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Invariant measures for the flow of a first order partial differential equation

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Abstract. We prove that the dynamical systems generated by first order partial differential equations are K-flows and chaotic in the sense of Auslander & Yorke.

0. *Introduction*

The purpose of this paper is to apply a Brownian motion to the problem of the existence of invariant measures for the dynamical systems generated by some first order partial differential equations.

§ 1 contains basic notation and definitions. In § 2 we define a flow describing the evolution of solutions of partial differential equations. In the last section we give a construction of an invariant measure for such a flow. This measure is positive on the open sets and non-trivial. The corresponding system is a K-flow and the flow is chaotic in the sense of Auslander & Yorke [**1**]. These theorems generalize the results of Lasota [**5**], [**6**], Brunovský and Komornik [**2**], [**3**] and Dawidowicz [**4**].

1. *Preliminaries*

Let X be a topological Hausdorff space and let $S_t: X \to X$, $t \in \mathbb{R}$, be a group of transformations. We call the group $\{S_t\}$ a *flow* if the mapping

$$\mathbb{R} \times X \ni (t, x) \mapsto S_t x \in X$$

is continuous in (t, x). By a *measure on* X we mean any probabilistic measure μ defined on the σ-algebra $\mathscr{B}(X)$ of Borel subsets of X. A measure μ is called *non-trivial* with respect to $\{S_t\}$, if $\mu(P) = 0$, where P denotes the set of all periodic points of $\{S_t\}$. Let X be a linear topological space, and let μ be a measure on X. We will say that μ is a *Gaussian measure* if each continuous linear functional on X has a Gaussian distribution.

Denote by $C^n(U, V)$ the space of n-times continuously differentiable functions defined on U with values in V, where U and V are non-empty intervals. Assume that $C^n(U, V)$ is equipped with the topology of uniform convergence (with derivatives of order $\leq n$) on compact subsets.

2. *Flow*

Consider the initial value problem

$$\begin{aligned} u_t + a(x)u_x &= b(x, u) \qquad \text{for } (t, x) \in D, \\ u(0, x) &= v(x) \qquad \text{for } x \in U_1. \end{aligned} \tag{E}$$

In this section and throughout the paper we shall assume that a and b are given functions satisfying

(1°) $a \in C^r(\bar{U}_1, \mathbb{R})$ for $r \geq 1$;

(2°) $a(x) \neq 0$ for $x \in U_1$ and $a(x) = 0$ for $x \in \partial U_1$;

(3°) there are constants K and L such that

$$|a(x)| \leq L + K|x| \qquad \text{for } x \in U_1;$$

(4°) $b \in C^r(U_1 \times \bar{U}_2, \mathbb{R})$,

(5°) $b(x, u) = 0 \quad$ for $(x, u) \in U_1 \times \partial U_2$,

(6°) there are continuous functions $M(x)$ and $N(x)$ such that

$$|b(x, u)| \leq M(x) + N(x)|u| \qquad \text{for } (x, u) \in U_1 \times U_2.$$

Here U_1 and U_2 are open intervals (bounded or not) of the real line, $D = \mathbb{R} \times U_1$, $\bar{U}_i$ denotes the closure of U_i in $\mathbb{R}$, and $\partial U_i = \bar{U}_i \setminus U_i$. These conditions will not be repeated in the statements of the theorems.

We denote by $\pi_t s$ the unique solution of the equation

$$x'(t) = a(x(t))$$

with the initial condition $x(0) = s$, $s \in U_1$. By $\psi(t, s, p)$ we denote the solution of the equation

$$y'(t) = b(\pi_t s, y(t))$$

with the initial condition $y(0) = p$, $p \in U_1$. From (1°)-(3°) it follows that π is defined for all $(t, s) \in \mathbb{R} \times U_1$ and possesses rth-order continuous partial derivatives. For given $x_0 \in U_1$ the function $t \mapsto \pi_t x_0$ is a C^r-diffeomorphism of $\mathbb{R}$ onto U_1. From (1°)-(6°) it follows that ψ is a C^r-mapping of $\mathbb{R} \times U_1 \times U_2$ into U_2. The functions π and ψ satisfy the following equalities

$$\pi_{s+t} x = \pi_s(\pi_t x), \tag{2.1}$$

$$\psi(s + t, x, y) = \psi(s, \pi_t x, \psi(t, x, y)) \tag{2.2}$$

for each $s, t \in \mathbb{R}$, $x \in U_1$ and $y \in U_2$. Let v be a continuously differentiable function from U_1 into U_2. Then there exists exactly one solution of (E), namely

$$u(t, x) = \psi(t, \pi_{-t} x, v(\pi_{-t} x)). \tag{2.3}$$

Let v be a continuous function from U_1 into U_2. Then $u(t, x)$ given by the formula (2.3) will be called a generalized solution of (E).

