J. Aust. Math. Soc. **94** (2013), 202–221
doi:10.1017/S1446788712000523

EXTREME POINTS OF INTEGRAL FAMILIES OF ANALYTIC FUNCTIONS

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(Received 13 August 2011; accepted 17 October 2012; first published online 14 March 2013)

Communicated by P. C. Fenton

Abstract

Extreme points of compact, convex integral families of analytic functions are investigated. Knowledge about extreme points provides a valuable tool in the optimization of linear extremal problems. The functions studied are determined by a two-parameter collection of kernel functions integrated against measures on the torus. For specific choices of the parameters many families from classical geometric function theory are included. These families include the closed convex hull of the derivatives of normalized close-to-convex functions, the ratio of starlike functions of different orders, as well as many others. The main result introduces a surprising new class of extreme points.

2010 *Mathematics subject classification*: primary 30C45; secondary 30E20.

Keywords and phrases: extreme points.

1. Main theorem

Let $\mathbb{D}$ and Γ, respectively, denote the open unit disk and the unit circle in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the space of functions analytic in $\mathbb{D}$ and let $\mathbb{T} = \Gamma \times \Gamma$ denote the torus. We investigate extreme points of the compact, convex families in $H(\mathbb{D})$ defined by, for $p, q > 0$,

$$F_{p,q} = \left\{ f_\mu(z) = \int_{\mathbb{T}} \frac{(1 - xz)^p}{(1 - yz)^q}\, d\mu(x, y) : \mu \text{ is a probability measure on } \mathbb{T} \right\}.$$

For $p = 1$, $q = 3$, the family $F_{1,3}$ is the closed convex hull of the derivatives of the normalized close-to-convex functions on $\mathbb{D}$ [4]. If $1 \le p \le q \le 2$, $p = 2(1 - \beta)$ and $q = 2(1 - \alpha)$, one obtains the closed convex hull of the ratio of two starlike functions of order α and β [1]. The closed convex hull of a number of other families of analytic functions can be identified with specific $F_{p,q}$ families [5, 6]. Solving linear extremal problems reduces to optimizing linear functionals over the extreme points of these families, which leads to the question of determining, for a given p and q, the extreme points of $F_{p,q}$.

For many calculations, it is convenient to make a change of variables: $(x, y) \to (xy, y)$. Let

$$k_{x,y}(z) = \frac{(1 - xyz)^p}{(1 - yz)^q} = \sum_{n=0}^{\infty} K_n(x, y) z^n.$$

With this choice of notation and change of variables,

$$F_{p,q} = \left\{ f_\mu(z) = \int_{\mathbb{T}} k_{x,y}(z)\, d\mu(x, y) : \mu \text{ is a probability measure on } \mathbb{T} \right\}.$$

We also define

$$I_{p,q} = \left\{ f_\mu(z) = \int_{\mathbb{T}} k_{x,y}(z)\, d\mu(x, y) : \mu \text{ is a complex Borel measure on } \mathbb{T} \right\}.$$

Then $I_{p,q}$ is the linear span of $F_{p,q}$ in $H(\mathbb{D})$.

Consider the curve $C_p = \{(1 - x)^p : |x| = 1\}$ and let E_p denote the closed convex hull of C_p in $\mathbb{C}$. The following are well-known facts.

(i) Every extreme point of $F_{p,q}$ is a kernel function $k_{x,y}$.

(ii) If $x \neq 1$ and $(1 - x)^p$ is an extreme point of E_p, then $k_{x,1}$ is an extreme point of $F_{p,q}$.

(iii) The family $F_{p,q}$ is closed under rotations. That is, if f is a function in $F_{p,q}$ and $|u| = 1$, then $g(z) = f(uz)$ is also in $F_{p,q}$. Any rotation of an extreme point is an extreme point.

By rotation, it follows that $k_{x,y}$ is an extreme point if and only if $k_{x,1}$ is an extreme point. If $0 < p \le 1$, the curve C_p encloses a convex region and every point on C_p is an extreme point of E_p. In this case, by (ii), $k_{x,1}$ is an extreme point of $F_{p,q}$ whenever $x \neq 1$. For $p > 1$, as one traverses the curve C_p in either direction starting at $2^p (x = -1)$ and ending up at the origin $(x = 1)$, there are two distinguished 'turning points'. These turning points occur when $|\arg(x)| = \pi(p - 1)/(p + 1)$ and correspond to the points on C_p where $\operatorname{Re}(1 - x)^p$ attains its minimum value. In this case, the convex set E_p is bounded by part of the curve C_p – the part traversed from one turning point through 2^p to the other turning point – together with the vertical line segment joining the two turning points. Thus, if $p > 1$, a point $(1 - x)^p$ is an extreme point of E_p if and only if $\pi(p - 1)/(p + 1) \le |\arg(x)| \le \pi$. Again by (ii), $k_{x,1}$ is an extreme point of $F_{p,q}$ for $|\arg(x)|$ in this interval and $x \neq 1$.

Two fundamental questions arise. The first question is whether any points $(1 - x)^p$, with $|\arg(x)| < \pi(p - 1)/(p + 1)$, yield kernel functions that are extreme points of $F_{p,q}$. The second question is whether the parameter q plays a role in the determination of extreme points. In this paper we address both of these questions. The main theorem answers the first affirmatively and the constructive approach in the proof utilizes the parameter q.

Theorem 1.1 (Main theorem). *For $p > 1$ and $q > 0$ there exist points $(1-x)^p$ on C_p with $|\arg(x)| < \pi(p-1)/(p+1)$ such that the corresponding kernel functions $k_{x,1}$ are extreme points of $F_{p,q}$.*

2. Preliminaries

Fix $p, q > 0$. Let $F = F_{p,q}$ and $I = I_{p,q}$. A *generalized functional* on F is a map $L : F \to \mathbb{C}$ that is continuous on F and linear on I. Let F^* denote the collection of all generalized functionals on F. If $f \in F$ and there exists $L \in F^*$, with $\operatorname{Re} L$ nonconstant on F, such that $\operatorname{Re} L(f) = \max_F \operatorname{Re} L$, then f is called a *generalized support point* of F. If f uniquely maximizes $\operatorname{Re} L$, then f is called a *generalized exposed point* of F.

Remark 2.1. The set of generalized support points of F corresponding to a generalized functional L is a compact, convex, extremal subset of F. Therefore it contains extreme points of F as a consequence of the Krein–Milman theorem. A generalized exposed point then is an extreme point of F.

Our approach is to associate the space of generalized functionals with a subspace of the space $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$ and to associate generalized support points with points where the real parts of certain functions in this subspace achieve their maximum value.

Let $B = \overline{Sp}\{K_n : n = 0, 1, 2, \ldots, \}$, that is, B is the uniformly closed linear span of the coefficient functions $K_n(x, y)$ in $C(\mathbb{T})$. If one writes

$$(1 - xyz)^p = \sum_{n=0}^{\infty} A_n(p)x^n y^n z^n \quad \text{and} \quad \frac{1}{(1 - yz)^q} = \sum_{n=0}^{\infty} B_n(q)y^n z^n,$$

then

$$A_n(-p) = B_n(q) \quad \text{and} \quad K_n(x, y) = \sum_{i=0}^{n} [A_i(p)x^i B_{n-i}(q)]y^n = K_n(x, 1)y^n.$$

For example, when $p = 2$ and $q = 1$,

$$K_0(x, y) = 1, \quad K_1(x, y) = (1 - 2x)y, \quad K_n(x, y) = (1 - x)^2 y^n, \quad n \geq 2,$$

and

$$B = \{a_0 + a_1(1 - 2x)y + (1 - x)^2 y^2 f(y) : a_0, a_1 \in \mathbb{C} \text{ and } f \in A_0\},$$

where A_0 is the classic disk algebra on $\mathbb{D}$. In general, B is not so easily described.

We use the following special case of an unpublished theorem in [7]. For completeness, we sketch the proof.

