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PROJECTIVE SCHUR ALGEBRAS OVER A FIELD OF POSITIVE CHARACTERISTIC

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If the characteristic of a field K is not zero then the Schur group $S(K) = 0$. In this paper we ask a similar question for the projective Schur group $PS(K)$ and prove that the subgroup of $PS(K)$ consisting of radical algebras is trivial. This disproves the conjecture that every projective Schur algebra is similar to a radical algebra.

1. INTRODUCTION

Let K be a field. A K-central simple algebra is called a Schur K-algebra if it is a homomorphic image of a group algebra KG for some finite group G. The Schur group $S(K)$ is the subgroup of the Brauer group $B(K)$ of K consisting of those classes in $B(K)$ which are represented by Schur algebras. If the characteristic of K is not zero then the Schur group $S(K)$ is trivial [**6**, proof of Corollary 7.11, p.148]. When K is a ring of positive characteristic, the same result that $S(K)$ is trivial holds [**7**, Proposition 1].

The Schur algebra has been generalised by Lorenz and Opolka [**9**] by replacing the group algebra by a twisted group algebra $K^\alpha G$, where $\alpha \in H^2(G, K^*)$ with G acting trivially on K^*. This algebra is called a projective Schur K-algebra. Aljadeff and Sonn [**1**] showed that the projective Schur algebra B can be characterised as a K-central simple algebra containing a group Γ in the group of units of A such that Γ/K^* is a finite group and $K(\Gamma) = B$. The *projective Schur group* $PS(K)$ is the subgroup of $B(K)$ consisting of classes containing projective Schur algebras. It is obvious that $S(K) \subset PS(K) \subset B(K)$.

When we say an algebra B in $B(K)$ is a radical algebra we mean it is a crossed product algebra of the form

$$(L/K, f) = \sum_{\sigma \in G} Lu_\sigma$$

where $L = K(A)$ is a finite Abelian radical Galois extension of K with Galois group G, that is, A is a subgroup of L^* such that A/K^* is a finite group and $f \in H^2(G, L^*)$

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has values in A such that $u_\sigma u_\tau = f(\sigma,\tau)u_{\sigma\tau}$ for $\sigma,\tau \in G$. Here K^* denotes K-$\{0\}$. Clearly a radical algebra is a projective Schur algebra. For the converse, it is proved in [**1**] that a projective Schur division algebra is a radical algebra, and it was conjectured in [**2**] that every projective Schur algebra is similar to a radical algebra.

In this article, we study projective Schur algebras over a field of characteristic $p > 0$ and define a subgroup of $PS(K)$ which contains $S(K)$. We show that the new group is trivial if the characteristic of K is positive while $PS(K) \neq 0$. Using these groups we disprove the conjecture in [**2**].

2. Projective Schur algebras and Radical algebras

We denote by $[B]$ an equivalence class of finite dimensional central simple K-algebras.

Consider the set of all radical algebras over K. Let $(L_1/K,\alpha_1)$ and $(L_2/K,\alpha_2)$ be two radical K-algebras. Let $\mathrm{Gal}(L_i/K) = G_i$ be Galois groups and let $L_i = K(A_i)$ for $i = 1,2$, where A_i is a subgroup of L_i^* such that A_i/K^* is a finite group. Then L_1L_2/K is a radical extension such that $L_1L_2 = K(A_1A_2)$ and it is a Galois extension, too. Denote the Galois group of L_1L_2 over K by G. Then there is an injection $G \to G_1 \times G_2$ which maps $\sigma \in G$ to its restriction $\left(\sigma|_{L_1}, \sigma|_{L_2}\right) = (\sigma_1,\sigma_2)$ for $\sigma_i \in G_i$ $(i = 1,2)$.

Define a map $\alpha : G \times G \to L_1L_2$ by

$$\alpha(\sigma,\tau) = \alpha_1\left(\sigma|_{L_1}, \tau|_{L_1}\right)\alpha_2\left(\sigma|_{L_2}, \tau|_{L_2}\right) = \alpha_1(\sigma_1,\tau_1)\alpha_2(\sigma_2,\tau_2)$$

for $\sigma,\tau \in G$ and $\sigma_i \in G_1$, $\tau_i \in G_2$. This makes the diagram commute:

$$\begin{array}{ccccc} & & (G_1 \times G_2) \times (G_1 \times G_2) & & \\ & \nearrow & & \searrow \alpha_1 \times \alpha_2 & \\ G \times G & & \longrightarrow & & L_1L_2 \end{array}$$

where $\alpha_1 \times \alpha_2$ is a 2-cocycle in $Z^2(G_1 \times G_2, (L_1L_2)^*)$ defined by

$$(\alpha_1 \times \alpha_2)((x_1,x_2),(y_1,y_2)) = \alpha_1(x_1,y_1)\alpha_2(x_2,y_2)$$

for $x_i, y_i \in G_i$ $(i = 1,2)$. It is routine to check that α is a 2-cocycle over G, and all the values of α belong to A_1A_2. We note that if $L_1 \cap L_2 = K$ then G is isomorphic to $G_1 \times G_2$ and α is equal to $\alpha_1 \times \alpha_2$.

