CAPACITABILITY OF ANALYTIC SETS

MASANORI KISHI

Introduction

Let Ω be a locally compact separable metric space and let Φ be a positive symmetric kernel. Then the inner and outer capacities of subsets of Ω are defined by means of Φ-potentials of positive measures in the following manner. We define the capacity $c(K)$ of a compact set K in a certain manner by means of Φ-potentials. By this set function we define the inner and outer capacities of a subset X of Ω as follows:

$$\text{cap}_i(X) = \sup c(K) \text{ for all compact } K \text{ contained in } X,$$
$$\text{cap}_e(X) = \inf \text{cap}_i(K) \text{ for all open } G \text{ containing } X.$$

A subset whose inner capacity coincides with its outer capacity is said to be capacitable. In this paper we discuss whether or not every analytic set is capacitable, where an analytic set is, by definition, the continuous image of a $K_{\sigma\delta}$ set in a compact space.

As for the classical capacities, for example, the α-capacities $(0 < \alpha \leqq 2)$ in the m-dimensional euclidean space R^m $(m \geqq 3)$, the problem of the capacitability was solved affirmatively by Choquet [3]. This result was extended by Aronszajn and Smith [1] as follows: every analytic set in R^m is capacitable with respect to the α-capacities for all α, $0 < \alpha < m$. Here the α-capacities are defined by the set function

$$c^{(\alpha)}(K) = \inf_{\mu \in \mathfrak{E}_K} \int U_\alpha^\mu \, d\mu, \tag{1}$$

where U_α^μ denotes the α-potential of a positive measure μ, that is,

$$U_\alpha^\mu(x) = \int \frac{1}{|x-y|^{m-\alpha}} \, d\mu(y), \qquad x \text{ and } y \in R^m,$$

and $\mathfrak{E}_K$ denotes the family of positive measures μ such that the α-potential U_α^μ $\geqq 1$ on K with a possible exception of a set E which is of measure zero with

Received November 7, 1959.

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

respect to any positive measure with finite α-energy. The α-capacities may be defined by the set function

$$(2) \qquad \tilde{c}^{(\alpha)}(K) = \sup_{\mu \in \mathfrak{G}_K} \mu(K),$$

where $\mathfrak{G}_K$ denotes the family of positive measures μ such that the carriers S_μ are contained in K and the α-potential $U_\alpha^\mu \leqq 1$ everywhere in R^m. For every α, $0 < \alpha \leqq 2$, the above two α-capacities are identical.

The capacitability of analytic sets contained in a compact set in a locally compact separable metric space is also assured if Φ satisfies Frostman's maximum principle [7]. In this case the capacities are understood in the sense of (2).

Recently Fuglede [5] studied consistent kernels Φ in a locally compact space Ω and he proved the Φ-capacitability of analytic sets under the assumption that every closed subset of Ω possesses a countable fundamental system of neighborhoods each of which is closed.

In §1 the definition of a capacity is given. A capacity is defined by a functional which fulfills four postulates. It is shown that every compact set is capacitable with respect to any capacity. In §2 we define and study admissible capacities, and in §3 we prove that every analytic set is capacitable with respect to any admissible capacity, provided that the kernel Φ satisfies a condition (*) stated in §2.3. In §§4 and 5 we study capacities defined by functions given in terms of total measures and energy integrals, respectively. It is shown that the former capacity is admissible if the kernel satisfies the condition (*) and the continuity principle and that every analytic set is capacitable with respect to this capacity. The condition (*) is related to the behavior of the kernel near the point at infinity and the continuity principle is related to the behavior of the kernel in the neighborhood of the diagonal set of $\Omega \times \Omega$. This principle is equivalent to the local boundedness principle, from which the continuity principle may be regarded as the weakest one in the potential theory from the viewpoint of the maximum principle. The capacity in §5 is admissible if the kernel is regular and of type positive and if it satisfies the continuity principle and the condition (*). The results in §4 are announced in [8].

§1. Definitions

1. Let Ω be a locally compact separable metric space. The space $\mathfrak{M}^+(\Omega)$·

of positive measures[1] on Ω is defined as follows: let $\mathfrak{C}(\Omega)$ be the vector space of real-valued (finite) continuous functions vanishing outside compact sets. For every function f of $\mathfrak{C}(\Omega)$ we put

$$\|f\| \equiv \sup_{P \in \Omega} |f(P)|.$$

A linear functional μ on $\mathfrak{C}(\Omega)$ is called a positive measure if it has the following properties:

(1) $\mu(f) \geqq 0$ for every $f \geqq 0$,

(2) for any compact set K, there is a finite number $M_K \geqq 0$ such that $|\mu(f)| \leqq M_K \|f\|$ for every function f of $\mathfrak{C}(\Omega)$ which vanishes outside K, i.e., the carrier S_f of f is contained in K, where M_K depends on μ and K, but not on f.

It is shown that every positive linear functional is a positive measure. In what follows we write $\int f d\mu$ instead of $\mu(f)$.

We say that a sequence $\{\mu_n\}$ of $\mathfrak{M}^+(\Omega)$ *converges vaguely* to μ of $\mathfrak{M}^+(\Omega)$ if $\int f d\mu = \lim_n \int f d\mu_n$ for every function f of $\mathfrak{C}(\Omega)$. The following selection theorem is well known.

SELECTION THEOREM. *Let $\{\mu_n\}$ be a sequence of $\mathfrak{M}^+(\Omega)$ such that μ_n-measures of every compact set K, $\mu_n(K)$[2], are bounded by a finite number $M(K)$ which depends only on K. Then we can choose a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ which converges vaguely to a positive measure μ.*

We say that a positive measure μ is carried by a closed set F if $\mu(\Omega - F) = 0$, that is, $\int f d\mu = 0$ for every function f of $\mathfrak{C}(\Omega)$ such that $S_f \subset \Omega - F$. We call the intersection of all closed sets which carries μ the carrier of μ, and we denote it by S_μ. We denote by $\mathfrak{M}_0^+(\Omega)$ the subspace of $\mathfrak{M}^+(\Omega)$, consisting of measures whose carriers are compact.

2. Now let Φ be a positive symmetric kernel, that is, a real-valued continuous function defined on the product space $\Omega \times \Omega$ such that

1 $0 < \Phi(P, Q) \leqq +\infty$,

2 $\Phi(P, Q)$ is finite except at most at the points of diagonal set of $\Omega \times \Omega$,

3 $\Phi(P, Q) = \Phi(Q, P)$.