Theorem 2.2. *F^* is isometrically isomorphic to B where, for $L \in F^*$, $\|L\| = \sup\{|L(f)| : f \in F\}$. The correspondence is given by $L_G \leftrightarrow G$ where, for each $G \in B$ and $f_\mu \in I$,*

$$L_G(f_\mu) = \int_{\mathbb{T}} G(x, y)\, d\mu(x, y).$$

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PROOF. Let $G \in B$. First we show that L_G is well defined. Suppose that $f_\mu = f_\nu \in I$. Then

$$\int_{\mathbb{T}} k_{x,y}(z)\, d\mu(x, y) = \int_{\mathbb{T}} k_{x,y}(z)\, d\nu(x, y), \quad z \in \mathbb{D}.$$

Hence

$$\sum_{n=0}^{\infty} \int_{\mathbb{T}} K_n(x, y)\, d\mu(x, y) z^n = \sum_{n=0}^{\infty} \int_{\mathbb{T}} K_n(x, y)\, d\nu(x, y) z^n$$

and

$$\int_{\mathbb{T}} K_n(x, y)\, d\mu(x, y) = \int_{\mathbb{T}} K_n(x, y)\, d\nu(x, y), \quad n = 0, 1, 2, \ldots.$$

Since $G \in B = \overline{Sp}\{K_n : n = 0, 1, 2, \ldots,\}$, $\int_{\mathbb{T}} G\, d\mu = \int_{\mathbb{T}} G\, d\nu$. Thus $L_G(f_\mu) = L_G(f_\nu)$ and L_G is well defined on F. Also, L_G is clearly linear on I.

To show that $L_G \in F^*$, it remains to show that L_G is continuous on F. Suppose that f_{μ_n} converges to f_μ in F. Let $b_n = L_G(f_{\mu_n}) = \int_{\mathbb{T}} G\, d\mu_n$ and let $b = L_G(f_\mu) = \int_{\mathbb{T}} G\, d\mu$. The set of probability measures on $\mathbb{T}$ is compact, convex and metrizable in the *weak** topology. Let λ be any weak* subsequential limit point of $\{\mu_n\}$. Apply the weak* convergence to $k_{x,y}(z)$ for each fixed z in $\mathbb{D}$ and use the fact that $f_{\mu_n}(z) \to f_\mu(z)$, that is, $\int_{\mathbb{T}} k_{x,y}(z)\, d\mu_n(x, y) \to \int_{\mathbb{T}} k_{x,y}(z)\, d\mu(x, y)$, to conclude that $\int_{\mathbb{T}} k_{x,y}(z)\, d\lambda(x, y) = \int_{\mathbb{T}} k_{x,y}(z)\, d\mu(x, y)$ for z in $\mathbb{D}$. Thus $L_G(f_\lambda) = L_G(f_\mu) = b$ is the unique subsequential limit point of $\{b_n\}$. Therefore $b_n \to b$, L_G is continuous on F and $L_G \in F^*$.

Conversely, we show that every $L \in F^*$ is of the form given in the theorem. Fix L in F^* and let $G(x, y) = L(k_{x,y})$. Since L is continuous on F, G is continuous on $\mathbb{T}$. We also need to show that $G \in B$. Suppose that $G \notin B$. Then, by a corollary to the Hahn–Banach theorem, there exists a complex Borel measure λ such that $\int_{\mathbb{T}} G\, d\lambda = 1$ and $\int_{\mathbb{T}} h\, d\lambda = 0$ for all h in B. For fixed $z \in \mathbb{D}$, $k_{x,y}(z) = \sum_{n=0}^{\infty} K_n(x, y) z^n$ and the series converges uniformly on $\mathbb{T}$, so that $k_{x,y}(z) \in \overline{Sp}\{K_n : n = 0, 1, 2, \ldots\} = B$. We thus have $f_\lambda(z) = \int_{\mathbb{T}} k_{x,y}(z)\, d\lambda(x, y) = 0$ for each $z \in \mathbb{D}$, that is, $f_\lambda \equiv 0$. Then

$$\begin{aligned}\int_{\mathbb{T}} G(x, y)\, d\lambda(x, y) &= \int_{\mathbb{T}} L(k_{x,y}(z))\, d\lambda(x, y) \\ &= L\Big(\int_{\mathbb{T}} k_{x,y}(z)\, d\lambda(x, y)\Big) = L(f_\lambda) = L(0) = 0,\end{aligned}$$

contradicting the choice of λ. Hence $G \in B$ and, if $f_\mu \in I$, then

$$\begin{aligned}L(f_\mu) &= L\Big(\int_{\mathbb{T}} k_{x,y}(z)\, d\mu(x, y)\Big) \\ &= \int_{\mathbb{T}} L(k_{x,y}(z))\, d\mu(x, y) = \int_{\mathbb{T}} G(x, y)\, d\mu(x, y) = L_G(f_\mu).\end{aligned}$$

Finally, to show that the correspondence is isometric,

$$\begin{aligned}\|L_G\| &= \sup\{|L_G(f)| : f \in F\}\\ &= \sup\left\{\left|\int_{\mathbb{T}} G\,d\mu\right| : \mu \text{ is a probability measure on } \mathbb{T}\right\}\\ &\le \|G\| = \sup\{|G(x,y)| : (x,y) \in \mathbb{T}\}.\end{aligned}$$

If (x_0, y_0) is such that $|G(x_0, y_0)| = \|G\|$, then $|L_G(k_{x_0,y_0})| = |G(x_0, y_0)| = \|G\|$ and the correspondence is an isometric isomorphism. □

A point $(x_0, y_0) \in \mathbb{T}$ is a *peak point* of a closed subspace B of $C(\mathbb{T})$ if there exists $f \in B$ such that $f(x_0, y_0) = 1$ while $|f(x, y)| < 1$ for all $(x, y) \in \mathbb{T}$ such that $(x, y) \neq (x_0, y_0)$. A point $(x_0, y_0) \in \mathbb{T}$ is a *peak point* of $\operatorname{Re} B$ if there exists $f \in B$ such that $\operatorname{Re} f(x_0, y_0) = 1$ while $\operatorname{Re} f(x, y) < 1$ for all $(x, y) \in \mathbb{T}$ such that $(x, y) \neq (x_0, y_0)$. In each case we say that the function *peaks* at the point (x_0, y_0).

Corollary 2.3. *(x_0, y_0) is a peak point of $\operatorname{Re} B$ if and only if k_{x_0,y_0} is a generalized exposed point of F.*

Proof. Suppose that (x_0, y_0) is a peak point of $\operatorname{Re} B$. Then there exists G in B such that $\operatorname{Re} G(x_0, y_0) = 1$ and $\operatorname{Re} G(x, y) < 1$ if $(x, y) \neq (x_0, y_0)$.

For any probability measure μ on $\mathbb{T}$,

$$\operatorname{Re} L_G(f_\mu) = \operatorname{Re} \int_{\mathbb{T}} G(x, y)\,d\mu(x, y) = \int_{\mathbb{T}} \operatorname{Re} G(x, y)\,d\mu(x, y).$$

If $f_\mu = k_{x_0,y_0}$ and μ is unit point mass at (x_0, y_0), then $\operatorname{Re} L_G(k_{x_0,y_0}) = 1$. If $f_\mu \neq k_{x_0,y_0}$, then μ is not unit point mass at (x_0, y_0) and $\operatorname{Re} L_G(f_\mu) < 1$. Thus k_{x_0,y_0} is a generalized exposed point.

Conversely, if f_μ is a generalized exposed point, then f_μ is an extreme point and $f_\mu = k_{x_0,y_0}$ for some (x_0, y_0). Suppose that L is such that $\operatorname{Re} L$ peaks at k_{x_0,y_0}. Let G be such that $L = L_G$. Then, using unit point mass at (x, y) to represent each kernel function $k_{x,y}$,

$$\begin{aligned}1 &= \operatorname{Re} L_G(k_{x_0,y_0}) = \operatorname{Re} G(x_0, y_0) \quad \text{and}\\ 1 &> \operatorname{Re} L_G(k_{x,y}) = \operatorname{Re} G(x, y), \quad \text{if } (x, y) \neq (x_0, y_0).\end{aligned}$$

Hence (x_0, y_0) is a peak point of $\operatorname{Re} B$. □

The final preliminary result concerns a method for constructing peaking functions for the space $\operatorname{Re} B$. The ideas in the proof originally come from Bishop [2], who was interested in constructing peak points for algebras of functions. A concise statement and proof of Bishop's result is given by [3, Theorem 11.1 in Ch. 2]. Rice [7] adapted the argument to spaces of functions and proved the following theorem.

