delta),\, t_1 t_2 \le 1} (S_1^*(b) + S_3^*(b) - S_1(b) - S_3(b))\,db,$$

where

$$S_i(b) = t_1^{2(s_1+s_2)} t_2^{2s_2} \sum_{x \in R_i(M)} \psi_p^{(1)} f(\rho(b)x),$$

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$$S_i^*(b) = C(M, \psi_p)\chi_2(b)^{-3/2} t_1^{2(s_1+2s_2)} t_2^{2s_2} \sum_{y \in R_i(M^*)} \psi_p^{(1)}(y) f^*(\rho^*(b)y),$$

for $i = 1, 2$. An element $x \in R_1(M)$ can be written as

$$x = l \begin{pmatrix} c^2 & cd \\ cd & d^2 \end{pmatrix},$$

with $l \in \mathbf{Q}$ and $c, d \in \mathbf{Z}$, $c \neq 0$ such that $(c, d) = 1$. We see

$$\rho(b)x = \begin{pmatrix} c^2 t_1^2 & ct_1(cu + dt_2) \\ ct_1(cu + dt_2) & (cu + dt_2)^2 \end{pmatrix},$$

and

$$\begin{aligned}
&\int_{B^+/\Gamma_\infty(\delta),\, t_1 t_2 \leq 1} S_1(b)\,db \\
&= \int_{B^+,\, t_1 t_2 \leq 1} t_1^{2s_1+2s_2} t_2^{2s_2} \\
&\quad \times \sum_{x \in \Gamma_\infty(\delta)\backslash R_1(M)} \psi_p^{(1)}(x) \,|\, c \,|^{-1} f\left(l \begin{pmatrix} c^2 t_1^2 & ct_1 t_2 u \\ ct_1 t_2 u & u^2 t_2^2 \end{pmatrix} \right) dt_1 dt_2 du \\
&= \frac{1}{2}\Big\{ \Big(\sum_{x \in \Gamma_\infty(\delta)\backslash R_1(M),\, x_1 > 0} \psi_p^{(1)}(x) \,|\, x_1 \,|^{-s_1-s_2} \Big) \\
&\qquad\qquad \times \int_{|x_1|^{-1/2} t_1^{1/2} t_2 \leq 1} t_1^{s_1+s_2-2} t_2^{2s_2-1} \int_{\mathbf{R}} f\left(\begin{pmatrix} t_1 & u \\ u & t_1^{-1} u^2 \end{pmatrix} \right) du dt_1 dt_2 \\
&\quad + \Big(\sum_{x \in \Gamma_\infty(\delta)\backslash R_1(M),\, x_1 < 0} \psi_p^{(1)}(x) \,|\, x_1 \,|^{-s_1-s_2} \Big) \\
&\quad \times \int_{|x_1|^{-1/2} t_1^{1/2} t_2 \leq 1} t_1^{s_1+s_2-2} t_2^{2s_2-1} \int_{\mathbf{R}} f\left(- \begin{pmatrix} t_1 & u \\ u & t_1^{-1} u^2 \end{pmatrix} \right) du dt_1 dt_2 \Big\} \\
&= \frac{1}{8s_2} \zeta_0(s_1, M, \psi_p^{(1)}) \Sigma^\varepsilon(f, s_1 - 1).
\end{aligned}$$

Here we note $\psi_p(-1) = (-1)^\varepsilon$. In the same way, we can prove

$$\begin{aligned}
&\int_{B^+/\Gamma_\infty(\delta),\, t_1 t_2 \leq 1} S_1^*(b)\,db \\
&\qquad = \frac{1}{8(s_1 + s_2 - 3/2)} C(M, \psi_p) \zeta_0(s_1, M^*, \psi_p^{(1)}) \Sigma^\varepsilon(f_p^*, s_1 - 1).
\end{aligned}$$