For an integer n, $0 \leq n \leq r$, we set $X = C^n(U_1, U_2)$ and $Y = C^n(\mathbb{R}, \mathbb{R})$. We shall consider solutions of equation (E) as the trajectories of the flow $\{S_t\}_{t \in \mathbb{R}}$ defined on X by the formula

$$(S_t v)(x) = u(t, x) = \psi(t, \pi_{-t} x, v(\pi_{-t} x)). \tag{2.4}$$

We now define a mapping $T: \mathbb{R} \times Y \to Y$ by

$$(T_t v)(s) = v(s - t).$$

It is clear that $\{T_t\}_{t \in \mathbb{R}}$ is a flow on Y.

THEOREM 1. *There exists a homeomorphism Q of X onto Y such that $Q \circ S_t = T_t \circ Q$ for each $t \in \mathbb{R}$.*

In order to prove this theorem we need the following lemma.

LEMMA 2.1. *Let V_1, V_2, W_1 and W_2 be open intervals of* $\mathbb{R}$. *Assume that $f: W_1 \times V_2 \to W_2$ and $g: W_1 \to V_1$ are C^n-maps, $n \geq 0$. Then the map $P: C^n(V_1, V_2) \to C^n(W_1, W_2)$ defined by $P(v)(x) = f(x, v(g(x)))$ is continuous.*

The proof of this lemma is simple, so it is omitted.

Proof of theorem 1. Given a point $x_0 \in U_1$. The map $t \to \pi_t x_0$ is a C^n-diffeomorphism of $\mathbb{R}$ onto U_1. Let $h: U_1 \to \mathbb{R}$ be the inverse of πx_0. Then h is a C^n-diffeomorphism of $\mathbb{R}$ onto U_1. Let p be a C^n-diffeomorphism of $\mathbb{R}$ onto U_2. Define the maps $Q: X \to Y$ and $N: Y \to X$ by

$$(Qv)(s) = p^{-1}(\psi(-s, \pi_s x_0, v(\pi_s x_0))) = p^{-1}((S_{-s}v)(x_0))$$

and

$$(Nv)(x) = \psi(h(x), x_0, p(v(h(x)))).$$

From lemma 2.1 the maps Q and N are continuous. Using (2.1) and (2.2) it is easy to verify that $N \circ Q = Q \circ N = I$. Thus Q is a homeomorphism of X onto Y. We verify that $Q \circ S_t = T_t \circ Q$,

$$(Q \circ S_t)(v)(x) = p^{-1}((S_{-x} \circ S_t v)(x_0)) = (Qv)(x-t) = (T_t \circ Q)(v)(x).$$

COROLLARY 1. *The set of all periodic points of $\{S_t\}$ is dense in X.*

Remark 1. From the definition of Q it follows that for every $s \in \mathbb{R}$ we have $(Qv)(s) = (Qw)(s)$ iff $v(\pi_s x_0) = w(\pi_s x_0)$.

Examples. (1) Let $b(x, u) = f(x) + g(x)u$ and $U_2 = \mathbb{R}$. Then $Qv = v_0 + Lv$, where L is a linear isomorphism from $C^n(U_1, \mathbb{R})$ onto $C^n(\mathbb{R}, \mathbb{R})$ and $v_0 \in C^n(\mathbb{R}, \mathbb{R})$.

(2) Let $a(x) = x$, $b(x, u) = \lambda u(1-u)$, $U_1 = \mathbb{R}^+$ and $U_2 = (0, 1)$. We take $x_0 = 1$ and $p(u) = e^u/(1+e^u)$. Then

$$(Qv)(s) = \ln v(e^s) - \ln [e^{\lambda s} - e^{\lambda s} v(e^s)].$$

3. *Measure on the space* $C^n(\mathbb{R}, \mathbb{R})$

Let w_t, $0 \leq t < \infty$ be a Brownian motion defined on a probability space (Ω, Σ, P). We may assume that the sample functions of w_t are continuous. Set $\xi_x^0 = e^{-x} w_{e^{2x}}$ for $x \in \mathbb{R}$. Then ξ_x^0 is a stationary Gaussian Markov process with mean value $E\xi_x^0 = 0$ and correlation function $E\xi_x^0 \xi_{x+h}^0 = e^{-|h|}$. The sample functions of ξ_x^0 are continuous. The process ξ_x^0 is not differentiable in the mean. From the law of the iterated logarithm it follows that