Thus these yield a subgroup of $PS(K)$ which contains $S(K)$ as follows.

THEOREM 1. *Assume the same notation as above. Then the set of similar classes of radical algebras over a field K form a group under the multiplication*

$$[(L_1/K,\alpha_1)]\,[(L_2/K,\alpha_2)] = [(L_1L_2/K,\alpha)].$$

We call the group a radical group and denote it by $\mathrm{Rad}(K)$. Because each Schur algebra is a radical algebra and each radical algebra is a projective Schur algebra, we have the inclusions:

$$S(K) \subset \mathrm{Rad}(K) \subset PS(K) \subset B(K).$$

The projective Schur group is much bigger than the Schur group. In fact $PS(K)$ is big enough to be $B(K)$ if K is a number field. It had been conjectured in [**10**] that $PS(K) = B(K)$ for all fields K, but Aljadeff and Sonn [**3**] disproved this. Instead, Aljadeff and Sonn conjectured in [**2**] that every projective Schur algebra is similar to a radical algebra, that is, $PS(K) = \mathrm{Rad}(K)$.

THEOREM 2. *Let K be a field and let B be a projective Schur division K-algebra. Then $[B]$ is contained in a Schur group $S(E)$, where E is a certain finite field extension of K.*

PROOF: By Aljadeff and Sonn [**1**, Theorem 1] B itself is an Abelian radical algebra $(L/K,f)$ where $L = K(A)$ is an Abelian radical Galois extension of K with Galois group $G = \mathrm{Gal}(L/K)$ and $f \in H^2(G,L^*)$ is the image of some elements in $H^2(G,A)$. Furthermore in [**5**, Theorem 4], a finite radical Galois extension field E of L has been constructed such that B is similar to a crossed product algebra $(E/K,\alpha)$ where $\alpha \in H^2(E/K,E^*)$ has a representing 2-cocycle whose values are contained in the group $\mu_u < E^*$ of the u-th root of unity for some $u > 0$. Because E is a finite Galois extension over K (which may not Abelian), $(E/K,\alpha) = E^\alpha H$ is the twisted group algebra with finite group $H = \mathrm{Gal}(E/K)$. Since all values of α are contained in $\mu_u < E^*$, the order of α is finite.

Consider a central group extension Γ of μ_u by H:

$$1 \to \langle \mathrm{im}\,\alpha\rangle = \mu_u \to \Gamma \to H \to 1.$$

The Γ may be thought as an α-covering group $H(\alpha)$ [**4**, Theorem 6, Corollary 3], so that it is of finite order $|H|\,o(\alpha) < \infty$.

With respect to the group algebra $E\Gamma$ and twisted group algebra $E^\alpha H$, it is known that there is an isomorphism $E\Gamma/E\Gamma\cdot I \cong E^\alpha H$ where I is the augmentation ideal of $E\Gamma$ [**8**, Theorem 3.2.8]. Thus we have

$$E\Gamma \xrightarrow{\pi} E^\alpha H = (E/K,\alpha) \sim (L/K,f) = B$$

where the map π is a surjection (here the notation $\sim$ means the similarity of two algebras), so that $[B]$ is contained in $S(E)$. ∎

COROLLARY 3. *Let K be a field of positive characteristic p. Then any class $[B]$ of a projective Schur division K-algebra B is trivial in $PS(K)$. Therefore Rad$(K) = 0$.*

PROOF: This proof is an immediate consequence of Theorem 2. If B is a projective Schur division K-algebra then the class $[B]$ is contained in a Schur group $S(E)$, where E is a certain finite field extension of K. Since the characteristic of K is $p > 0$, the characteristic of E is p. Due to the fact that $S(E) = 0$, the algebra class $[B]$ is trivial and hence $\mathrm{Rad}(K) = 0$. ∎

In [**3**, Theorem 3.2] it has been shown that there is an element in $PS(K(t))_p$ of order p^r if characteristic $K = p > 0$ where $K(t)$ is a function field with a mild condition on K. But Corollary 3 shows that $\mathrm{Rad}(K(t)) = 0$ in this case. This means that $\mathrm{Rad}(K(t))$ is a proper subgroup of $PS(K(t))$. Therefore the following conjecture does not hold:

CONJECTURE. [2] $PS(K) = \mathrm{Rad}(K)$.

REMARK. If the characteristic of K is zero then the above conjecture still remains open.

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