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NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A GENERALIZED STIELTJES INTEGRAL

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Abstract

In Russell (1973) a Riemann-type necessary and sufficient condition was given for the existence of $\int_a^b f(d^k g/dx^{k-1})$ (defined also in Russell (1975)) when f was bounded and g was k-convex in $[a', b']$. In this paper we present necessary and sufficient conditions for the existence of a particular Stieltjes-type integral without imposing a convexity condition upon g. These conditions are used to obtain an additivity result for the integral over adjoining intervals without any additional restrictions being imposed upon the functions involved.

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Introduction

We shall follow the notation of Hildebrandt (1963), and speak of norm-integrals and σ-integrals, the first being defined as a limit when the maximum length of sub-intervals of a subdivision tends to zero, while the latter is defined as a limit under successive refinements of a subdivision. We shall accordingly prefix the integrals with the symbols 'N' or 'σ'.

It is well known in the theory of Riemann–Stieltjes integration that the additive property

$$(N)\int_a^b f\,dg = (N)\int_a^c f\,dg + (N)\int_c^b f\,dg, \tag{1}$$

where $a<c<b$, does not hold unconditionally. For example, if both integrals on the right-hand side of (1) exist, the integral on the left does not exist if f and g have a common discontinuity at c. Such restrictions can be removed by considering

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σ-integrals in which case the additivity result (1) is free of constraints and can be stated: 'If any two of the integrals $(\sigma)\int_a^b f\,dg$, $(\sigma)\int_a^c f\,dg$, $(\sigma)\int_c^b f\,dg$ exist, so does the third, and

$$(\sigma)\int_a^b f\,dg = (\sigma)\int_a^c f\,dg + (\sigma)\int_c^b f\,dg.\text{'}$$

In Russell (1973) an analogue of (1) was established, but again restrictions upon the behaviour of the functions f and g in a neighbourhood of c had to be imposed in order to achieve 'additivity'. The integrals involved were norm-integrals. Even if second-order σ-integrals are introduced, additivity is not achieved. This is illustrated by the following example: If

$$f(x) = 1 \quad \text{and} \quad g(x) = |x| \quad \text{for all } x,$$

then

$$(\sigma)\int_{-1}^{1} f\frac{d^2g}{dx} = (\sigma)\int_{-1}^{0} f\frac{d^2g}{dx} = (\sigma)\int_{0}^{1} f\frac{d^2g}{dx} = 2.$$

If we introduce a slightly modified second-order σ-integral, an additivity result can be obtained without imposing extra restrictions upon f and g. To achieve this somewhat more desirable result we first obtain necessary and sufficient conditions for the existence of the integral, denoted by $(\sigma)\int_a^b f(d^2g/dx)$, and show that they exclude the possibility of f being discontinuous and g non-differentiable at the same point.

1. Notation and preliminaries

As mentioned previously we will be concerned with a second-order Riemann–Stieltjes σ-integral. In order to define such an integral we need a particular type of subdivision of the closed interval $[a, b]$.

DEFINITION 1. Γ *subdivisions*. We will denote by $\Gamma(x_{-1}, x_0, \ldots, x_n, x_{n+1})$, or often more briefly by Γ, a subdivision of the interval $[a, b]$ of the form

$$a' \leqslant x_{-1} < a = x_0 < x_1 < \ldots < x_n = b < x_{n+1} \leqslant b',$$

where a' and b' are fixed, and $a' < a < b < b'$. For convenience, such a set of points will be called a Γ subdivision of $[a, b]$, even though it is not strictly a subdivision of $[a, b]$.

We will have need to speak of synchronized Γ subdivisons which are defined as follows:

DEFINITION 2. *Synchronized* Γ *subdivisions.* Let $a<c<b$, so that $[a,c]$ and $[c,b]$ are adjoining closed intervals. Let

$$\Gamma_1(x_{-1}, x_0, \ldots, x_m. x_{m+1}) \quad \text{and} \quad \Gamma_2(y_{-1}, y_0, \ldots, y_n, y_{n+1})$$

be, respectively, Γ subdivisions of $[a,c]$ and $[c,b]$. If $x_{m-1}=y_{-1}$ and $x_{m+1}=y_1$, we will say that Γ_1 and Γ_2 are synchronized.

DEFINITION 3. *Refinements.* We will say that Γ_1 is a refinement of Γ_2, and write $\Gamma_1 \geqslant \Gamma_2$, if every point of $\Gamma_2 \cap [a,b]$ beongs to $\Gamma_1 \cap [a,b]$, and if $x_{-1}^{(1)}$, $x_{m+1}^{(1)} \in \Gamma_1$ and $x_{-1}^{(2)}$, $x_{n+1}^{(2)} \in \Gamma_2$ satisfy the conditions $a' \leqslant x_{-1}^{(2)} \leqslant x_{-1}^{(1)} < a$, $b < x_{m+1}^{(1)} \leqslant x_{n+1}^{(2)} \leqslant b'$.