On the contribution of $R_3(M)$, we have

$$\int_{B^+/\Gamma_\infty(\delta),\ t_1t_2\le 1} S_3(b)\,db$$
$$= \int_{\mathbf{R}_+\times\mathbf{R}_+,\ t_1t_2\le 1}\int_0^{\delta t_2} du\, t_1^{2s_1+2s_2-2} t_2^{2s_2-1} \sum_{l=-\infty}^{\infty} \psi_p(\lambda' l) f\left(\begin{pmatrix} 0 & 0 \\ 0 & \lambda' l t_2^2 \end{pmatrix}\right)$$
$$= 2\delta\psi_p(-\lambda')(p\lambda')^{-1}G(\psi_p)\int_{\mathbf{R}_+\times\mathbf{R}_+,\ t_1t_2\le 1} dt_1 dt_2\, t_1^{2s_1+2s_2-2}t_2^{2s_2-1}$$
$$\times \int_{\mathbf{R}^2} dx_2\, dx_3 \sum_{n=-\infty}^{\infty} \psi_p(n) f^*\left(\begin{pmatrix} n(\lambda' p t_2^2)^{-1} & x_2 \\ x_2 & x_3 \end{pmatrix}\right)$$
$$= \frac{1}{2s_1+2s_2-1}(p\lambda')^{s_1-1}\delta\psi_p(-\lambda')G(\psi_p)L(s_1,\psi_p)$$
$$\times \{\Phi_1^\varepsilon(f^*, s_1-1, 0) + \Phi_2^\varepsilon(f^*, s_1-1, 0)\}.$$

In the same way, we obtain

$$\int_{B^+/\Gamma_\infty(\delta),\ t_1t_2\le 1} S_3^*(b)\,db$$
$$= \frac{1}{4(s_2-1)}\delta\lambda^{1-s_1}\psi_p(\lambda)C(M,\psi_p)G(\psi_p)L(s_1,\psi_p)$$
$$\times \{\Phi_1^\varepsilon(f, s_1-1, 0) + \Phi_2^\varepsilon(f, s_1-1, 0)\}.$$

This completes the proof of Lemma 13.

We have $W_2^2(\psi) = G(\psi_p)G(\psi)G(\psi\psi_p)$. Hence in the course of the proof we proved the following equality, for which we have not found an elementary proof.

COROLLARY 14. *Let ψ be a character modulo p for an odd prime p such that $\psi^2 \ne \psi_0$. Then one has*

$$G(\psi^2) = p^{-1}\psi_p(-1)\psi(4)G(\psi_p)G(\psi)G(\psi\psi_p).$$

From this lemma and (3.1), (3.2), we can deduce the following lemma in the same way as in the proof of Lemma 2.4 of [5].

LEMMA 15. *Let $F_p^*(x, y) = F^*(x/p, y)$ for $F \in \mathcal{S}(\tilde V(\mathbf{R}))$.*
(1) *If $\psi^2 \ne \psi_0$, and* $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2 > 1$, *then*

$$Z(F, \tilde L, \psi^{(2)}, s_1, s_2) = Z_+^{(2)}(F, \tilde L, \psi^{(2)}, s_1, s_2)$$
$$+ C(L, \psi^{(2)})Z_+^{(2)}(F_p^*, \tilde L^*, (\bar\psi\psi_p)^{(2)}, s_1, 3/2 - s_1 - s_2).$$

(2) *If $\psi = \psi_p$ and* $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2 > 1$, *then*