[1] As to the theory of measures and integrals in a locally compact space, see Bourbaki [2].

[2] $\mu_n(K) = \int \chi_K d\mu_n$, where χ_K denotes the characteristic function of K.

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

The potential $U^{\mu}(P)$ of a measure $\mu \in \mathfrak{M}^{+}(\Omega)$ is defined by

$$U^{\mu}(P) = \int \Phi(P, Q)\, d\mu(Q).$$

Then $U^{\mu}(P)$ is lower semi-continuous in Ω and continuous in $\Omega - S_{\mu}$. We denote by I_{μ} the set of points P at which $U^{\mu}(P) = +\infty$. This is a G_{δ} set. It may happen that $I_{\mu} = \Omega$ or $I_{\mu} = \phi$.

3. Now we shall define the inner and outer capacities of subsets of Ω. The capacities are defined by means of a functional on the space $\mathfrak{M}^{+}(\Omega)$.

Let c be a functional defined on $\mathfrak{M}^{+}(\Omega)$ for which the following four postulates are fulfilled:

(c. 1) $0 \leqq c(\mu) \leqq +\infty$, $c(0) = 0$,

(c. 2) $c(\mu + \nu) \leqq c(\mu) + c(\nu)$,

(c. 3) c is lower semi-continuous with respect to the vague convergence, that is, $c(\mu) \leqq \underline{\lim}_{n}\, c(\mu_n)$ if $\mu_n \to \mu$ vaguely,

(c. 4) for any positive number t, $c(t\mu) = t \cdot c(\mu)$.

By means of this functional c the inner and outer capacities are defined. Let $\mathfrak{F}$ be the family of all positive measures μ such that $c(\mu)$ is finite, and let $\mathfrak{P}$ be the class of all subsets E of Ω which is contained in some I_{μ}, $\mu \in \mathfrak{F}$. Each element of $\mathfrak{P}$ is called a *polar set*. For any subset X of Ω we put[3)]

$$\mathfrak{F}_X = \{\mu \in \mathfrak{F};\ U^{\mu} \geqq 1 \text{ on } X \text{ except } E \in \mathfrak{P}\},$$

where the statement, $U^{\mu} \geqq 1$ on X except $E \in \mathfrak{P}$, means that the set $E = \{P \in X;\ U^{\mu}(P) < 1\}$ is a polar set. We define the following set functions:

$$c(X) = \begin{cases} \inf_{\mu \in \mathfrak{F}_X} c(\mu) & \\ +\infty & \text{if } \mathfrak{F}_X \text{ is empty,} \end{cases}$$

$\mathrm{cap}_i(X) = \sup c(K)$ where K ranges over the class of all compact sets contained in X,

$\mathrm{cap}_e(X) = \inf \mathrm{cap}_i(G)$ where G ranges over the class of all open sets containing X.

These set functions are increasing and the following inequalities are valid:

$$\mathrm{cap}_i(X) \leqq c(X) \text{ and } \mathrm{cap}_i(X) \leqq \mathrm{cap}_e(X).$$

3) We put $\mathfrak{F}_{\phi} = \mathfrak{F}$.

The set functions $\operatorname{cap}_i(X)$ and $\operatorname{cap}_e(X)$ are called the ***inner*** and ***outer capacities*** of X, respectively. When these set functions are defined by means of a function c which fulfills the four postulates, we say that a ***capacity*** is defined in Ω. A set X is said to be ***capacitable*** when its inner capacity $\operatorname{cap}_i(X)$ coincides with its outer capacity $\operatorname{cap}_e(X)$, and we denote by $\operatorname{cap}(X)$ the common value. Evidently every open set is capacitable. In §1.7 we shall show the capacitability of compact sets.

4. By simple examples we can show that the four postulates are independent of each other.

Example 1. $c(\mu) = -\mu(K)$, where K is a fixed compact subset of Ω. This functional fulfills all the postulates but (c.1).

Example 2. Let Ω be the 1-dimensional euclidean space and $\Phi(x, y) = |x-y|+1$, and let

$$c(\mu) = \sqrt{\iint \Phi(x, y)\, d\mu(x)\, d\mu(y)}.$$

Then this functional fulfills all the postulates but (c.2).

Example 3. Let P_0 be a fixed point in Ω, and let

$$c(\mu) = \begin{cases} \mu(\Omega) & \text{if } S_\mu \text{ contains } P_0 \\ 0 & \text{if } S_\mu \text{ does not contain } P_0. \end{cases}$$

Then this functional fulfills all the postulates but (c.3).

Example 4. Let ω be a fixed open set in Ω, and let

$$c(\mu) = \begin{cases} 1 & \text{if } S_\mu \cap \omega \neq \phi \\ 0 & \text{if } S_\mu \cap \omega = \phi. \end{cases}$$

Then this functional fulfills all the postulates but (c.4).

5. The following lemma is a direct consequence of our postulates.

LEMMA 1. *If $c(\mu_n)$ $(n=1, 2, \ldots)$ tends to zero and if $\mu \equiv \sum \mu_n$ is a positive measure, then there is a subsequence $\{\mu_{n_k}\}$ such that $c(\mu')$ is finite, where $\mu' = \sum_{k=1}^{\infty} \mu_{n_k}$.*

An important property of the class $\mathfrak{P}$ of polar sets is stated in

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

THEOREM 1. *The class $\mathfrak{P}$ of polar sets is countably additive, that is, if E_n $(n=1, 2, \ldots)$ is polar sets, then $E=\bigcup_1^\infty E_n$ is also a polar set.*

Proof. First we remark that $E=E_1\cup E_2$ is a polar set if each E_i $(i=1, 2)$ is a polar set. In fact, let E_i be contained in I_{μ_i}, $\mu_i\in\mathfrak{F}$ $(i=1, 2)$. Then $c(\mu_1+\mu_2)$ is finite by (c.2) and E is contained in $I_{\mu_1+\mu_2}$, and hence it is a polar set.