$$\lim_{|x| \to \infty} \frac{|\xi_x^0|}{|x|} = 0 \tag{3.1}$$

with probability 1. We assume that all sample functions of ξ_x^0 satisfy (3.1). Denote by H the closed linear subspace of $L^2(\Omega)$ spanned by all variables ξ_x^0, $x \in \mathbb{R}$. The joint distribution of the random functions $\zeta_1, \ldots, \zeta_n \in H$ is Gaussian. Denote by $\mathcal{F}_T$ the σ-algebra of events generated by the process w_t on the set T.

Starting from the random process ξ_x^0 we can define, by induction, some new process. Set for $k\geq 0$

$$\xi_x^{k+1}=\int_{-\infty}^{x} e^{s-x}\xi_s^k\,ds. \tag{3.2}$$

By (3.1) the integral (3.2) exists for every $\omega\in\Omega$ and consequently the sample functions of ξ_x^n possess nth-order continuous derivatives. The integral (3.2) exists also in the mean and consequently ξ_x^n is a stationary Gaussian process. Write $\eta_x=\xi_x^n$. Then for every $\omega\in\Omega$ we have

$$\sum_{k=0}^{n}\binom{n}{k}\eta_x^{(k)}=\xi_x^0. \tag{3.3}$$

The process ξ_x^n is not $(n+1)$-times differentiable in the mean.

LEMMA 3.1. *Let $\eta_x=\xi_x^n$, $n\geq 1$. The joint distribution of $(\eta_x, \eta'_x,\dots,\eta_x^{(n-1)}, \xi_x^0)$ is non-degenerate for every x.*

Proof. The distribution of $(\eta_x, \eta'_x,\dots,\xi_x^0)$ is Gaussian and it is independent of x. Assume that it is degenerate. Then for every x

$$a_0\eta_x+a_1\eta'_x+\cdots+a_{n-1}\eta_x^{(n-1)}+a_n\xi_x^0=0 \qquad \text{a.e.} \tag{3.4}$$

with at least one $a_k\neq 0$. Set $p=\max\{k\colon a_k\neq 0\}$. The process $a_0\eta_x+a_1\eta'_x+\cdots+a_n\xi_x^0$ is not $(n+1-p)$-times differentiable. This contradicts the equality (3.4).

Set $Y=C^n(\mathbb{R},\mathbb{R})$ and $\eta_x=\xi_x^n$, $n\geq 0$. Let $\mathcal{B}(Y)$ be the σ-algebra of Borel subsets of Y. The σ-algebra $\mathcal{B}(Y)$ is generated by the sets of the form $\{\varphi\in Y\colon(\varphi(x),\dots,\varphi^{(n)}(x))\in B\}$, where $x\in\mathbb{R}$ and B is a Borel subset of $\mathbb{R}^{n+1}$. Thus for every $A\in\mathcal{B}(Y)$ we have $\{\eta\in A\}\in\Sigma$. We obtain a probability measure μ on $\mathcal{B}(Y)$ by setting $\mu(A)=P(\eta\in A)$ for all $A\in\mathcal{B}(Y)$.

Now we shall investigate the properties of μ. The following known property of Wiener measure will be used in the next lemma:

(3.5) If $\varphi\colon[0,a]\to\mathbb{R}$ is a continuous function and $\varphi(0)=0$, then for every $\varepsilon>0$ we have

$$P(|w_t-\varphi(t)|<\varepsilon \quad \text{for } t\in[0,a])>0.$$

LEMMA 3.2. *$\mu(U)>0$ for each non-empty open set U.*

Proof. If $n=0$, then the proof follows immediately from (3.5). We assume $n\geq 1$. We may assume, without loss of generality, that U is of the form

$$U=\{\varphi\in C^n(\mathbb{R},\mathbb{R})\colon|\varphi^{(k)}(x)-\varphi_0^{(k)}(x)|<\varepsilon \quad \text{for } x\in[a,b] \text{ and } 0\leq k\leq n\},$$

where $\varphi_0\in C^n(\mathbb{R},\mathbb{R})$, $\varepsilon>0$ and $a, b\in\mathbb{R}$. According to the definition of μ we have $\mu(U)=P(A)$, where

$$A=\{\omega\in\Omega\colon|\eta_x^{(k)}-\varphi_0^{(k)}(x)|<\varepsilon \quad \text{for } x\in[a,b] \text{ and } 0\leq k\leq n\}.$$