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$$
\begin{aligned}
Z(F, \tilde{L}, \phi_p^{(1)}, s_1, s_2) &= Z_+^{(2)}(F, \tilde{L}, \phi_p^{(1)}, s_1, s_2)\\
&+ C(L, \phi_p^{(1)}) Z_+^{(2)}(F_p^*, \tilde{L}^*, \phi_p^{(1)}, s_1, 3/2 - s_1 - s_2)\\
&- \frac{1}{8s_2}\zeta(2s_1)\Big(\sum_{i=1}^{\nu}\zeta_0(s_1, \rho(\sigma_i)^{-1}L, \phi_p^{(1)})\Big)\tilde{\Sigma}^{\varepsilon}(F, s_1 - 1)\\
&+ \frac{1}{8(s_1 + s_2 - 3/2)} C(L, \phi_p^{(1)})\zeta(2s_1)\Big(\sum_{i=1}^{\nu}\zeta_0(s_1, \rho(\sigma_i)^{-1}L^*, \phi_p^{(1)})\Big)\\
&\qquad\qquad \times \tilde{\Sigma}^{\varepsilon}(F_p^*, s_1 - 1)\\
&- \frac{1}{2s_1 + 2s_2 - 1} p^{s_1 - 1} G(\phi_p) L(s_1, \phi_p)\zeta(2s_1)\Big(\sum_{i=1}^{\nu}\phi_p(-\lambda_i^*)\delta_i\lambda_i^{*1-s_1}\Big)\\
&\qquad \times \{\Psi_1^{\varepsilon}(F^*, s_1 - 1, 0) + \Psi_2^{\varepsilon}(F^*, s_1 - 1, 0)\}\\
&+ \frac{1}{4(s_2 - 1)} C(L, \phi_p^{(1)}) G(\phi_p) L(s_1, \phi_p)\zeta(2s_1)\Big(\sum_{i=1}^{\nu}\phi_p(\lambda_i)\delta_i\lambda_i^{1-s_1}\Big)\\
&\qquad \times \{\Psi_1^{\varepsilon}(F, s_1 - 1, 0) + \Psi_2^{\varepsilon}(F^*, s_1 - 1, 0)\}.
\end{aligned}
$$

By some calculations, we can prove

$$
\sum_{i=1}^{\nu}\zeta_0(s, \rho(\sigma_i)^{-1}L) = 2\Big(\sum_{i=1}^{\nu}\delta_i\lambda_i^{*s}\Big)\zeta(s)\zeta(2s-1)/\zeta(2s),
$$

$$
\sum_{i=1}^{\nu}\zeta_0(s, \rho(\sigma_i)^{-1}L, \phi_p^{(1)}) = 2\Big(\sum_{i=1}^{\nu}\phi_p(\lambda_i^*)\delta_i\lambda_i^{*s}\Big)L(s, \phi_p)\zeta(2s-1)/\zeta(2s).
$$

We set for $M = L, L^*$

$$
\tilde{M}_1' = (M_1' \oplus \mathbf{Z}^2) \cap \tilde{V}_1, \quad \tilde{M}_1'' = (M_1'' \oplus \mathbf{Z}^2) \cap \tilde{V}_1, \quad \tilde{M}_2' = M_2 \oplus \mathbf{Z}^2.
$$

Then $\tilde{M}_1'$, M_1'' are $\tilde{\rho}(\tilde{\Gamma})$-stable subset of $\tilde{M}$. We set

$$
\zeta_i(s, M_i', \varphi) = \sum_{x \in \Gamma\backslash M_i',\, x_1 > 0 \text{ if } i=2} \varphi(x)\mu(x)\,|\det x|^{-s}
$$

$$
\zeta_i(s, M_1'', \varphi) = \sum_{x \in \Gamma\backslash M_1''} \varphi(x)\mu(x)\,|\det x|^{-s}.
$$

Then $\zeta_1(s, M, \varphi) = \zeta_1(s, M_1', \varphi) + \zeta_1(s, M_1'', \varphi)$, and $\zeta_2(s, M, \varphi) = \zeta_2(s, M_2', \varphi)$. We note $\zeta_1(s, M_1', \varphi) = 0$ if $\varepsilon = -1$. The following two lemmas can be proved in the same way as Lemma 2.5 and Lemma 2.8 of [5].

LEMMA 16. *Let φ be one of the characters defined above, and let $i = 1,2$, $\varepsilon = 1,2$. If* $\mathrm{Re}\, s_1 > 1$ *and* $\mathrm{Re}\, s_2 > 3/2$, *one has*

$$
\begin{aligned}
Z(F, \tilde{M}_i', \varphi, s_1, s_2) &= Z_+^{(1)}(F, \tilde{M}_i', \varphi, s_1, s_2)\\
&\qquad + \hat{Z}_+^{(1)}(\hat{F}, \tilde{M}_i', \varphi, 1 - s_1, s_1 + s_2 - 1/2)
\end{aligned}
$$

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$$+\frac{1}{2(s_1-1)}\zeta_i(s_2+1/2,\, M_i',\, \varphi)\Psi_i^{\varepsilon}(F,\, 0,\, s_2-1)$$
$$-\frac{1}{2s_1}\zeta_i(s_2,\, M_i',\, \varphi)\Psi_i^{\varepsilon}(F,\, 0,\, s_2-3/2).$$