DEFINITION 4. *The integral.* Consider a $\Gamma(x_{-1}, x_0, \ldots, x_n, x_{n+1})$ subdivision of $[a,b]$, and suppose that f and g are functions defined on $[a',b']$. The integral $(\sigma)\int_a^b f(d^2g/dx)$ is the real number L, if it exists uniquely, such that for each $\varepsilon>0$ there is a Γ_ε subdivision with the property: whenever $\Gamma \geqslant \Gamma_\varepsilon$ and $x_{i-1} \leqslant \xi_i \leqslant x_{i+1}$ for $i=1,2,\ldots,n-1$, then

$$\Bigg| L - \Bigg\{ \tfrac{1}{2} f(a)\, V_2(g;\, x_{-1}, x_0, x_1) + \sum_{i=1}^{n-1} f(\xi_i)\, V_2(g;\, x_{i-1}, x_i, x_{i+1}) + \tfrac{1}{2} f(b)\, V_2(g;\, x_{n-1}, x_n, x_{n+1}) \Bigg\} \Bigg| < \varepsilon$$

whenever $\Gamma \geqslant \Gamma_\varepsilon$.

For convenience, we shall often write the triple $\{x_{i-1}, x_i, x_{i+1}\}$ as T_i, and write the approximating sums for the integral as

$$\sum_{\Gamma} f V_2(g;\, T) \quad \text{or} \quad \sum_{i=0}^{n} f V_2(g;\, T_i),$$

where

$$V_2(g;\, T_i) = \delta_i \left[\frac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i} \quad \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} \right],$$

and where $\delta_i = +1$ when $i = 1, \ldots, n-1$, and $\delta_i = \frac{1}{2}$ when $i=0$ and n.

REMARK. If the integral exists, it is clear from Definitions 3 and 4 that it is independent of a' and b'.

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DEFINITION 5. We define the oscillation function $\omega f V_2(g; [a,b])$ to be equal to

$$\sup_{\Gamma_1,\Gamma_2} \left| \sum_{\Gamma_1} f V_2(g; T) - \sum_{\Gamma_2} f V_2(g; T) \right|,$$

where the supremum is taken over all Γ_1 and Γ_2 subdivisions of $[a,b]$ and the associated ξ_i's as in Definition 4.

Finally, for convenience, we include the well-known definition of oscillation of a function over an interval.

DEFINITION 6. The oscillation of f on a closed interval $I = [a,b]$ is defined to be

$$\operatorname{osc}(f; I) = \sup_{x,y \in I} |f(x) - f(y)|.$$

2. Necessary and sufficient conditions for integrability

We begin with a Cauchy-type necessary and sufficient condition.

THEOREM 1. *A necessary and sufficient condition that $(\sigma) \int_a^b f(d^2 g/dx)$ exists is that for each $\varepsilon > 0$ there is a Γ_ε subdivision of $[a,b]$ such that whenever $\Gamma_1 \geq \Gamma_\varepsilon$ and $\Gamma_2 \geq \Gamma_\varepsilon$,*

$$\left| \sum_{\Gamma_1} f V_2(g; T) - \sum_{\Gamma_2} f V_2(g; T) \right| < \varepsilon. \tag{2}$$

PROOF. The necessity of the condition follows in the usual way.

For the sufficiency, we assume that for each $\varepsilon > 0$ there exists a Γ_ε subdivision such that

$$\left| \sum_{\Gamma_1} f V_2(g; T) - \sum_{\Gamma_2} f V_2(g; T) \right| < \varepsilon$$

whenever $\Gamma_1 \geq \Gamma_\varepsilon$ and $\Gamma_2 \geq \Gamma_\varepsilon$. We construct a sequence $\{\Gamma_n\}$ of subdivisions such that $\Gamma_n \geq \Gamma_{n-1}$, and whenever $\Gamma' \geq \Gamma_n$ and $\Gamma'' \geq \Gamma_n$,

$$\left| \sum_{\Gamma'} f V_2(g; T) - \sum_{\Gamma''} f V_2(g; T) \right| < \frac{1}{n}.$$

Hence,

$$\left| \sum_{\Gamma_{n+1}} f V_2(g; T) - \sum_{\Gamma_{n+m}} f V_2(g; T) \right| < \frac{1}{n}$$

for all m and n. Consequently $\{\sum_{\Gamma_n} f V_2(g; T)\}$ is a Cauchy sequence of real numbers, and so has a limit L, say. Hence, for each $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\left| \sum_{\Gamma_n} f V_2(g; T) - L \right| < \varepsilon \quad \text{whenever } n > N(\varepsilon).$$