Now suppose that E_n $(n=1, 2, \ldots)$ are contained in I_{μ_n}, $\mu_n\in\mathfrak{F}$. By the above remark we may suppose that the sequence $\{E_n\}$ is increasing. By the familiar diagonal method we can choose suitable positive numbers t_n so that $\sum t_n\mu_n$ is a positive measure. We may suppose that $c(t_n\mu_n)$ tends to zero by (c.4). Then by Lemma 1 there is a subsequence $\{t_{n_k}\mu_{n_k}\}$ such that $c(\mu)$ is finite, where $\mu=\sum_{k=1}^{\infty}t_{n_k}\mu_{n_k}$. At each point P of $E=\cup E_n=\cup E_{n_k}$ the potential $U^\mu(P)$ is infinite, because $U^\mu(P)\geqq t_{n_k}U^{\mu_{n_k}}(P)$. Hence E is a polar set.

THEOREM 2. *Every polar set E is of outer capacity zero.*

Proof. Suppose that E is contained in I_μ, $\mu\in\mathfrak{F}$. For every n we put

$$G_n=\{P\in\Omega;\ U^\mu(P)>n\}.$$

Then the open set G_n contains E and $\frac{1}{n}\mu$ belongs to the family $\mathfrak{F}_{G_n}$. Hence

$$\text{cap}_e(E)\leqq\text{cap}\,(G_n)\leqq c(G_n)\leqq c\left(\frac{1}{n}\mu\right).$$

Therefore by the postulate (c.4), $\text{cap}_e(E)=0$.

6. Now we shall prove that the outer capacity is a countably sub-additive set function. First we prove

LEMMA 2. *For any finite family $\{G_n\}$ $(n=1, 2, \ldots, N)$ of open sets* $\text{cap}\,(\bigcup_1^N G_n)\leqq\sum_1^N\text{cap}\,(G_n)$.

Proof. It is sufficient to verify the inequality for the case $N=2$. Suppose that $\text{cap}\,(G_1\cup G_2)$ is finite. Then for any positive number ε there exists a compact set $K\subset G_1\cup G_2$ such that $\text{cap}\,(G_1\cup G_2)-\varepsilon\leqq\text{cap}_i\,(K)$. It is easily verified that there exist two compact sets K_1 and K_2 such that $K=K_1\cup K_2$ and $K_n\subset G_n$ $(n=1, 2)$. For each K_n we have a measure $\mu_n\in\mathfrak{F}_{K_n}$ such that $c(\mu_n)<\text{cap}_i\,(K_n)+\varepsilon/2$. Then the measure $\mu=\mu_1+\mu_2$ belongs to $\mathfrak{F}_K$ and

$$\begin{aligned}\text{cap}_i\,(K)&\leqq c(\mu)\leqq c(\mu_1)+c(\mu_2)\\&\leqq\text{cap}_i\,(K_1)+\text{cap}_i\,(K_2)+\varepsilon.\end{aligned}$$

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

Therefore we obtain that

$$\begin{aligned}\text{cap}\,(G_1 \cup G_2) &\leqq \text{cap}_i\,(K_1) + \text{cap}_i\,(K_2) + 2\varepsilon \\ &\leqq \text{cap}\,(G_1) + \text{cap}\,(G_2) + 2\varepsilon,\end{aligned}$$

so that

$$\text{cap}\,(G_1 \cup G_2) \leqq \text{cap}\,(G_1) + \text{cap}\,(G_2).$$

In the case that $\text{cap}\,(G_1 \cup G_2) = +\infty$, we can show the inequality in the same way.

Using this lemma we prove

THEOREM 3. *For any sequence* $\{X_n\}$ $(n=1, 2, \ldots)$ *of arbitrary sets* $\text{cap}_e\,(\bigcup_1^\infty X_n) \leqq \sum_1^\infty \text{cap}_e\,(X_n)$.

Proof. First we prove the inequality in the case that all X_n are open sets. If $\text{cap}\,(\bigcup_1^\infty X_n)$ is finite, there exists, for any positive number ε, a compact set $K \subset \bigcup_1^\infty X_n$ such that $\text{cap}\,(\bigcup_1^\infty X_n) - \varepsilon \leqq \text{cap}_i\,(K)$. Since K is compact, it is covered by a finite subfamily $\{X_n\}$ $(n=1, 2, \ldots, N)$. Then by Lemma 2 we have

$$\text{cap}\,(\bigcup_1^N X_n) \leqq \sum_1^N \text{cap}\,(X_n).$$

Therefore

$$\text{cap}\,(\bigcup_1^\infty X_n) - \varepsilon \leqq \sum_1^N \text{cap}\,(X_n) \leqq \sum_1^\infty \text{cap}\,(X_n)$$

and hence

$$\text{cap}\,(\bigcup_1^\infty X_n) \leqq \sum_1^\infty \text{cap}\,(X_n).$$

If $\text{cap}(\bigcup_1^\infty X_n)$ is infinite, we can show the inequality in the same way.

Next let $\{X_n\}$ be an arbitrary sequence of subsets of Ω. We may suppose that each $\text{cap}_e\,(X_n)$ is finite. Then for any positive number ε, there are open sets $G_n \supset X_n$ such that

$$\text{cap}_e\,(X_n) + \frac{\varepsilon}{2^n} \geqq \text{cap}\,(G_n).$$

Then the open set $G = \bigcup_1^\infty G_n$ contains $\bigcup_1^\infty X_n$ and we obtain by the preceding argument that

$$\text{cap}_e\,(\bigcup^\infty X_n) \leqq \text{cap}\,(G) \leqq \sum_1^\infty \text{cap}\,(G_n) \leqq \sum_1^\infty \text{cap}_e\,(X_n) + \varepsilon,$$

and hence

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

$$\text{cap}_e(\bigcup_1^\infty X_n) \leqq \sum_1^\infty \text{cap}_e(X_n).$$

7. We close this section by showing the capacitability of compact sets. First we prove

LEMMA 3. *For every set X,* $\text{cap}_e(X) \leqq c(X)$.

Proof. We may suppose that $c(X)$ is finite. Then for any positive number ε there is a positive measure μ such that $U^\mu \geqq 1$ on X except $E \in \mathfrak{P}$ and $c(X) + \varepsilon \geqq c(\mu)$. For every n we put

$$G_n = \left\{P \in \Omega;\ U^\mu(P) > \frac{n-1}{n}\right\}.$$

Then open sets G_n contain $X - E$ and hence

$$\text{cap}_e(X - E) \leqq \text{cap}_e(G_n) \leqq c\left(\frac{n}{n-1}\mu\right).$$

Therefore by Theorems 2 and 3 and the postulate (c. 4) we obtain

$$\text{cap}_e(X) \leqq c(X) + \varepsilon$$

and hence

$$\text{cap}_e(X) \leqq c(X).$$

THEOREM 4. *Every compact set is capacitable.*

Proof. Since $\text{cap}_i(K) = c(K)$ for every compact set, our assertion follows immediately from Lemma 3.