Theorem 2.4. *Let X be a compact metric space, E be a closed subset of X, B be a closed subspace of $C(X)$ and $x_0 \in E$. Suppose that there exist numbers c and M,*

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with $0 < c < 1 \leq M$*, such that, for any neighborhood* U *of* E *in* X*, there exists* $H \in B$ *such that:*

(a) $\|H\| \leq M$;
(b) $\operatorname{Re} H(x) < c,\ x \notin U$; *and*
(c) $\operatorname{Re} H(x) < 1 = \operatorname{Re} H(x_0)$*, for all* $x \in E$ *with* $x \neq x_0$.

Then there exists $G \in B$ *such that* $\operatorname{Re} G(x_0) > \operatorname{Re} G(x)$ *for all* $x \in X$ *with* $x \neq x_0$.

We need a modified version of this theorem. We give the proof of the modified version, the details of which are guided by the arguments of Bishop [2] and Rice [7].

THEOREM 2.5. *Let* X *be a compact metric space,* E *be a closed subset of* X, B *be a closed subspace of* $C(X)$ *and* $x_0 \in E$*. Suppose that there exist numbers* c *and* M*, with* $0 < c < 1 \leq M$*, functions* $H_n \in B$, $n \in \mathbb{N}$*, and* $H : X \to \mathbb{C}$ *satisfying:*

(a) $\|H_n\| \leq M$ *for every* n;
(b) H_n *converges uniformly to* H *on* E;
(c) $\operatorname{Re} H_n(x_0) = 1$ *for every* n;
(d) $1 = \operatorname{Re} H(x_0) > \operatorname{Re} H(x)$ *for all* $x \in E$ *with* $x \neq x_0$;
(e) H_n *converges to* H *uniformly on compact subsets of* $X \setminus E$; *and*
(f) $\operatorname{Re} H(x) < c,\ x \in X \setminus E$.

Then, for a given relatively open neighborhood W *of* x_0 *in* E*, there exists* $G \in B$ *such that*

$$\left\{x \in X : \operatorname{Re} G(x) = \max_{x' \in X} \operatorname{Re} G(x')\right\} \subseteq W.$$

PROOF. Given any open neighborhood U of E, the hypotheses allow one to find $h \in B$ satisfying:

(i) $\|h\| \leq M$;
(ii) $\operatorname{Re} h < c$ on $X \setminus U$;
(iii) $\operatorname{Re} h(x_0) = 1$; and
(iv) $\operatorname{Re} h < 1$, on $E \setminus W$.

Let $\{U_n\}_{n=0}^{\infty}$ be a decreasing sequence of neighborhoods of E with $U_0 = X$ and $E = \bigcap_{n=0}^{\infty} U_n$. Since $1 > (M-1)/(M-c) \geq 0$, we can choose $0 < s < 1$ such that $(M-1) - s(M-c) < 0$.

Choose a sequence $\{\epsilon_n\}_{n=0}^{\infty}$ of positive numbers, decreasing to 0, satisfying $\epsilon_{n-1}(1 - s^n) + s^n(M - 1 - s(M-c)) < 0$, $n \geq 1$.

Claim. There exists a decreasing sequence of open sets $\{V_j\}_{j=0}^{\infty}$ such that $V_0 = X$, $E \cup W = \bigcap_{j=0}^{\infty} V_j$ and a sequence of functions $\{h_j\}_{j=0}^{\infty} \subseteq B$ satisfying:

(1) $\operatorname{Re} h_j(x) < \operatorname{Re} h_j(x_0) = 1,\ x \in E \setminus W,\ j \geq 0$;
(2) $\|h_j\| \leq M$;
(3) $\operatorname{Re} h_j(x) < c,\ x \notin V_j,\ j \geq 1$; and
(4) $\operatorname{Re} h_i(x) < 1 + \epsilon_{j-1};\ x \in V_j \setminus W,\ i = 0, 1, 2, \ldots, j-1,\ j \geq 1$.

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Take $V_0 = X$ and let h_0 correspond to V_0. That is, choose $h_0 \in B$ satisfying (i)–(iv) above with $U = V_0$.

Define

$$Y_1 = \{x \in U_1 \cap V_0 : \operatorname{Re} h_0(x) < 1 + \epsilon_0\} \quad \text{and} \quad V_1 = Y_1 \cup W.$$

Let h_1 correspond to V_1. Then $V_1 \subseteq V_0$, (1)–(3) follow from the properties of h_1 and (4) follows from the definition of Y_1.

Inductively, assume $V_0, \ldots, V_j$ and $h_0, \ldots, h_j$ have been chosen. Define

$$Y_{j+1} = \{x \in U_{j+1} \cap V_j : \operatorname{Re} h_i(x) < 1 + \epsilon_j, i = 0, 1, 2, \ldots, j\}$$

and

$$V_{j+1} = Y_{j+1} \cup W.$$

Let h_{j+1} correspond to V_{j+1}. Since $Y_{j+1} \subseteq V_j$, we have $V_{j+1} \subseteq V_j$. By the choice of h_{j+1} it satisfies properties (1)–(3) and the definition of Y_{j+1} yields property (4). Since $E \cup W \subseteq U_j \cup W$ for all j and $\bigcap_{j=0}^{\infty} U_j = E$, we have $E \cup W = \bigcap_{j=0}^{\infty} V_j$. This establishes our claim.

To complete the proof of Theorem 2.5, let

$$G = (1 - s) \sum_{j=0}^{\infty} s^j h_j.$$

Then $G \in B$ and $G(x_0) = 1$. If $x \in E \setminus W$, then $\operatorname{Re} G(x) < 1$ since $h_j(x) < 1$ for every j. If $x \notin E$, then there exists a maximal $l \geq 0$ such that $x \in V_l \setminus V_{l+1}$.

If $l = 0$, then $x \in V_0$ but $x \notin V_j$ for all $j \geq 1$, and

$$\begin{aligned} \operatorname{Re} G(x) &= (1 - s)\Big[\operatorname{Re} h_0(x) + \sum_{j=1}^{\infty} s^j \operatorname{Re} h_j(x)\Big] \\ &< (1 - s)\Big[M + \sum_{j=1}^{\infty} s^j c\Big] = M + s(c - M) < 1 \quad \text{by the choice of s.} \end{aligned}$$

If $l \geq 1$,

$$\begin{aligned} \operatorname{Re} G(x) &= (1 - s)\Big[\sum_{j=0}^{l-1} s^j \operatorname{Re} h_j(x) + s^l \operatorname{Re} h_l(x) + \sum_{j=l+1}^{\infty} s^j \operatorname{Re} h_j(x)\Big] \\ &< (1 - s)\Big[(1 + \epsilon_{l-1})\frac{1 - s^l}{1 - s} + M \cdot s^l + c \cdot \frac{s^{l+1}}{1 - s}\Big] \\ &= 1 + \epsilon_{l-1}(1 - s^l) + s^l[M - 1 - s(M - c)] < 1. \end{aligned}$$

Hence

$$\operatorname{Re} G(x) < 1 \quad \text{if } x \notin W, \quad \operatorname{Re} G(x_0) = 1 \text{ and}$$
$$\Big\{x \in X : \operatorname{Re} G(x) = \max_{x' \in X} \operatorname{Re} G(x')\Big\} \subseteq W \quad \text{as claimed.} \qquad \square$$

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REMARK 2.6. If, in Theorem 2.5, each H_n satisfies

$$1 = \operatorname{Re} H_n(x_0) > \operatorname{Re} H_n(x), \qquad x \in E, x \neq x_0,$$

then there is no need for the relatively open neighborhood W of x_0 in E and one can conclude that there exists $G \in B$ such that $\operatorname{Re} G$ peaks at x_0.