If $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 > 1$, *one has the formula obtained from the above formula by replacing* $F, \hat{F}, s_1, s_2$ *on the right hand side by* $\hat{F}, F, 1-s_1, s_2+s_2-1/2$ *respectively.*

LEMMA 17. *Let* $F \in \mathcal{S}(\tilde{V}(\mathbf{R}))$. *If* $\mathrm{Re}\, s_1 > 1$, $\mathrm{Re}\, s_2 > 3/2$, *one has*

$$\begin{aligned}
Z(F, \tilde{M}_1'', \varphi, s_1, s_2) &= Z_+^{(1)}(F, \tilde{M}_1'', \varphi, s_1, s_2) \\
&+ \hat{Z}_+^{(1)}(\hat{F}, \tilde{M}_1'', \varphi, 1-s_1, s_1+s_2-1/2) \\
&+ \frac{1}{(s_1-1)^2} 2^{-3}\eta(s_2+1/2, M, \varphi)\Psi_1^{\varepsilon}(F, 0, s_2-1) \\
&+ \frac{1}{s_1^2} 2^{-3}\eta(s_2, M, \varphi)\Psi_1^{\varepsilon}(\hat{F}, 0, s_2-3/2) \\
&+ \frac{1}{2(s_1-1)}\{\phi_1(s_2+1/2)\Psi_1^{\varepsilon}(F, 0, s_2-1) \\
&\qquad + \frac{1}{4}\eta(s_2+1/2, M, \varphi)\frac{\partial\Psi_1^{\varepsilon}}{\partial s_1}(F, 0, s_2-1)\} \\
&+ \frac{1}{2s_1}\Big\{\phi_2(s_2)\Psi_1^{\varepsilon}(\hat{F}, 0, s_2-3/2) \\
&\qquad + \frac{1}{4}\eta(s_2, M, \varphi)\Big(\frac{\partial\Psi_1^{\varepsilon}}{\partial s_1}(\hat{F}, 0, s_2-3/2) \\
&\qquad\qquad - \frac{\partial\Psi_1^{\varepsilon}}{\partial s_2}(\hat{F}, 0, s_2-3/2)\Big)\Big\}.
\end{aligned}$$

Here

$$\phi_1(s) = \frac{2C-\log 2}{4}\eta(s, M, \varphi) + \frac{1}{8}\eta'(s, M, \varphi) + \zeta_1(s, M_1'', \varphi),$$
$$\phi_2(s) = \frac{2C-\log 2}{4}\eta(s, M, \varphi) - \frac{1}{8}\eta'(s, M, \varphi) + \zeta_1(s, M_1'', \varphi).$$

If $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 > 1$, *one has the formula obtained from the above formula replacing* $F, \hat{F}, s_1, s_2$ *by* $\hat{F}, F, 1-s_2, s_1+s_2-1/2$, *respectively.*

Proof of Proposition 7. By the lemmas proved above, we can prove Proposition 7 as in [5]. We give a sketch for it. Lemma 8 and Lemma 9 (3) imply that $(s_2-1)(s_1+s_2-3/2)\xi_i(s_1, s_2, \tilde{L}, \varphi)$ can be extended to a holomorphic function of (s_1, s_2) in $D_1 = \{(s_1, s_1) \mid \mathrm{Re}\, s_1 > 1\}$. Lemma 8, Lemma 9 (1), (2), Lemma 16, and Lemma 17 imply that $(s_1 - 1)^2 \xi_i(s_1, s_2, \tilde{L}, \varphi)$ can be extended to the demain $D_2 = \{(s_1, s_2) \mid \mathrm{Re}(s_1 + s_2) > 3/2, \mathrm{Re}\, s_2 > 3/2\}$. Since the convex hull of the domain $D_1 \cup D_2$ is the whole $\mathbf{C}^2$, $(s_1 - 1)^2(s_2 - 1)(s_1 + s_2 - 3/2)\xi_i(s_1, s_2, \tilde{L}, \varphi)$ can be continued to an entire function on $\mathbf{C}^2$. The functional equations (a) and (b) follow respectively from Lemma 15 and Lemma 16, Lemma 17. The assertions (c) and (c)′ follows from Lemma 15. The assertions (b), (b) (c) and (c)′ imply (d) and (d)′. Finally, (e), (e)′ (f), and (f)′ can be derived from Lemma 16 and Lemma 17.