§ 2. Admissible capacities

1. We say that a capacity is *admissible* if it fulfills the following postulates:

(a. 1) Every compact set K does not belong to $\mathfrak{P}$ provided that there is a positive measure $\mu \not\equiv 0$ such that S_μ is contained in K and U^μ is continuous in Ω,

(a. 2) Every compact set is a polar set or the converse of (a. 1) is valid.

(a. 3) Every potential is *quasi continuous* in Ω, that is, for any positive number ε, there is an open set G_ε such that $\text{cap}(G_\varepsilon) \leqq \varepsilon$ and the restriction of U^μ to $\Omega - G_\varepsilon$ is finite and continuous,

(a. 4) If $c(X_n) \leqq M$ for $n = 1, 2, \ldots,$ then for each X_n, there exists

$\mu_n \in \mathfrak{F}_{X_n}$ such that $c(\mu_n)$ is sufficiently near $c(X_n)$ and the total measure $\mu_n(\Omega)$ is bounded by a constant depending only on M.

2. By simple examples we can show that any of these postulates does not follow from the postulates in § 1.

Example 5. Let $\Omega = R^1$ and let φ be a positive symmetric and continuous function defined on R^1 such that $\varphi(0)$ is finite and $\int^{+\infty} \varphi(x)dx = +\infty$. We put $\Phi(x, y) = \varphi(x-y)$ and $c(\mu) = 0$ for every $\mu \in \mathfrak{M}^+(\Omega)$. Then every finite closed interval is a polar set and the potential of a positive measure of $\mathfrak{M}_0^+(\Omega)$ is finite and continuous in Ω. Thus the capacity induced by the functional c fulfills the postulates (a.2) and (a.3) but neither (a.1) nor (a.4).

Example 6. Let P_0 be a fixed point of Ω and let $\Phi(P_0, P_0) = +\infty$. We put $c(\mu) = U^\mu(P_0)$ for every measure μ of $\mathfrak{M}^+(\Omega)$. Then every polar set does not contain the point P_0 and hence the compact set $\{P_0\}$ is not a polar set. Thus the capacity induced by this functional c fulfills neither (a.2) nor (a.3).

3. Now we assume that the kernel Φ satisfies the condition

(*) for any compact set K and for any positive number ε, there is a compact set L such that

$$\Phi(P, Q) < \varepsilon \quad \text{on} \quad K \times (\Omega - L).$$

For the later use we prove several lemmas.

LEMMA 4. *Suppose that a potential U^γ of a positive measure γ of $\mathfrak{M}_0^+(\Omega)$ is continuous in Ω and positive measures μ_n converge vaguely to μ. If the total measures $\mu_n(\Omega)$ are bounded, then* $\int U^\gamma d\mu = \lim_n \int U^\gamma d\mu_n$.

Proof. Since $\mu_n \to \mu$ vaguely, $U^\mu(P) \leqq \underline{\lim}_n U^{\mu_n}(P)$ at every point P of Ω, and hence $\int U^\gamma d\mu = \int U^\gamma d\gamma \leqq \underline{\lim} \int U^{\mu_n} d\gamma = \underline{\lim} \int U^\gamma d\mu_n$. Therefore it is sufficient to show that $\int U^\gamma d\mu \geqq \overline{\lim}_n \int U^\gamma d\mu_n$. For any positive number ε there is a compact set L, by the condition (*), such that $\Phi(P, Q) < \varepsilon$ on $S_\gamma \times (\Omega - L)$. Then

$$\int_{\Omega - L} U^\gamma d\mu_n = \int_{\Omega - L} \int_{S_\gamma} \Phi(P, Q)\, d\gamma(Q)\, d\mu_n(P)$$
$$< \varepsilon \mu_n(\Omega - L) \cdot \gamma(S_\gamma) < \varepsilon M$$

for every n. We put

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

$$f = \begin{cases} U^{\gamma} & \text{on } L \\ 0 & \text{on } \Omega - L, \end{cases}$$

then f is upper semi-continuous in Ω and we have

$$\overline{\lim_n} \int_L U^{\gamma} d\mu_n = \overline{\lim_n} \int f d\mu_n \leqq \int f d\mu = \int_L U^{\gamma} d\mu.$$

Therefore

$$\begin{aligned} \int U^{\gamma} d\mu \geqq \int_L U^{\gamma} d\mu &\geqq \overline{\lim_n} \int_L U^{\gamma} d\mu_n \\ &\geqq \overline{\lim_n} \int U^{\gamma} d\mu_n - \overline{\lim_n} \int_{\Omega - L} U^{\gamma} d\mu_n \\ &\geqq \overline{\lim_n} \int U^{\gamma} d\mu_n - \varepsilon M. \end{aligned}$$

Consequently we obtain $\int U^{\gamma} d\mu \geqq \overline{\lim_n} \int U^{\gamma} d\mu_n$.

Remark. When all S_{μ_n} are contained in a fixed compact set, this proposition follows immediately from the definition of the vague convergence. In general, this does not hold unless Φ satisfies the condition $(*)$.

LEMMA 5. *If a set E belongs to $\mathfrak{P}$, then $\nu(E) = 0$ for any positive measure ν whose potential is continuous in Ω.*

Proof. Contrary to the assertion we suppose that $\nu(E) > 0$ and the potential U^{ν} is continuous in Ω. Then there is a compact set $K \subset E$ such that $\nu(K) > 0$. Let ν' be the restriction of ν to K. Then $U^{\gamma'}$ is continuous in Ω, since U^{ν} is continuous in Ω. Thus there exists a positive measure $\nu' \not\equiv 0$ carried by a compact polar set whose potential is continuous in Ω. This contradicts the postulate (a. 1).