3. Proof of the main theorem

The proof of the main theorem will follow from Lemmas 3.2–3.4 below and Theorem 2.5. For Lemma 3.2 we will need some geometric information about the curve $C_p = \{(1-x)^p : |x| = 1\}$. For these calculations parametrize C_p with $x = -e^{i\varphi}$, $-\pi < \varphi \le \pi$. We refer to the part of C_p where $0 < \varphi < \pi$ as the *upper branch* of C_p and the part where $-\pi < \varphi < 0$ as the *lower branch*. Double angle formulas yield

$$(1 + e^{i\varphi})^p = 2^p[\cos(\varphi/2)]^p e^{ip\varphi/2}.$$

The tangent vector is given by

$$p(1 + e^{i\varphi})^{p-1} ie^{i\varphi} = p2^{p-1}[\cos(\varphi/2)]^{p-1} ie^{i(p+1)\varphi/2}.$$

If η is equal to the argument of the tangent vector, then

$$\eta = \eta(\varphi) = \varphi(p+1)/2 + \pi/2,$$

$\eta(\varphi)$ is strictly increasing on the interval $(-\pi, \pi)$ and C_p is locally convex there. Since

$$|(1 + e^{i\varphi})^p| = 2^p[\cos(\varphi/2)]^p,$$

starting at 2^p ($\varphi = 0$), the two symmetric branches spiral in different directions around and down to the origin with decreasing modulus. With this parametrization the points where $|\varphi| = 2\pi/(p+1)$ are the turning points introduced in Section 1.

Claim 3.1. Fix φ_0 with $2\pi/(p+1) < |\varphi_0| < \min(\pi, 2\pi/p)$. Then the tangent line at $(1 + e^{i\varphi_0})^p$ intersects C_p at exactly two points, excluding the point of tangency. If $\varphi_0 > 0$, then both of these points lie on the lower branch and, if $\varphi_0 < 0$, then both lie on the upper branch.

We sketch the proof and, due to symmetry, we only consider $\varphi_0 > 0$. The argument of the tangent vector at $(1 + e^{i\varphi_0})^p$ satisfies $3\pi/2 < \eta(\varphi_0) < \min(\pi + p\pi/2, 3\pi/2 + \pi/p) < 2\pi$. First fix $1 < p \le 2$. Then $2\pi/(p+1) < \varphi_0 < \pi$, the curve C_p is a simple closed curve and, geometrically, is essentially the same as the cardioid C_2. The restricted location of the point $(1 + e^{i\varphi_0})^p$ on the upper branch between the turning point and the origin, the symmetry, the local convexity of C_p and the inclination of the tangent line, as determined by the argument of the tangent vector, yield the result, just as with C_2.

Now fix $p > 2$. The point where $|\varphi| = 2\pi/p$ is common to both branches and is the point where both cross the negative real axis for the first time. Consequently

$$C_p^* = \{(1 + e^{i\varphi})^p : -2\pi/p \le \varphi \le 2\pi/p\}$$

is a simple closed curve and is, geometrically, essentially the same as the cardioid C_2. As in the case $1 < p \le 2$, the location of the point $(1 + e^{i\varphi_0})^p$ on C_p^* and the geometric properties of C_p^* yield that the tangent line, excluding the point of tangency, intersects C_p^* in exactly two points, both on the lower branch.

The local convexity and symmetry of C_p show that the tangent line can possibly have additional intersection points on C_p only after both branches, starting at 2^p, have made a complete revolution about the origin, that is, when $|\varphi| = 4\pi/p$. If $p < 4$, both branches terminate at the origin before there is a complete revolution about the origin, so there are exactly two intersection points as claimed. Thus we can assume $p \ge 4$ and we consider points $(1 + e^{i\varphi})^p$ where $4\pi/p \le |\varphi| \le \pi$. The tangent line at $(1 + e^{i\varphi_0})^p$ can be described by $(1 + e^{i\varphi_0})^p(1 + Rie^{i\varphi_0/2})$, $-\infty < R < \infty$. Then a straightforward calculation shows that $2^p[\cos(\varphi_0/2)]^{p+1}$ is the minimum modulus of points on the tangent line at $(1 + e^{i\varphi_0})^p$. Then, for $4\pi/p \le |\varphi| \le \pi$,

$$\begin{aligned}
|(1 + e^{i\varphi})^p| = 2^p[\cos(\varphi/2)]^p &\le 2^p[\cos(2\pi/p)]^p = 2^p[2(\cos(\pi/p))^2 - 1]^p \\
&\le 2^p[\cos(\pi/p)]^{2p} \\
&< 2^p[\cos(\pi/p)]^{p+1} \\
&< 2^p[\cos(\varphi_0/2)]^{p+1}.
\end{aligned}$$

Hence the tangent line avoids this part of C_p and the claim follows for all $p > 1$.

Our eventual goal is to apply Theorem 2.5 and, to this end, we need to construct appropriate functions H and H_n.

Lemma 3.2. *For a given* $p > 1$ *and* $q > 0$*, there exist* $x_0 = e^{i\theta_0}$*, where* $0 < |\theta_0| < \pi(p-1)/(p+1)$*, and* $C \in \mathbb{C}$ *such that*

$$H(x, y) = \begin{cases} -\overline{x}_0(q - px)y & \text{on } \Gamma \times \Gamma \setminus \{1\} \\ -\overline{x}_0(q - px) - C(1 - x)^p & \text{on } \Gamma \times \{1\} \end{cases}$$

satisfies:

(1) $\sup \operatorname{Re} H(x, y) < \operatorname{Re} H(x_0, 1)$, $y \ne 1$*; and*
(2) $\operatorname{Re} H(x, 1) < \operatorname{Re} H(x_0, 1)$, $x \ne x_0$.

Proof. First observe that, if we can find x_0 and C that satisfy the lemma, then by symmetry $\overline{x}_0$ and $\overline{C}$ will also satisfy the lemma with an appropriately modified H. Thus, it suffices to assume that $\theta_0 < 0$. Let

$$\theta^* = -\pi(p-1)/(p+1) \quad \text{and} \quad \theta_0 = \theta^* + \epsilon, \epsilon > 0.$$

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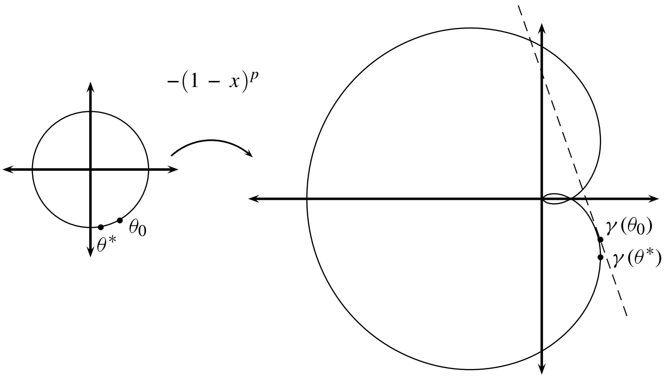

FIGURE 1. Illustration of Lemma 3.2 for the curve $\gamma = -C_p$.

Reflect the curve C_p through the origin and parametrize $\gamma = -C_p = \{-(1-x)^p : |x| = 1\}$ with $x = e^{i\theta}$ and

$$\gamma(\theta) = -(1-e^{i\theta})^p = e^{-i\pi}(1+e^{i(\theta+\pi)})^p, \quad -\pi < \theta \le \pi.$$

Then $\gamma(\theta^*)$ is a reflected turning point in the lower half plane. With this parametrization of γ, the tangent line at $\gamma(\theta_0)$ satisfies Claim 3.1 for small $\epsilon > 0$ and the two intersection points lie on the part of γ in the upper half plane (see Figure 1).