Appendix

In the appendix we give the formulas of Lemma 15 in the cases of $\phi_p^{(r)}(r = 0,2)$ and $\phi_0^{(r)}(r = 0,1,2)$. Let the notation be as in Section 2 and Section 3. In the same way as in the case of Lemma 15, we can prove the following.

LEMMA 15. (3) *Let L and L^* be as in Introduction, and set $\Psi_i(F, s_1, s_2) = \Psi_i^2(F, s_1, s_2)$ for $i = 1,2$ and $\tilde{\Sigma}(F, s) = \tilde{\Sigma}^2(F, s)$. Let $\phi = \phi_p$, or ϕ_0. If* $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_1 > 1$, *one has*

$$
\begin{aligned}
Z(F, \tilde{L}, \phi_0^{(2)}, s_1, s_2) &= Z_+^{(2)}(F, \tilde{L}, \phi_0^{(2)}, s_1, s_2) \\
&\quad + v(L)^{-1} p^{-3}\{w(2,0) Z_+^{(2)}(F_p^*, \tilde{L}^*, \phi_p^{(0)}, s_1, 3/2 - s_1 - s_2) \\
&\qquad + w(2,2) Z_+^{(2)}(F_p^*, \tilde{L}^*, \phi_p^{(2)}, s_1, 3/2 - s_1 - s_2)\} \\
&\quad + \frac{1}{8(s_1 + s_2 - 3/2)} v(L)^{-1} p^{-3} w(2,0) \zeta(2s_1) \\
&\qquad\qquad \times \sum_{i=1}^{\nu} \zeta_0(s_1, \rho(\sigma_i)^{-1} L^*) \tilde{\Sigma}(F^*, s_1 - 1) \\
&\quad + \frac{1}{4(s_1 - 1)} v(L)^{-1} p^{-3} w(2,0) \zeta(s_1) \zeta(2s_1) \Big(\sum_{i=1}^{\nu} \delta_i \lambda_i^{1-s_1}\Big) \\
&\qquad \times \{\Psi_1(F, s_1 - 1, 0) + \Psi_2(F, s_1 - 1, 0)\},
\end{aligned}
$$

$$
\begin{aligned}
Z(F, \tilde{L}, \phi_0^{(0)} + \phi_0^{(1)}, s_1, s_2) &= Z_+^{(2)}(F, \tilde{L}, \phi_0^{(0)} + \phi_0^{(1)}, s_1, s_2) \\
&\quad + v(L)^{-1} p^{-3}\{w(0,0) Z_+^{(2)}(F_p^*, \tilde{L}^*, \phi_p^{(0)}, s_1, 3/2 - s_1 - s_2) \\
&\qquad + w(0,2) Z_+^{(2)}(F_p^*, \tilde{L}^*, \phi_p^{(2)}, s_1, 3/2 - s_1 - s_2)\} \\
&\quad - \frac{1}{8s_2} \zeta(2s_1) \Big(\sum_{i=1}^{\nu} \zeta_0(s_1, \rho(\sigma_i)^{-1} L)\Big) \tilde{\Sigma}(F, s_1 - 1) \\
&\quad + \frac{1}{8(s_1 + s_2 - 3/2)} v(L)^{-1} p^{-3} w(0,0) \zeta(2s_1)
\end{aligned}
$$