LEMMA 6. *Let $\{K_n\}$ be an increasing sequence of compact sets and X be the union $\cup K_n$. If positive measures $\mu_n \in \mathfrak{F}_{K_n}$ converges vaguely to μ and the total measures $\mu_n(\Omega)$ are bounded, then $U^{\mu} \geqq 1$ on X except $E \in \mathfrak{P}$.*

Proof. For any positive integers k and m we put

$$E_k^m = \left\{ P \in K_k \ ; \ U^{\mu}(P) \leqq 1 - \frac{1}{m} \right\}.$$

Since U^{μ} is lower semi-continuous, E_k^m is compact. We shall show that there is not any positive measure $\gamma \not\equiv 0$ such that S_{γ} is contained in E_k^m and U^{γ} is

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

continuous in Ω. Contrary suppose that there is such a measure γ carried by E_k^m. Then by Lemma 4

$$\left(1-\frac{1}{m}\right)\gamma(E_k^m) \geqq \int U^{\gamma} d\gamma = \lim \int U^{\mu_n} d\gamma > \gamma(E_k^m),$$

where the last inequality follows from Lemma 5. This is impossible. Therefore there is not any positive measure $\gamma \not\equiv 0$ on E_k^m whose potential is continuous in Ω. Hence by the postulate (a.2), E_k^m is a polar set.

Now we put

$$E^m = \left\{P \in X;\ U^{\mu}(P) < 1 - \frac{1}{m}\right\}$$

$$E = \{P \in X;\ U^{\mu}(P) < 1\}.$$

Then $E = \cup E^m = \cup \cup E_k^m$. As each E_k^m is a polar set, E is also a polar set by Theorem 1. Consequently $U^{\mu} \geqq 1$ on X except $E \in \mathfrak{P}$.

4. Lemma 7. *For every K_σ set X, $c(X) = \text{cap}_i(X)$.*

Proof. By virtue of the inequality $c(X) \geqq \text{cap}_i(X)$ stated in §1.3 it is sufficient to prove the inequality $c(X) \leqq \text{cap}_i(X)$ under the assumption that the right hand side is finite. Let $\{K_n\}$ be an increasing sequence of compact sets and $X = \cup K_n$, and let ε be an arbitrary positive number. Then by the postulate (a.4) there is a measure $\mu_n \in \mathfrak{F}_{K_n}$ such that $c(\mu_n) \leqq c(K_n) + \varepsilon$ and $\mu_n(\Omega)$ is bounded by a constant depending only on $\text{cap}_i(X)$. Then by Selection theorem there is a subsequence $\{\mu_{n_k}\}$ which converges vaguely to a positive measure μ. This measure μ belongs to $\mathfrak{F}_X$ by Lemma 6. Hence by the lower semi-continuity of the functional c we have

$$c(X) \leqq c(\mu) \leqq \varliminf_k c(\mu_{n_k}) + \varepsilon \leqq \text{cap}_i(X) + \varepsilon.$$

Consequently $c(X) \leqq \text{cap}_i(X)$.

Corollary 1. *For every open set G, $c(G) = \text{cap}(G)$.*

Corollary 2. *Every K_σ set is capacitable.*

Proof. By Lemma 3, $c(X) \geqq \text{cap}_e(X)$ and hence $\text{cap}_i(X) = \text{cap}_e(X)$ for every K_σ set X.

Corollary 3. *For arbitrary set X, $c(X) = \text{cap}_e(X)$.*

Proof. By virtue of Lemma 3 it is sufficient to show that $c(X) \leqq \text{cap}_e(X)$.

For every open set G containing X it holds by Corollary 2 that

$$c(X) \leqq c(G) = \text{cap}(G)$$

and hence
$$c(X) \leqq \text{cap}_e(X).$$

5. Theorem 5. *Every set of outer capacity zero is a polar set.*

Proof. Let a set X be of outer capacity zero. Then there is a sequence $\{G_n\}$ of open sets such that $G_n \supset X$ and $\text{cap}(G_n) < \frac{1}{n}$, and hence by Corollary 1 of the preceding lemma there is a sequence $\{\mu_n\}$ of positive measures such that $U^{\mu_n} \geqq 1$ in G_n except $E_n \in \mathfrak{P}$ and $c(\mu_n) < \frac{2}{n}$. Then by the postulates (c.2) and (c.4) we can choose a subsequence $\{\mu_{n_k}\}$ so that $\mu = \sum_{k=1}^{\infty} \mu_{n_k}$ is a positive measure belonging to the family $\mathfrak{F}$. The potential U^μ is infinite at every point $X - \bigcup E_{n_k}$ and hence $X - \bigcup E_{n_k}$ is a polar set. Hence X is a polar set.

Summing up Theorems 2 and 5 we have

Theorem 6. *Suppose that Φ satisfies the condition* (*) *and the capacity is admissible. Then a set X is of outer capacity zero if and only if it is a polar set.*

§ 3. Capacitability of analytic sets with respect to admissible capacities

1. In this section we assume that the kernel Φ satisfies the condition (*) and the capacity is admissible. First we prove

Theorem 7.[4] *Suppose that a sequence $\{\mu_n\}$ of positive measures of $\mathfrak{F}$ converges vaguely to μ and that the total measures $\mu_n(\Omega)$ are bounded. Then $U^\mu = \underline{\lim}\, U^{\mu_n}$ in Ω except $E \in \mathfrak{P}$.*

Proof. By the postulate (a.3) and Theorem 3 there is an open set G_ε, for any positive number ε, such that $\text{cap}(G_\varepsilon) \leqq \varepsilon$ and the restrictions of U^{μ_n} to $\Omega - G_\varepsilon$ are finite and continuous. We put

$$V_n = \inf(U^{\mu_n}, U^{\mu_{n+1}}, \ldots),$$
$$V = \lim V_n = \underline{\lim}\, U^{\mu_n}.$$

Then the functions V_n are upper semi-continuous as functions on $\Omega - G_\varepsilon$, and V_n converges increasingly to V at every point of Ω. Since $\mu_n \to \mu$ vaguely, $U^\mu(P) \leqq V(P)$ at every point P of Ω. Hence it is sufficient to prove that

[4] Cf [6].