Recall that $\eta(\varphi)$, the argument of the tangent vector for the curve $(1+e^{i\varphi})^p$, is given by $\eta(\varphi) = \varphi(p+1)/2 + \pi/2$ so that the argument of the tangent vector at the point $\gamma(\theta_0)$ is given by $\epsilon(p+1)/2 + \pi/2$. Let $\alpha = \alpha(\theta_0) = -\epsilon(p+1)/2$. Then the rotated curve $\gamma_r(\theta) = e^{i\alpha}\gamma(\theta)$ has a vertical tangent at $\gamma_r(\theta_0)$. Define $T(\theta) = T(\theta_0, \theta) = \mathrm{Re}\,\gamma_r(\theta)$. By the choice of α, the local convexity of γ and Claim 3.1, $T(\theta)$ achieves a local maximum at θ_0, $T(\theta)$ achieves a unique maximum for some value of θ, say $\theta = \beta$, and there exist two distinguished angles $\varphi_1 = \varphi_1(\theta_0)$, $\varphi_2 = \varphi_2(\theta_0)$, $\varphi_1, \varphi_2 > 0$, such that $T(\beta - \varphi_1) = T(\beta + \varphi_2) = T(\theta_0)$. Also $T(\theta) < T(\theta_0)$ if $\theta \notin [\beta - \varphi_1, \beta + \varphi_2]$ (see Figure 2).

Define $g(\theta) = g(\theta_0, \theta) = \mathrm{Re}\,[-\overline{x}_0(q - px)]$ so that $g(\theta)$ has a unique maximum at θ_0. Another straightforward calculation yields that

$$T(\theta_0, \theta_0) = 2^p\{\cos[\epsilon/2 + \pi/(p+1)]\}^{p+1}.$$

Then, as ϵ decreases to 0:

(i) $1 + \cos(\theta_0)$ decreases to $1 + \cos(\theta^*)$;
(ii) $\alpha = \alpha(\theta_0)$ increases to 0;
(iii) $T(\theta_0) = T(\theta_0, \theta_0)$ increases to $T(\theta^*, \theta^*) = T(\theta^*, -\theta^*)$;
(iv) $g(\theta_0) = g(\theta_0, \theta_0)$ converges to $g(\theta^*, \theta^*) > g(\theta^*, -\theta^*)$;
(v) β converges to $-\theta^*$;
(vi) the interval $[\beta - \varphi_1, \beta + \varphi_2]$ decreases to the point $\{-\theta^*\}$.

Let $R^* = q(1 + \cos(\theta^*))/T(\theta^*, \theta^*)$ and fix R with $R^* < R < 2R^*$. Let $\sigma = g(\theta^*, \theta^*) - g(\theta^*, -\theta^*) > 0$. Then we can chose ϵ sufficiently close to 0 so that:

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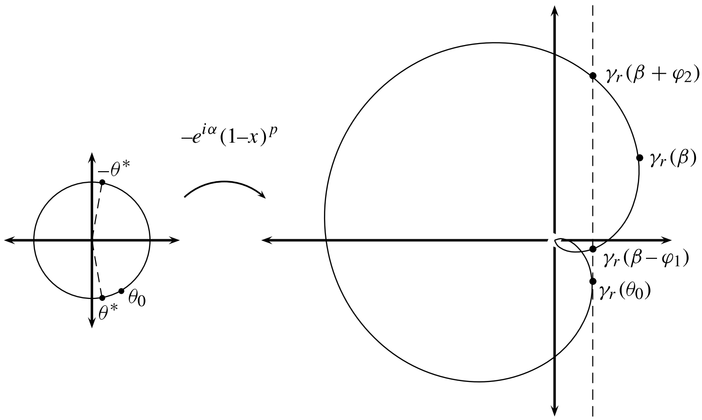

FIGURE 2. Illustration of the curve $\gamma_r(\theta)$.

(a) $g(\theta_0) - \sup\{g(\theta) : \theta \in [\beta - \varphi_1, \beta + \varphi_2]\} > \sigma/2$ by (iv)–(vi);
(b) $|T(\beta) - T(\theta_0)| < \sigma/(8R^*)$ by (iii) and (v);
(c) $|T(\theta) - T(\beta)| < \sigma/(8R^*)$, $\theta \in [\beta - \varphi_1, \beta + \varphi_2]$ by (v) and (vi);
(d) $R > q(1 + \cos(\theta_0))/T(\theta_0, \theta_0) > R^*$ by (i) and (iii).

So fix θ_0 satisfying (a)–(d), fix $\alpha = \alpha(\theta_0)$, let $C = \mathrm{R}e^{i\alpha}$ and let $f(\theta) = RT(\theta_0, \theta) = RT(\theta)$. Then $\mathrm{Re}\, H(x, 1) = g(\theta) + f(\theta)$. We claim that (1) and (2) in the lemma obtain with this choice of θ_0 and C.

Since $|\overline{x}_0(q - px)y| \le q + p$, to achieve (1) we want $q + p < f(\theta_0) + g(\theta_0) = R \cdot T(\theta_0) - q\cos(\theta_0) + p$, which, since $T(\theta_0, \theta_0) > 0$, is equivalent to $R > q(1 + \cos(\theta_0))/T(\theta_0, \theta_0)$, which is (d).

Next we want $f(\theta_0) + g(\theta_0) > f(\theta) + g(\theta)$ for $-\pi < \theta \le \pi$ and $\theta \ne \theta_0$. Since $g(\theta)$ peaks at θ_0, for $\theta \notin [\beta - \varphi_1, \beta + \varphi_2]$ we have $T(\theta) < T(\theta_0)$ and, hence, $f(\theta_0) + g(\theta_0) > f(\theta) + g(\theta)$. If $\theta \in [\beta - \varphi_1, \beta + \varphi_2]$, then by (a) we have $g(\theta_0) > \sigma/2 + g(\theta)$. Also by (b) and (c) we have $|T(\theta) - T(\theta_0)| < \sigma/(4R^*)$ so that $|f(\theta) - f(\theta_0)| < \sigma R/(4R^*) < \sigma/2$. Hence

$$f(\theta_0) + g(\theta_0) > (f(\theta) - \sigma/2) + (\sigma/2 + g(\theta)) = f(\theta) + g(\theta),$$

which establishes Lemma 3.2.

LEMMA 3.3. *Let $\{f_n\}$ be a sequence of continuous functions that converges to $f(x) = (1 - x)^p$ uniformly on Γ. Let $(1 + y)^n/2^n = \sum_{i=0}^n b_{n_i} y^i$ on Γ and $g_n(x, y) = y^n \sum_{i=0}^n b_{n_i} f_{n+i}(x) y^i$ on $\mathbb{T}$. Then $g_n(x, y)$ converges to 0 uniformly on compact subsets of $\Gamma \times \Gamma \setminus \{1\}$ and uniformly to $(1 - x)^p$ on $\Gamma \times \{1\}$.*

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Proof. We have $b_{n_i} > 0$ and $\sum_{i=0}^{n} b_{n_i} = 1$. Rewrite

$$\begin{aligned} g_n(x, y) &= y^n \sum_{i=0}^{n} b_{n_i} y^i f_n(x) + y^n \sum_{i=0}^{n} b_{n_i} y^i (f_{n+i}(x) - f_n(x)) \\ &= y^n \Big(\frac{1+y}{2}\Big)^n f_n(x) + y^n \sum_{i=0}^{n} b_{n_i} y^i (f_{n+i}(x) - f_n(x)). \end{aligned} \tag{3.1}$$

Let $\epsilon > 0$. Since $\{f_n\}$ is uniformly Cauchy on Γ, $|f_{n+i} - f_n| < \epsilon$ on Γ for large n. Therefore

$$\Big| y^n \sum_{i=o}^{n} b_{n_i} y^i (f_{n+i}(x) - f_n(x)) \Big| < \sum_{i=o}^{n} b_{n_i} \epsilon = \epsilon \quad \text{on } \mathbb{T} \text{ for large } n.$$

Thus the second term in (3.1) converges to 0 uniformly on $\mathbb{T}$. By hypothesis, $f_n(x)$ converges to $f(x) = (1-x)^p$ uniformly on $\Gamma \times \{1\}$ and, since $(1+y)^n/2^n$ converges to 0 uniformly on compact subsets of $\Gamma \setminus \{1\}$, the first term converges to 0 uniformly on compact subsets of $\Gamma \times \Gamma \setminus \{1\}$. Hence $g_n(x, y)$ converges as stated in the lemma.