Let ψ_0 be a continuous function satisfying the following equation

$$\sum_{k=0}^{n}\binom{n}{k}\varphi_0^{(k)}(x)=e^{-x}\psi_0(e^{2x}).$$

Set $\alpha=e^{2a}$ and $\beta=e^{2b}$. By (3.3) and from the continuous dependence on the initial values and the parameter it follows that there exists $\delta>0$ such that $A_\delta\cap B_\delta\subset A$, where the sets A_δ and B_δ are given by formulae

$$A_\delta = \{\omega \in \Omega : |\eta_a^{(k)} - \varphi_0^{(k)}(a)| < \delta \quad \text{for } 0 \le k \le n-1\},$$
$$B_\delta = \{\omega \in \Omega : |w_t - \psi_0(t)| < \delta \qquad \text{for } t \in [\alpha, \beta]\}.$$

It is easy to verify that the random variables $\eta_a^{(k)}$ are $\mathcal{F}_{\le\alpha}$-measurable and, therefore, $A_\delta \in \mathcal{F}_{\le\alpha}$. We also have $B_\delta \in \mathcal{F}_{\ge\alpha}$. By lemma 3.1 the joint distribution of $(\eta_a, \ldots, \eta_a^{(n-1)}, \xi_a^0)$ is Gaussian and non-degenerate. Consequently the joint distribution of $(\eta_a, \ldots, \eta_a^{(n-1)}, w_\alpha)$ is also non-degenerate. Thus we have $P(A|\mathcal{F}_{=\alpha}) > 0$ almost everywhere on B_δ. According to (3.5) we have $P(B_\delta) > 0$. From the Markov property of Brownian motion it follows that

$$P(A_\delta \cap B_\delta) = \int_{B_\delta} P(A_\delta | \mathcal{F}_{=\alpha})\, dP > 0,$$

which completes the proof.

LEMMA 3.3. *The measure μ is invariant under $\{T_t\}$.*

Proof. The invariance of μ follows directly from the stationarity of the process $(\eta_x, \eta'_x, \ldots, \eta_x^{(n)})$.

LEMMA 3.4. *The flow $\{T_t\}$ on $(Y, \mathcal{B}(Y), \mu)$ is a K-flow.*

Proof. Let $\mathcal{B}_0$ be the smallest σ-algebra containing the sets of the form

$$A = \{\varphi \in C^n(\mathbb{R}, \mathbb{R}) : (\varphi(x), \ldots, \varphi^{(n)}(x)) \in B\}, \tag{3.6}$$

where $x \le 0$ and B is a Borel subset of $\mathbb{R}^{n+1}$. The σ-algebra $T_t\mathcal{B}_0$ is the smallest σ-algebra containing the sets of the form (3.6) with $x \le t$. Thus $\mathcal{B}_0 \subset T_t\mathcal{B}_0$ for $t > 0$, and $\mathcal{B}(Y)$ is the smallest σ-algebra containing all the σ-algebras $T_t\mathcal{B}_0$ for $t \in \mathbb{R}$. It remains to verify that the σ-algebra $\bigcap_{t\in\mathbb{R}} T_t\mathcal{B}_0$ contains only sets of measure zero or one. Let $A \in \bigcap_t T_t\mathcal{B}_0$. Now we define E by $E = \{\omega \in \Omega : \eta_\cdot(\omega) \in A\}$. Then $\mu(A) = P(E)$ and $E \in \bigcap_t \mathcal{A}_{\le t}$, where $\mathcal{A}_{\le t}$ is the smallest σ-algebra generated by the random variables η_x, $x \le t$. Since $\mathcal{A}_{\le t} \subset \mathcal{F}_{\le e^{2t}}$, this implies $E \in \bigcap_{t>0} \mathcal{F}_{\le t}$. Thus, according to the Blumenthal's zero-or-one law, $P(E)$ is one or zero. This completes the proof.