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

$E=\{P\in\Omega;\ U^{\mu}(P)<V(P)\}$ is a polar set. We put

$$E_n^{(k)}(\varepsilon)=\left\{P\in\Omega-G_\varepsilon;\ U^{\mu}(P)+\frac{1}{k}\leqq V_n(P)\right\}\qquad(k,\ n=1,2,\ldots)$$
$$E_n(\varepsilon)=\{P\in\Omega-G_\varepsilon;\ U^{\mu}(P)<V_n(P)\}\qquad(n=1,2,\ldots)$$
$$E_n=\{P\in\Omega;\ U^{\mu}(P)<V_n(P)\}\qquad(n=1,2,\ldots).$$

Then each $E_n^{(k)}(\varepsilon)$ is a closed set and $E_n(\varepsilon)=\bigcup_{k=1}^{\infty}E_n^{(k)}(\varepsilon)$ and $E=\bigcup_{1}^{\infty}E_n$. We shall prove that all $E_n(\varepsilon)$ are polar sets. Contrary to the assertion we suppose that $E_{n_0}(\varepsilon)$ is not a polar set. Then there exists $E_{n_0}^{(k_0)}(\varepsilon)$ which is not a polar set and hence a compact subset K of $E_{n_0}^{(k_0)}(\varepsilon)$ which is not a polar set. Then by the postulate (a. 2) there exists a positive measure $\gamma\not\equiv 0$ such that S_γ is contained in K and the potential U^{γ} is continuous in Ω. Then by Lemma 4

$$0<\frac{1}{k_0}(E_{n_0}^{(k_0)}(\varepsilon))\leqq\int(V_{n_0}-U^{\mu})\,d\gamma$$
$$\leqq\int(V-U^{\mu})\,d\gamma<\underline{\lim}\int U^{\mu_n}d\gamma-\int U^{\mu}d\gamma=0,$$

which is absurd. Thus we obtain that each $E_n(\varepsilon)$ is a polar set. Hence $\mathrm{cap}_e(E_n(\varepsilon))=0$ by Theorem 6, and $\mathrm{cap}_e(E_n)=0$, since

$$\mathrm{cap}_e(E_n)\leqq\mathrm{cap}_e(E_n(\varepsilon))+\mathrm{cap}(G_\varepsilon)\leqq\varepsilon$$

by Theorem 3. Consequently by Theorem 6, E_n is a polar set and hence E is a polar set.

THEOREM 8. *Let $\{X_n\}$ be an increasing sequence of arbitrary sets and $X=\cup X_n$. Then* $\mathrm{cap}_e(X)=\lim\mathrm{cap}_e(X_n)$.

Proof. By virtue of Corollary 3 of Lemma 7 it is sufficient to prove that $c(X)=\lim c(X_n)$. Since $c(X_n)\leqq c(X_{n+1})\leqq c(X)$ and hence $\lim c(X_n)\leqq c(X)$, it is sufficient to prove $c(X)\leqq\lim c(X_n)$ under the assumption that $\lim c(X_n)$ is finite. By the postulate (a. 4), for any positive integer n, there is a measure $\mu_n\in\mathfrak{F}_{X_n}$ such that $c(\mu_n)\leqq c(X_n)+\frac{1}{n}\leqq\lim c(X_n)+1$ and $\mu_n(\Omega)$ is bounded. Then by Selection Theorem a subsequence $\{\mu_{n_k}\}$ converges vaguely to a positive measure μ. This measure μ belongs to $\mathfrak{F}_X$ by Theorem 7. Consequently

$$c(X)\leqq c(\mu)\leqq\underline{\lim}_{k}\,c(\mu_{n_k})\leqq\lim c(X_n).$$

This completes the proof.

2. By virtue of Theorems 4 and 8 we can apply Choquet's method [3] to prove the capacitability of analytic sets and we obtain

THEOREM 9. *Suppose that* Φ *satisfies the condition* $(*)$ *and the capacity is admissible. Then every analytic set is capacitable.*

§ 4. The continuity principle and admissible capacities

1. We say that a kernel Φ satisfies the *continuity principle*[5] provided that the following statement is valid for every measure μ of $\mathfrak{M}_0^+(\Omega)$: if a potential U^μ is finite and continuous as a function on the carrier S_μ, then it is continuous in Ω. From the viewpoint of the maximum principle, the continuity principle is equivalent to the local boundedness principle, that is, the following proposition is valid.

THEOREM 10.[6] *Φ satisfies the continuity principle if and only if it has the following propety: if a potential U^μ of a measure μ of $\mathfrak{M}_0^+(\Omega)$ is bounded on S_μ, then for any neighborhood ω of S_μ with compact closure there is a finite number $M = M(\mu, \omega) > 0$ such that*

$$\sup_{P \in \omega} U^\mu(P) \leqq M \cdot \sup_{P \in S_\mu} U^\mu(P).$$

By this theorem the continuity principle may be regarded as the weakest one in the potential theory from the viewpoint of the maximum principle.

2. Now we put $c(\mu) = \mu(\Omega)$ for every measure μ of $\mathfrak{M}^+(\Omega)$. It is easily seen that the functional c fulfills the four postulates in § 1, and it defines a capacity. We shall examine the four postulates in § 2. The postulate (a. 4) is obviously fulfilled.

LEMMA 8. *If Φ satisfies the condition $(*)$, then the capacity fulfills* (a. 1).

Proof. Let a compact set K be contained in I_μ, $\mu(\Omega) < +\infty$, and suppose that there is a positive measure $\nu \not\equiv 0$ such that $S_\nu \subset K$ and U^ν is continuous in Ω. Since Φ satisfies the condition $(*)$ and $\nu(\Omega)$ is finite, $U^\nu(P)$ is bounded from above by a finite number M. Hence

$$+\infty = \int U^\mu d\nu = \int U^\nu d\mu < M \cdot \mu(\Omega) < +\infty,$$

---

[5] Φ is called regular by Choquet [4] if it satisfies the continuity principle.
[6] Cf. Ohtsuka [9].

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

which is a contradiction.

LEMMA 9. *If Φ satisfies the continuity principle, then the capacity fulfills* (a.2).

Proof. Let K be a compact set, and suppose that there is not any positive measure $\nu \not\equiv 0$ such that S_ν is contained in K and U^ν is continuous in Ω. Then by the continuity principle there is an Evans-Selberg potential U^μ on K,[7] that is S_μ is contained in K and $I_\mu = K$. Thus K is a polar set.

LEMMA 10. *If Φ satisfies the continuity principle, then every potential U^μ of a measure μ of $\mathfrak{F}$ is quasi continuous. Namely* (a.3) *is fulfilled.*

This is shown in the same way as in the proof of Theorem 2.1 in [7].