Lemma 3.4. *For $p > 1$ and $q > 0$, $K_n(x, 1)/B_n(q)$ converges to $f(x) = (1-x)^p$ uniformly on Γ.*

Proof. We have

$$f(x) = (1-x)^p = \sum_{n=0}^{\infty} A_n(p) x^n,$$

where

$$A_0(p) = 1 \quad \text{and} \quad A_n(p) = (-1)^n \frac{p(p-1)\cdots(p-n+1)}{n!}, \quad n \geq 1.$$

Let $p = s + r$, $0 \leq r < 1$, $s = 1, 2, 3, \ldots$. Let $M = p(p-1)\cdots(p-s+1) = (s+r)(s+r-1)\cdots(1+r)$. Then, for $n = 1, 2, 3, \ldots, s+1$,

$$|A_n(p)| = \frac{(s+r)(s+r-1)\cdots(s+r-n+1)}{n!}.$$

For $n \geq s+2$,

$$\begin{aligned} |A_n(p)| &= \frac{M \cdot r(1-r)(2-r)\cdots(n-(s+1)-r)}{n!} \\ &= M \cdot r \cdot \Big(\frac{1-r}{1}\Big) \cdot \Big(\frac{2-r}{2}\Big) \cdots \Big(\frac{n-(s+1)-r}{n-(s+1)}\Big) \cdot \frac{1}{n-s} \cdots \frac{1}{n}. \end{aligned}$$

Since $(s+r) - j \geq 1$ for $j = 0, 1, 2, \ldots, s-1$, it follows that, for all n, $|A_n(p)| \leq M$ and, for $n \geq 2$, $|A_n(p)| \leq M/(n-1)n$.

Let $f(x) = T_N(x) + R_N(x)$ where

$$T_N(x) = \sum_{i=0}^{N} A_i(p) x^i \quad \text{and} \quad R_N(x) = \sum_{i=N+1}^{\infty} A_i(p) x^i.$$

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For $n > N$, consider

$$\left|\frac{K_n(x,1)}{B_n(q)} - f(x)\right| = \left|\sum_{i=0}^{n} A_i(p)x^i \frac{B_{n-i}(q)}{B_n(q)} - T_N(x) - R_N(x)\right|$$
$$\leq \left|\sum_{i=0}^{N} A_i(p)x^i\left(\frac{B_{n-i}(q)}{B_n(q)} - 1\right)\right| + \left|\sum_{i=N+1}^{n} A_i(p)x^i \frac{B_{n-i}(q)}{B_n(q)}\right| + |R_N(x)|.$$

Let $\epsilon > 0$. Choose N_1 such that $\sum_{i=N_1+1}^{\infty} M/(i^2 - i) < \epsilon/3$. Then $|R_N(x)| < \epsilon/3$ if $N \geq N_1$.

Let $n = i + k$ so that $B_{n-i}(q)/B_n(q) = B_k(q)/B_{k+i}(q)$. Since $B_{k+1}(q) = B_k(q)(k+q)/(k+1)$ and $B_0(q) = 1$, $B_{n-i}(q)/B_n(q) \leq 1$ for $0 \leq i \leq n$, $q \geq 1$.

If $0 < q < 1$, then

$$\frac{B_k(q)}{B_{k+i}(q)} = \frac{(k+1)(k+2)\cdots(k+i)}{(k+q)(k+q+1)\cdots(k+q+i-1)} < \left(\frac{1}{q}\right)\left(\frac{2}{q+1}\right)\cdots\left(\frac{i}{q+i-1}\right).$$

Recall that the gamma function satisfies

$$\Gamma(q) = \lim_{i\to\infty} \frac{(i-1)!(i-1)^q}{q(q+1)\cdots(q+i-1)}.$$

Choose N_2 so that, if $i > N_2$, then

$$\frac{(i-1)!(i-1)^q}{q(q+1)\cdots(q+i-1)} < 2\Gamma(q)$$

and hence

$$\frac{B_{n-i}(q)}{B_n(q)} < \frac{i!}{q(q+1)\cdots(q+i-1)} < \frac{2\Gamma(q)\cdot i}{(i-1)^q}.$$

Choose N_3 so that, if $m > N_3$, then

$$\sum_{i=m}^{\infty} \frac{1}{(i-1)^{1+q}} < \frac{\epsilon}{6\Gamma(q)M}.$$

Fix $N = \max\{N_1, N_2, N_3\}$. Then $|R_N(x)| < \epsilon/3$. For $n > N$,

$$\text{if } q \geq 1, \quad \left|\sum_{i=N+1}^{n} A_i(p)x^i \frac{B_{n-i}(q)}{B_n(q)}\right| \leq \sum_{i=N+1}^{n} |A_i(p)| \leq \sum_{i=N+1}^{\infty} \frac{M}{(i-1)i} < \frac{\epsilon}{3}$$

and,

$$\text{if } 0 < q < 1, \quad \left|\sum_{i=N+1}^{n} A_i(p)x^i \frac{B_{n-i}(q)}{B_n(q)}\right| \leq \sum_{i=N+1}^{n} \frac{M}{(i-1)i}\cdot\frac{2\Gamma(q)i}{(i-1)^q}$$
$$= 2M\Gamma(q)\sum_{i=N+1}^{n} \frac{1}{(i-1)^{1+q}} < \frac{\epsilon}{3}.$$

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Now observe that, for each fixed i, $B_k(q)/B_{k+i}(q)$ converges to 1 as $k \to \infty$. Hence there exists $N_0 \geq N$ such that, if $n \geq N_0$, then $|1 - B_{n-i}(q)/B_n(q)| < \epsilon/(3NM)$ for each i, $0 \leq i \leq N$. Consequently

$$\left|\sum_{i=0}^{N} A_i(p)x^i\left(1 - \frac{B_{n-i}(q)}{B_n(q)}\right)\right| < NM\frac{\epsilon}{3NM} = \frac{\epsilon}{3} \quad \text{if } n \geq N_0.$$

Thus, given $\epsilon > 0$, there exists N_0 such that

$$\left|\frac{K_n(x, 1)}{B_n(q)} - f(x)\right| < \epsilon \quad \text{on } \Gamma, \text{ if } n \geq N_0.$$

That is, $K_n(x, 1)/B_n(q)$ converges to $f(x)$ uniformly on Γ.

Proof (Main theorem). In Theorem 2.5 let $X = \mathbb{T}$, $E = \Gamma \times \{1\}$ and $B = \overline{Sp}\{K_n : n = 0, 1, 2, \ldots, \}$. Replace $x_0 = e^{i\theta_0}$ with $x_1 = e^{i\theta_1}$ and let x_1, C and H be as in Lemma 3.2. Let $f_n(x) = K_n(x, 1)/B_n(q)$ in Lemma 3.3 and choose a relatively open neighborhood W of $(x_1, 1)$ in $\Gamma \times \{1\}$. Finally, let

$$\begin{aligned} H_n(x, y) &= -\bar{x}_1 K_1(x, y) - C\sum_{i=0}^{n} \frac{b_{n_i}}{B_{n+i}(q)} K_{n+i}(x, y) \\ &= -\bar{x}_1(q - px)y - Cy^n \sum_{i=0}^{n} b_{n_i} f_{n+i}(x) y^i, \end{aligned}$$

and normalize H and H_n with $\widetilde{H} = H/\mathrm{Re}\, H(x_1, 1)$ and $\widetilde{H}_n = H_n/\mathrm{Re}\,{}_n(x_1, 1)$. We verify that the hypotheses of Theorem 2.5 are satisfied with the functions $\widetilde{H}_n$ and $\widetilde{H}$:

$$\begin{aligned} \left|\sum_{i=0}^{n} b_{n_i} f_{n+i}(x) y^i\right| &\leq \left|\sum_{i=0}^{n} b_{n_i}(f_{n+i}(x) - f_n(x)) y^i\right| + \left|\sum_{i=0}^{n} b_{n_i} y^i f_n(x)\right| \\ &= \left|\sum_{i=0}^{n} b_{n_i}(f_{n+i}(x) - f_n(x)) y^i\right| + \left|\left(\frac{1+y}{2}\right)^n f_n(x)\right|. \end{aligned}$$