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$$\times \left(\sum_{i=1}^{\nu} \zeta_0(s_1, \rho(\sigma_i)^{-1}L^*)\right)\tilde{\Sigma}(F^*, s_1 - 1)$$

$$+ \frac{1}{4(s_2-1)} v(L)^{-1}p^{-3}w(0,0)\zeta(s_1)\zeta(2s_1)\left(\sum_{i=1}^{\nu} \delta_i\lambda_i^{1-s_1}\right)$$
$$\times \{\Psi_1(F, s_1 - 1, 0) + \Psi_2(F, s_1 - 1, 0)\}$$

$$- \frac{1}{2s_1 + 2s_2 - 1} \zeta(s_1)\zeta(2s_1)\left(\sum_{i=1}^{\nu} \delta_i\lambda_i^{*1-s_1}\right)$$
$$\times \{\Psi_1(F^*, s_1 - 1,0) + \Psi_2(F^*, s_1 - 1,0)\},$$

$$v(L)^{-1}p^{-3}\{w(2,0)Z(F_p^*, \tilde{L}^*, \phi_p^{(0)}, s_1, s_2) + w(2,2)Z(F_p^*, \tilde{L}^*, \phi_p^{(2)} s_1, s_2)\}$$
$$= v(L)^{-1}p^{-3}\{w(2,0)Z_+(F_p^*, \tilde{L}^*, \phi_p^{(0)}, s_1, 3/2 - s_1 - s_2)$$
$$+ w(2,2)Z_+(F_p^*, \tilde{L}^*, \phi_p^{(2)} s_1, 3/2 - s_1 - s_2)\}$$
$$+ Z_+^{(2)}(F, \tilde{L}, \phi_0^{(2)} s_1, 3/2 - s_1 - s_2)$$
$$- \frac{1}{8s_2} v(L)^{-1}p^{-3}w(2,0)\zeta(2s_1)\left(\sum_{i=1}^{\nu} \zeta_0(s_1, \rho(\sigma_i)^{-1}L^*)\right)\tilde{\Sigma}(F^*, s_1 - 1)$$
$$- \frac{1}{2(2s_1 + 2s_2 - 1)} v(L)^{-1}p^{-3}w(2,0)\zeta(s_1)\zeta(2s_1)\left(\sum_{i=1}^{\nu} \delta_i\lambda_i^{1-s_1}\right)$$
$$\times \{\Psi_1(F, s_1 - 1, 0) + \Psi_2(F, s_1 - 1, 0)\}$$

$$v(L)^{-1}p^{-3}\{w(0,0)Z(F_p^*, \tilde{L}^*, \phi_p^{(0)}, s_1, s_2) + w(0,2)Z(F_p^*, \tilde{L}^*, \phi_p^{(2)} s_1, s_2)\}$$
$$= v(L)^{-1}p^{-3}\{w(0,0)Z_+^{(2)}(F_p^*, \tilde{L}^*, \phi_p^{(0)}, s_1, s_2)$$
$$+ w(0,2)Z_+^{(2)}(F_p^*, \tilde{L}^*, \phi_p^{(2)} s_1, s_2)\}$$
$$+ Z_+^{(2)}(F, \tilde{L}, \phi_0^{(0)} + \phi_0^{(1)} s_1, 3/2 - s_1 - s_2)$$
$$+ \frac{1}{8(s_1 + s_2 - 3/2)} \zeta(2s_1)\left(\sum_{i=1}^{\nu} \zeta_0(s_1, \rho(\sigma_i)^{-1}L)\right)\tilde{\Sigma}(F, s_1 - 1)$$
$$- \frac{1}{8s_2} v(L)^{-1}p^{-3}w(0,0)\zeta(2s_1)\left(\sum_{i=1}^{\nu} \zeta_0(s_1, \rho(\sigma_i)^{-1}L^*)\right)\tilde{\Sigma}(F^*, s_1 - 1)$$
$$- \frac{1}{2(2s_1 + 2s_2 - 1)} v(L)^{-1}p^{-3}w(0,0)\zeta(s_1)\zeta(2s_1)\left(\sum_{i=1}^{\nu} \delta_i\lambda_i^{1-s_1}\right)$$
$$\times \{\Psi_1(F, s_1 - 1, 0) + \Psi_2(F, s_1 - 1, 0)\}$$
$$+ \frac{1}{2(s_1 - 1)} \zeta(s_1)\zeta(2s_1)\left(\sum_{i=1}^{\nu} \delta_i\lambda_i^{*1-s_1}\right)$$
$$\times \{\Psi_1(F^*, s_1 - 1, 0) + \Psi_2(F, s_1 - 1, 0)\}.$$

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