3. From the above considerations we obtain that the capacity is admissible and hence by Theorem 9 every analytic set is capacitable.

THEOREM 11. *If Φ satisfies the condition* (*) *and the continuity principle, then every analytic set is capacitable with respect to the capacity induced by the functional*

$$c(\mu) = \mu(\Omega) \qquad \text{for all } \mu \in \mathfrak{M}^+(\Omega).$$

4. In the classical potential theory the capacity is defined as follows: for every compact set K we put

$$\mathfrak{G}_K = \{\mu \in \mathfrak{M}_0^+(\Omega);\ S_\mu \subset K \text{ and } U^\mu \leqq 1 \text{ in } \Omega\},$$
$$g(K) = \sup_{\mu \in \mathfrak{G}_K} \mu(K),$$

and for an arbitrary set X we put

$\mathrm{cap}_i(X) = \sup g(K)$, where K ranges over the class of all compact sets contained in X,

$\mathrm{cap}_e(X) = \inf \mathrm{cap}_i(G)$, where G ranges over the class of all open sets containing X.

This capacity coincides with the capacity defined in § 4.2, provided that Φ satisfies Frostman's maximum principle.[8] This is verified, for example, by Theorem 3.3 in [7]. From Theorem 11 follows

[7] Cf. Ugaheri [10].

[8] We say that Φ satisfies Frostman's maximum principle, if a potential U^μ of a measure μ of $\mathfrak{M}_0^+(\Omega)$ is not greater than 1 on S_μ, then $U^\mu \leqq 1$ everywhere in Ω.

THEOREM 12. *Suppose that Φ satisfies the condition* $(*)$ *and the continuity principle. If an analytic set X is of inner capacity zero with respect to the above capacity, then it is of outer capacity zero.*

Proof. It is sufficient to prove that if X is of inner capacity zero with respect to the above capacity, then it is of inner capacity zero with respect to the capacity defined in §4.2. Let K be a compact set such that $g(K)=0$. Then there is not any positive measure $\nu \not\equiv 0$ such that S_ν is contained in K and U^ν is continuous in Ω since Φ satisfies the condition $(*)$. Hence by Lemma 9, K is a polar set and hence K is of outer capacity zero with respect to the capacity defined in §4.2 by Theorem 2. This completes the proof.

§5. Capacity defined in connection with energy

1. In this section we discuss the capacitability with respect to the capacity induced by a functional defined by the square root of energy, that is,

$$c(\mu)=\sqrt{\int U^\mu d\mu}, \qquad \text{for every } \mu \in \mathfrak{M}^+(\Omega).$$

This functional fulfills the postulates (c.1), (c.3) and (c.4). We assume that Φ is of type positive, that is, for any pair of positives measures μ and ν the inequality $\left(\int U^\mu d\nu\right)^2 \leqq \int U^\mu d\mu \cdot \int U^\nu d\nu$ holds. Then the functional fulfills (c.2) and it defines a capacity. Moreover we assume that Φ satisfies the condition $(*)$ and the continuity principle and that it is ***regular***. Here Φ is said to be regular, when for any point P_0 and for any neighborhood $\omega(P_0)$ of P_0, there exist a positive constant A, depending only on P_0, and a positive measure λ such that

$$(**) \qquad \begin{gathered} S_\lambda \subset \omega(P_0), \quad \lambda(\omega(P_0))=1, \quad \int U^\lambda d\lambda < +\infty \\ \text{and } U^\lambda(P) \leqq A\cdot\Phi(P, P_0) \text{ in } \Omega. \end{gathered}$$

It is well known that the kernel of the α-potential $(0<\alpha<m)$ in the m-dimensional euclidean space is regular.

First we prove

LEMMA 11. *Suppose that Φ is regular. If $U^\mu \geqq 1$ nearly everywhere*[9] *on X,*

[9] We say that a property holds nearly everywhere on X, if the set of points of X where the property fails to hold is of measure zero with respect to every positive measure with finite energy.

then $U^{\mu} \geqq 1$ *at every inner point of* X.

Proof. Let P_0 be an inner point of X, and let $\{\omega_n(P_0)\}$ be a fundamental base of neighborhoods of P_0. Then by the properties $(**)$ we obtain easily that $U^{\mu}(P_0) = \lim_n \int U^{\mu} d\lambda_n$ and hence $U^{\mu}(P_0) \geqq 1$, since every λ_n is of finite energy.

2. Let K be a compact set, on which a measure $\mu \in \mathfrak{M}_0^+(\Omega)$ with finite energy is carried. Then we can find a measure μ_K which minimizes $\int U^{\mu} d\mu - 2\mu(K)$ among all measures $\mu \in \mathfrak{M}_0^+(\Omega)$ carried by K. The potential U^{μ_K} of this measure has the following properties:

$$U^{\mu_K} \geqq 1 \quad \text{nearly everywhere on } K,$$
$$U^{\mu_K} = 1 \quad \mu_K - \text{a.e.}$$

The measure μ_K is called a *capacitary measure* of K. We put $e(K) = \int U^{\mu_K} d\mu_K = \mu_K(K)$. When there is not any measure with finite energy on K, we put $e(K) = 0$. Since Φ is of type positive, $e(K)$ equals inf $\int U^{\nu} d\nu$ taken over all measures $\gamma \in \mathfrak{M}^+(\Omega)$ with finite energy such that $U^{\nu} \geqq 1$ nearly everywhere on K. Using this functional e we put for an arbitrary set X

$$e_i(X) = \sup e(K) \quad \text{over all compact sets } K \subset X$$
$$e_0(X) = \inf e_i(G) \quad \text{over all open sets } G \supset X.$$