By Lemma 3.4, f_n converges to $(1 - x)^p$ uniformly on Γ. It follows that there exists $M \geq 1$ such that $\|\widetilde{H}_n\| < M$ for all n. From Lemma 3.3, $\widetilde{H}_n$ converges to $\widetilde{H}$ uniformly on $\Gamma \times \{1\}$. The normalization yields $\mathrm{Re}\, \widetilde{H}_n(x_1, 1) = \mathrm{Re}\, \widetilde{H}(x_1, 1) = 1$. From Lemma 3.2, $\mathrm{Re}\, \widetilde{H}(x, 1) < \mathrm{Re}\, \widetilde{H}(x_1, 1)$, $x \neq x_1$, and there exists c such that $\sup \mathrm{Re}\, \widetilde{H}(x, y) < c < 1 = \mathrm{Re}\, \widetilde{H}(x_1, 1)$ on $\Gamma \times \Gamma \setminus \{1\}$. From Lemma 3.3, $\widetilde{H}_n$ converges to $\widetilde{H}$ uniformly on compact subsets of $\Gamma \times \Gamma \setminus \{1\}$. Thus all of the hypotheses of Theorem 2.5 are fulfilled and there exists $G \in B$ such that

$$K = \left\{(x, y) \in \mathbb{T} : \mathrm{Re}\, G(x, y) = \max_{(x', y') \in \mathbb{T}} \mathrm{Re}\, G(x', y')\right\} \subseteq W.$$

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If one considers the generalized functional L_G corresponding to G and the set of generalized support points

$$\Sigma = \Big\{k_{x,y} : \mathrm{Re}\, L_G(k_{x,y}) = \max_{k_{x',y'} \in F} \mathrm{Re}\, L_G(k_{x',y'})\Big\},$$

then $f_\mu \in \Sigma$ if and only if the closed support of μ is contained in K. According to our Remark 2.1, there exist extreme points in Σ and, hence, pairs $(x_0, 1) \in W$ such that $k_{x_0,1}$ is an extreme point of F. Moreover, since W is an arbitrary relatively open neighborhood of $(x_1, 1)$, $x_1 = e^{i\theta_1}$, and $|\theta_1| < \pi(p-1)/(p+1)$, there exist extreme points $k_{x_0,1}$, $x_0 = e^{i\theta_0}$, with x_0 arbitrarily close to x_1 and satisfying $|\theta_0| < \pi(p-1)/(p+1)$ as claimed. □

Corollary 3.5. *For $p > 1$ and $q > 0$ there exist $\delta > 0$ and a dense subset J_e of $J = \{x = e^{i\theta} : \pi(p-1)/(p+1) - \delta < |\theta| < \pi(p-1)/(p+1)\}$ such that $k_{x,1}$ is an extreme point of F for each $x \in J_e$.*

Proof. By the symmetry of the curve $(1-x)^p$, the arguments in the proof of Lemma 3.2 demonstrate that:

(i) if $x_0 = e^{i\theta_0}$ satisfies Lemma 3.2, then so does $\bar{x}_0 = e^{-i\theta_0}$;
(ii) if $(1-x_0)^p$ is sufficiently close to the turning point $(1-x^*)^p$, then x_0 and $\bar{x}_0$ both satisfy Lemma 3.2.

Therefore we can choose δ so that, if $x_0 \in J$ as defined above, then x_0 satisfies Lemma 3.2. The proof of the main theorem then shows that any relatively open neighborhood of a point in J contains a point x such that $k_{x,1}$ is an extreme point of F. Take $J_e = \{x \in J : k_{x,1}$ is an extreme point of $F\}$. □

Remark 3.6. According to Remarks 2.1 and 2.6, if the functions H_n in Theorem 2.5 actually peak on E, then there is no need to consider relatively open neighborhoods and one could conclude that there is an interval beyond each turning point that yields extreme points of F. This is actually the case when p is an integer, $p = m \geq 2$.

Theorem 3.7. *Fix $q > 0$ and $p = m$ $(m = 2, 3, 4, \dots)$. Then there exist functions $H_n \in \overline{Sp}\{K_n : n = 0, 1, 2, \dots, \}$ such that:*

(i) $H_n(x, 1) = H(x, 1) = -\bar{x}_0(q - mx) - C(1-x)^m$;
(ii) $H_n(x, y)$ converges to $H(x, y) = -\bar{x}_0(q - mx)y$ uniformly on compact subsets of $\Gamma \times \Gamma \setminus \{1\}$.

(See the function $H(x, y)$ in Lemma 3.2.)

We will need the following lemma in the proof of Theorem 3.7.

Lemma 3.8. *Fix $q > 0$ and $p = m$. Let $n^* = \min(n, m)$. Then*

$$K_n(x, 1) = \sum_{l=0}^{n^*} \binom{m}{l} B_{n-l}(q - m + l)(1-x)^l.$$

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PROOF. We have

$$\frac{(1-xz)^m}{(1-z)^q} = \sum_{n=1}^{\infty} K_n(x, 1)z^n.$$

Let $t = 1 - x$, $x = 1 - t$.

$$(1-xz)^m = (1-(1-t)z)^m = (1-z+tz)^m = \sum_{l=0}^{m} \binom{m}{l} t^l z^l (1-z)^{m-l}.$$

So

$$\begin{aligned}\frac{(1-xz)^m}{(1-z)^q} &= \sum_{l=0}^{m} \binom{m}{l} t^l z^l (1-z)^{m-l-q} \\ &= \sum_{l=0}^{m} \binom{m}{l} t^l z^l \sum_{k=0}^{\infty} A_k(m-l-q) z^k \\ &= \sum_{l=0}^{m} \binom{m}{l} t^l \sum_{k=0}^{\infty} A_k(m-l-q) z^{k+l}.\end{aligned}$$

When $m - l - q$ is a nonnegative integer, this series is a finite sum.
Let $n = k + l$, $k = n - l$. Then

$$\begin{aligned}\frac{(1-xz)^m}{(1-z)^q} &= \sum_{l=0}^{m} \binom{m}{l} t^l \sum_{n=l}^{\infty} A_{n-l}(m-l-q) z^n \\ &= \sum_{n=0}^{\infty} \Big[\sum_{l=0}^{n^*} \binom{m}{l} A_{n-l}(m-l-q) t^l \Big] z^n.\end{aligned}$$

Since $A_{n-l}(m-l-q) = B_{n-l}(q-m+l)$,

$$K_n(x, 1) = \sum_{l=0}^{n^*} \binom{m}{l} B_{n-l}(q-m+l)(1-x)^l. \qquad \square$$

PROOF OF THEOREM 3.7. For fixed $n \geq m$, let

$$H_n(x, y) = \sum_{j=0}^{m-1} D_j(n) K_j(x, y) - C \sum_{i=0}^{n} \frac{b_{n_i}}{B_{n+i-m}(q)} K_{n+i}(x, y),$$

where b_{n_i} is as in Lemma 3.3 and where $D_0(n), D_1(n), \ldots, D_{m-1}(n)$ are to be determined. Since $K_j(x, y) = K_j(x, 1)y^j$ for all $j = 0, 1, 2, \ldots$, by Lemma 3.8,

$$\begin{aligned}H_n(x, y) = &\sum_{j=0}^{m-1} D_j(n) \Big[\sum_{l=0}^{j} \binom{m}{l} B_{j-l}(q-m+l)(1-x)^l \Big] y^j \\ &- C \sum_{i=0}^{n} \frac{b_{n_i}}{B_{n+i-m}(q)} \Big[\sum_{l=0}^{m} \binom{m}{l} B_{n+i-l}(q-m+l)(1-x)^l \Big] y^{n+i}.\end{aligned}$$

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Let

$$\alpha_{j,l} = \binom{m}{l} B_{j-l}(q-m+l), \qquad j = 0, 1, 2, \ldots, m-1, l = 0, 1, 2, \ldots, j,$$

and

$$\beta_{n,l}(y) = \binom{m}{l} \sum_{i=0}^{n} \frac{b_{n_i} B_{n+i-l}(q-m+l)}{B_{n+i-m}(q)} y^i.$$