LEMMA 3.5. *μ is a Gaussian measure.*

Proof. The map L defined by $L(\varphi) = (\varphi^{(n)}, \varphi(0), \varphi'(0), \ldots, \varphi^{(n-1)}(0))$ is a linear isomorphism between $C^n(\mathbb{R}, \mathbb{R})$ and $C^0(\mathbb{R}, \mathbb{R}) \times \mathbb{R}^n$. From this, and by the Riesz representation theorem every continuous linear functional defined on $C^n(\mathbb{R}, \mathbb{R})$ is of the form

$$f(\varphi) = \int_a^b \varphi^{(n)}(x)\, dg(x) + c_0\varphi(a) + \cdots + c_{n-1}\varphi^{(n-1)}(a),$$

where g is a function of bounded variation defined on some interval $[a, b]$, and $\{c_0, \ldots, c_{n-1}\}$ is a sequence of real numbers. From the definition of μ it follows that f has the same distribution as the random variable ζ defined by

$$\zeta = \int_a^b \eta_x^{(n)}\, dg(x) + c_0\eta_a + \cdots + c_{n-1}\eta_a^{(n-1)}.$$

The random variable ζ belongs to H, thus ζ has a Gaussian distribution. This completes the proof.

LEMMA 3.6. *The measure μ is non-trivial.*

Proof. We denote by $P(t)$ the set of all periodic points of $\{T_t\}$ with period t ($P(0)$ denotes the set of all stationary points of $\{T_t\}$). Let $P(\le t)$ be the set of all periodic points with period $s \in (0, t]$. Then $P(t)$ and $P(\le t)$ are closed invariant subsets of Y. Suppose that $\mu(P) > 0$, where P is the set of all periodic points of $\{T_t\}$. Then $\mu(P(\le t)) > 0$ for some $t > 0$. By ergodicity of $\{T_t\}$ it follows that $\mu(P(\le t)) = 1$ if $\mu(P(\le t)) \ne 0$. Let $t_0 = \inf\{t > 0: \mu(P(\le t)) = 1\}$. Then $\mu(P(t_0)) = 1$. We may assume, without loss of generality, that $t_0 > 0$. Let A be a Borel subset of Y such that $0 < \mu(A) < 1$. Then the set $A \cap P(t_0)$ is invariant under T_{t_0} and $0 < \mu(A \cap P(t_0)) < 1$. This contradicts the total ergodicity of μ. □

We may summarize results of this section as follows.

THEOREM 2. *For every $n \ge 0$ there exists a probability measure μ defined on the σ-algebra of Borel subsets of $C^n(\mathbb{R}, \mathbb{R})$ satisfying the following conditions:*

(a) *μ is invariant under $\{T_t\}$;*

(b) *$\mu(U) > 0$ for each non-empty open set U;*

(c) *$\{T_t\}$ is a K-flow on $(C^n(\mathbb{R}, \mathbb{R}), \mathcal{B}(C^n(\mathbb{R}, \mathbb{R})), \mu)$;*

(d) *μ is non-trivial;*

(e) *μ is a Gaussian measure.*

THEOREM 3. *Let (X, S_t) be the flow defined in § 2. Then there exists a probability measure m defined on $\mathcal{B}(X)$ such that $(X, \mathcal{B}(X), m, S_t)$ satisfies the conditions* (a)–(d) *of theorem* 2. *Moreover, if $b(x, u)$ is of the form $b(x, u) = f(x) + g(x)u$ and $U_2 = \mathbb{R}$, then m is a Gaussian measure.*

Proof. According to theorem 1 there exists a homeomorphism Q between X and $C^n(\mathbb{R}, \mathbb{R})$ such that $Q \circ S_t = T_t \circ Q$. Thus, we can define a measure m on $\mathcal{B}(X)$ by $m(A) = \mu(Q(A))$. The conditions (a), (b), (c) and (d) are a direct consequence of theorem 2. If $b(x, u) = f(x) + g(x)u$ and $U_2 = \mathbb{R}$, then Q is of the form $Qv = Lv + v_0$, where L is a linear isomorphism from X onto $C^n(\mathbb{R}, \mathbb{R})$. Thus, according to theorem 2(e), the measure m is Gaussian.

COROLLARY 2 (chaos). *The flow $\{S_t\}$ satisfies the following two conditions:*

(a) *every point $v \in X$ is unstable;*

(b) *there exists $v \in X$ such that the trajectory of v is dense in X.*

Remark 2. The construction of the measure m given in the proof of theorem 3 may be repeated as well replacing ξ_x^0 by $\xi_{\lambda x}^0$ with arbitrary $\lambda > 0$. It is interesting that the measures m_{λ_1} and m_{λ_2} corresponding to different λ_1 and λ_2 are mutually singular.

Remark 3. Let ζ_x, $x \in U_1$, be the process given by the formula $\zeta_\cdot(\omega) = Q^{-1}\xi_\cdot^n(\omega)$. Then $m(A) = P(\zeta \in A)$ for each Borel subset A of X. From remark 1 it follows that ζ_x, $x \in U_1$ is a Markov process for $n = 0$.

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