3. LEMMA 12. *For every open set* G, $c(G)^2 = e_i(G) = e_0(G)$.

Proof. The equality $e_i(G) = e_0(G)$ is obvious. We write $e(G)$ instead of $e_i(G)$ or $e_0(G)$. First we prove $c(G)^2 \geqq e(G)$. For any positive number ε there exists a measure μ of $\mathfrak{M}^+(\Omega)$ such that $\int U^{\mu} d\mu < c(G)^2 + \varepsilon$ and $U^{\mu} \geqq 1$ in G except $E \in \mathfrak{P}$, where $\mathfrak{P}$ is the class of polar sets with respect to the functional c. Since E is of measure zero with respect to every positive measure with finite energy, $U^{\mu} \geqq 1$ everywhere in G by Lemma 11. Now let K be an arbitrary compact set contained in G. Then we obtain $e(K) \leqq \int U^{\mu} d\mu < c(G)^2 + \varepsilon$. Hence $e(G) \leqq c(G)^2$. Next we prove $e(G) \geqq c(G)^2$. Let $e(G)$ be finite and $\{K_n\}$ be an increasing sequence of compact sets and $G = \cup K_n$, and let μ_n be a capacitary measure of K_n. Since the total measures $\mu_n(K_n)$ are bounded, a subsequence $\{\mu_{n_k}\}$ converges vaguely to a positive measure μ. It is easily seen that $U^{\mu} \geqq 1$ nearly everywhere in G, because Φ satisfies the continuity principle, and hence

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

by Lemma 11, $U^{\mu} \geqq 1$ everywhere in G and μ belongs to $\mathfrak{F}_G$. Thus we obtain $c(G)^2 \leqq \int U^{\mu} d\mu \leqq \lim \int U^{\mu_{n_k}} d\mu_{n_k} \leqq e(G)$.[10]

From this lemma follows

LEMMA 13. *For any set X, $c(X)^2 = e_0(X)$.*[11]

Proof. The inequality $c(X)^2 \leqq e_0(X)$ is immediate. We prove the converse. Let X be a set such that $c(X)^2 < +\infty$. Then for any positive member ε there is a measure $\mu \in \mathfrak{M}^+(\Omega)$ such that $\int U^{\mu} d\mu < c(X)^2 + \varepsilon$ and $U^{\mu} \geqq 1$ on X except $E \in \mathfrak{P}$, where E is contained in some I_{ν}, $\int U^{\nu} d\nu < +\infty$. For any positive integer n we put

$$G_n = \left\{P \in \Omega;\ U^{\mu}(P) > \frac{n}{n-1}\right\}$$

$$G'_n = \{P \in \Omega;\ U^{\nu}(P) > n\}.$$

Then $G_n \supset X - E$, $G'_n \supset E$ and hence $G_n \cup G'_n \supset X$. Since $\frac{n-1}{n}\mu$ and $\frac{1}{n}\nu$ belong to $\mathfrak{F}_{G_n}$ and $\mathfrak{F}_{G'_n}$ respectively, we have by Lemma 12 that

$$e_0(X) \leqq c(G_n \cup G'_n)^2 \leqq \text{ the energy of } \frac{n-1}{n}\mu + \frac{1}{n}\nu$$

$$\leqq \frac{(n-1)^2}{n^2}\{c(X)^2 + \varepsilon\} + \frac{1}{n^2}\int U^{\nu} d\nu + \frac{2(n-1)}{n^2}\sqrt{\int U^{\mu} d\mu \cdot \int U^{\nu} d\nu}.$$

Here we make n tend to infinity and we obtain $e_0(X) \leqq c(X)^2 + \varepsilon$ and hence $e_0(X) \leqq c(X)^2$.

4. Now we can prove

LEMMA 14. *The capacity induced by the functional c fulfills the postulate* (a. 4).

Proof. Let X be a set such that $c(X) < +\infty$ and ε be an arbitrary positive number. Since $e_0(X)$ is finite, there is an open set $G \supset X$ such that $e(G) < e_0(X) + \varepsilon = c(X)^2 + \varepsilon$. Let μ be a capacitary measure of G. Then $e(G) = \int U^{\mu} d\mu = \mu(\Omega)$ and $U^{\mu} \geqq 1$ in G. Consequently μ belongs to $\mathfrak{F}_X$ and $\int U^{\mu} d\mu < c(X)^2 + \varepsilon$ and $\mu(\Omega) < c(X)^2 + \varepsilon$. Therefore we conclude that the capacity induced by the functional c fulfills the postulate (a. 4).

[10] μ is called a capacitary measure of G.

[11] Cf. Aronszajn-Smith [1].

https://doi.org/10.1017/S0027763000007583 Published online by Cambridge University Press

LEMMA 15. *The capacity induced by the functional c fulfills the postulates* (a.1), (a.2) *and* (a.3).

Proof. Suppose that there is a positive measure $\mu \not\equiv 0$ carried by a compact set K such that U^{μ} is continuous in Ω. Then $c(K)$ is positive and by Theorem 2, K is not a polar set, and hence the capacity fulfills (a.1). Now suppose that a compact set K is not a polar set. Then it is shown that there is a positive measure $\mu \not\equiv 0$ carried by K such that $\int U^{\mu} d\mu$ is finite. Then by the continuity principle there is a positive measure $\nu \not\equiv 0$ carried by K such that U^{ν} is continuous in Ω. Hence the capacity fulfills (a.2). We can prove by the same argument as in the proof of Theorem 2.1 in [7] that the capacity fulfills (a.3).

From this lemma and Theorem 9 we obtain

THEOREM 13. *Suppose that a regular kernel Φ is of type positive and satisfies the condition* (*) *and the continuity principle. Then every analytic set is capacitable with respect to the capacity induced by the functional c in* § 6.1.

REFERENCES

[1] N. Aronszajn-K. T. Smith: Functional spaces and functional completion, Ann. Inst. Fourier, **6** (1956), 125–185.
[2] N. Bourbaki: Intégration, Paris, 1952.
[3] G. Choquet: Theory of capacities, Ann. Inst. Fourier, **5** (1955), 131-295.
[4] G. Choquet: Les noyaux réguliers en théorie du potentiel, C. R. Acad. Sci., Paris, **243** (1956), 635–638.
[5] B. Fuglede: On the theory of potentials in locally compact spaces, to appear.
[6] M. Kishi: Inferior limit of a sequence of potentials, Proc. Japan Acad., **33** (1957), 314-319.
[7] M. Kishi: Capacities of borelian sets and the continuity of potentials, Nagoya Math. Journ., **12** (1957), 195–219.
[8] M. Kishi: On the capacitability of analytic sets, Proc. Japan Acad., **35** (1959), 158-160.
[9] M. Ohtsuka: Les relations entre certains principes en théorie du potentiel, Proc. Japan Acad., **33** (1957), 37–40.
[10] T. Ugaheri: On the general capacities and potentials, Bull. Tokyo Inst. Tech., **4** (1953), 149–179.

Mathematical Institute
Nagoya University