Then

$$\begin{aligned} H_n(x, y) = \sum_{j=0}^{m-1} D_j(n) \Big[\sum_{l=0}^{j} \alpha_{j,l}(1-x)^l \Big] y^j \\ - Cy^n \sum_{l=0}^{m-1} \beta_{n,l}(y)(1-x)^l - Cy^n(1-x)^m \Big(\frac{1+y}{2}\Big)^n \end{aligned}$$

and

$$H_n(x, 1) = \sum_{j=0}^{m-1} D_j(n) \Big[\sum_{l=0}^{j} \alpha_{j,l}(1-x)^l \Big] - C \sum_{l=0}^{m-1} \beta_{n,l}(1)(1-x)^l - C(1-x)^m.$$

To achieve (i) in Theorem 3.7, we want to choose $D_0(n), \ldots, D_{m-1}(n)$ so that

$$H_n(x, 1) = -\overline{x}_0(q-m) - \overline{x}_0 m(1-x) - C(1-x)^m.$$

This generates the following system of equations to be satisfied:

$$\begin{aligned} \sum_{j=0}^{m-1} D_j(n)\alpha_{j,0} &= -\overline{x}_0(q-m) + C\beta_{n,0}(1). \\ \sum_{j=1}^{m-1} D_j(n)\alpha_{j,1} &= -\overline{x}_0 m + C\beta_{n,1}(1). \\ \sum_{j=l}^{m-1} D_j(n)\alpha_{j,l} &= C\beta_{n,l}(1), \qquad l = 2, 3, \ldots, m-1. \end{aligned}$$

Note that the coefficient matrix associated with the variables $D_0(n), D_1(n), \ldots, D_{m-1}$ (n) is a triangular matrix with entries down the main diagonal given by

$$\alpha_{l,l} = \binom{m}{l} B_0(q-m+l) = \binom{m}{l}, \qquad l = 0, 1, \ldots, m-1.$$

Hence this system of equations has unique solutions $D_j(n)$, $j = 0, 1, 2, \ldots, m-1$. With these choices in the definition of H_n, we obtain (i) in Theorem 3.7. It remains to show that $H_n(x, y)$ converges to $-\overline{x}_0(q-mx)y$ uniformly on compact subsets

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of $\Gamma \times \Gamma \setminus \{1\}$. We first note that $(1+y)^n/2^n$ converges to 0 uniformly on compact subsets of $\Gamma \times \{1\}$. Also,

$$\begin{aligned}
\beta_{n,l}(y) &= \binom{m}{l}\Big[\sum_{i=0}^{n} \frac{b_{n_i} B_{n+i-l}(q-m+l)}{B_{n+i-m}(q)} y^i\Big] \\
&= \binom{m}{l}\Big[\sum_{i=0}^{n} \frac{b_{n_i}(q-m+l)(q-m+l+1)\cdots(q-1)}{(n+i-l)(n+i-l-1)\cdots(n+i-m+1)} y^i\Big] \\
&= \binom{m}{l} r(m,q,l)\Big[\sum_{i=0}^{n} \frac{b_{n_i}}{(n+i-l)(n+i-l-1)\cdots(n+i-m+1)} y^i\Big],
\end{aligned}$$

where $r(m,q,l) = (q-m+l)(q-m+l+1)\cdots(q-1)$. Then

$$\begin{aligned}
|\beta_{n,l}(y)| &\le \binom{m}{l} |r(m,q,l)| \sum_{i=0}^{n} \frac{b_{n_i}}{(n+i-l)(n+i-l-1)\cdots(n+i-m+1)} \\
&\le \binom{m}{l} |r(m,q,l)| \frac{1}{n-m+1} \sum_{i=0}^{n} b_{n_i} \\
&= \frac{\binom{m}{l}|r(m,q,l)|}{n-m+1}.
\end{aligned}$$

Therefore $\beta_{n,l}(y)$ converges to 0 uniformly on $|y| = 1$ for $l = 0, 1, 2, \ldots, m-1$. If we let $n \to \infty$ in our system of equations, then $D_j = \lim_{n\to\infty} D_j(n)$ exists for each $j = 0, 1, 2, \ldots, m-1$ and

$$\begin{aligned}
\sum_{j=0}^{m-1} D_j \alpha_{j,0} &= -\bar{x}_0(q-m), \\
\sum_{j=1}^{m-1} D_j \alpha_{j,1} &= -\bar{x}_0 m
\end{aligned}$$

and

$$\sum_{j=l}^{m-1} D_j \alpha_{j,l} = 0, \qquad l = 2, 3, \ldots, m-1.$$

Since

$$\begin{aligned}
\alpha_{1,1} &= \binom{m}{1} B_0(q-m+1) = m, \\
\alpha_{0,0} &= \binom{m}{0} B_0(q-m) = 1, \\
\alpha_{1,0} &= \binom{m}{0} B_1(q-m) = q-m,
\end{aligned}$$

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solving these equations yields $D_j = 0$, $j = 2, 3, \ldots, m-1$, $D_1 = -\bar{x}_0$ and $D_0 = 0$. Therefore the first sum of $H_n(x, y)$ converges uniformly to $-\bar{x}_0(q - mx)y$ and the second sum converges to 0 uniformly on compact subsets of $\Gamma \times \Gamma \setminus \{1\}$. This establishes (ii) in Theorem 3.7. □

Corollary 3.9. *For fixed $q > 0$ and $p = m$ ($m = 2, 3, \ldots$), there exists $\delta > 0$ such that, if*

$$x \in J = \{e^{i\theta} : \pi(m-1)/(m+1) - \delta < |\theta| < \pi(m-1)/(m+1)\},$$

then $k_{x,1}$ is an extreme point of F.

Proof. Theorem 3.7, the proof of Lemma 3.2 and Remark 3.6 yield the result. □

It is almost certainly the case that, for any $p > 1$ and $q > 0$, there is an interval J, as in Corollary 3.9, with $k_{x,1}$ extreme when $x \in J$. The success of the argument when $p = m$ ($m = 2, 3, \ldots$) depends on Lemma 3.2 and the ability to 'capture' the term $(1 - x)^m$ in the functions H_n as shown in Theorem 3.7. When p is not an integer, one probably needs a version of Lemma 3.2 involving partial sums of $(1 - x)^p$.

Of course the ultimate goal is to precisely determine all extreme points for any p and q. In conclusion, we make the following conjecture.

Conjecture 3.10. *For fixed p and q, there exists an angle $\widetilde{\theta} = \widetilde{\theta}(p, q)$ separating extreme points from nonextreme points as follows.*

(a) *If $0 < p \le 1$, then $\widetilde{\theta} = 0$, $k_{x,1}$ is extreme if $x \ne 1$ and $k_{1,1}$ is not extreme.*
(b) *If $p > 1$, then $0 < \widetilde{\theta} < \pi(p-1)/(p+1)$ and, for $x = e^{i\theta}$, $k_{x,1}$ is extreme if $\widetilde{\theta} \le |\theta| \le \pi$ and $k_{x,1}$ is not extreme if $0 \le |\theta| < \widetilde{\theta}$.*

In the special case $p = 2$, $q = 1$, further analysis suggests that the critical angle $\widetilde{\theta}$ should yield an extreme point. According to (ii) in Section 1, for $0 < p \le 1$ the kernel function $k_{x,1}$ is extreme if $x \ne 1$, independent of q. It is also well known that $k_{1,1}$ is not extreme for any p and q, provided $q \ge p$. Thus part (a) of Conjecture 3.10 is correct if $0 < p \le 1$ and $q \ge p$. Recent work on identifying kernel functions that are not extreme shows that $k_{1,1}$ is not extreme if $q \ge 1$, independent of p. Moreover, if $q \ge 1$ and $p = m$ ($m = 2, 3, 4, \ldots$), not only is $k_{1,1}$ not extreme but $k_{x,1}$ is also not extreme for values of x near 1. Paired with Corollary 3.9 we thus have extreme points corresponding to values of x beyond the turning points and nonextreme points in intervals near the origin, providing strong evidence for the validity of part (b) of Conjecture 3.10, at least in the more tractable case of integer valued p. These nonextreme points results will be addressed in a planned companion paper.

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