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Furthermore,

$$\Big|\sum_{\Gamma_n} fV_2(g;\, T)-\sum_{\Gamma} fV_2(g;\, T)\Big|<\varepsilon$$

whenever $n-1>\varepsilon^{-1}$ and $\Gamma\geqslant\Gamma_{n-1}$. If we now choose $n>\max[N(\varepsilon), 1+\varepsilon^{-1}]$, and define $\Gamma_\varepsilon=\Gamma_{n-1}$, then it follows that $|\sum_\Gamma fV_2(g;\, T)-L|<2\varepsilon$ whenever $\Gamma\geqslant\Gamma_\varepsilon$. This concludes the proof.

REMARK. Each summation in (2) is of course multi-valued because of the choice of ξ_i in Definition 4. The proof, however, remains valid for all such choices of ξ_i.

LEMMA 1. *Let Γ_1 and Γ_2 be two Γ subdivisons of $[a, b]$ such that $\Gamma_2\geqslant\Gamma_1$. Then*

$$\Big|\sum_{\Gamma_1} fV_2(g;\, T)-\sum_{\Gamma_2} fV_2(g;\, T)\Big|\leqslant\sum_{i=1}^{n}\omega fV_2(g;\, [x_{i-1}, x_i]),$$

where the $x_i\in\Gamma_1$.

PROOF. To keep the details simple we consider a particular Γ_1 subdivision. The particular case will exhibit all properties of the general case. Accordingly, let Γ_1 be the subdivision $x_{-1}, x_0, x_1, x_2, x_3$, where $x_{-1}<a=x_0<x_1<x_2=b<x_3$. Let $\Gamma_2\geqslant\Gamma_1$ be obtained by inserting l and m extra points of subdivision in (x_0, x_1) and (x_1, x_2) respectively. Hence Γ_2 consists of points $y_{-1}, y_0, \dots, y_{l+m+3}$, where

$$y_{-1}<y_0=a<y_1<\dots<y_{l+1}=x_1<\dots<y_{l+m+2}=x_2<y_{l+m+3}=x_3.$$

Then, suppressing the arguments of f for convenience, we have

$$\begin{aligned}
&\sum_{\Gamma_1} fV_2(g;\, T)-\sum_{\Gamma_2} fV_2(g;\, T)\\
&\quad=\tfrac12 fV_2(g;\, x_{-1}, x_0, x_1)+fV_2(g;\, x_0, x_1, x_2)+\tfrac12 fV_2(g;\, x_1, x_2, x_3)\\
&\qquad-\tfrac12 fV_2(g;\, y_{-1}, y_0, y_1)-\sum_{i=1}^{l+m+1} fV_2(g;\, y_{i-1}, y_i, y_{i+1})\\
&\qquad\qquad-\tfrac12 fV_2(g;\, y_{l+m+1}, y_{l+m+2}, y_{l+m+3})\\
&\quad=\Big[\tfrac12 fV_2(g;\, x_{-1}, x_0, x_1)+\tfrac12 fV_2(g;\, x_0, x_1, x_2)-\tfrac12 fV_2(g;\, y_{-1}, y_0, y_1)\\
&\qquad-\sum_{i=1}^{l} fV_2(g;\, y_{i-1}, y_i, y_{i+1})-\tfrac12 fV_2(g;\, y_l, y_{l+1}, y_{l+2})\Big]\\
&\qquad+\Big[\tfrac12 fV_2(g;\, x_0, x_1, x_2)+\tfrac12 fV_2(g;\, x_1, x_2, x_3)-\tfrac12 fV_2(g;\, y_l, y_{l+1}, y_{l+2})\\
&\qquad-\sum_{i=l+2}^{l+m+1} fV_2(g;\, y_{i-1}, y_i, y_{i+1})-\tfrac12 fV_2(g;\, y_{l+m+1}, y_{l+m+2}, y_{l+m+3})\Big]\\
&\quad=\Big[\sum_{\Gamma_1'} fV_2(g;\, T)-\sum_{\Gamma_2'} fV_2(g;\, T)\Big]+\Big[\sum_{\Gamma_1''} fV_2(g;\, T)-\sum_{\Gamma_2''} fV_2(g;\, T)\Big],
\end{aligned}$$

where Γ_1' and Γ_2' are Γ subdivisions of $[x_0, x_1]$ and Γ_1'' and Γ_2'' are Γ subdivisions of $[x_1, x_2]$. Hence,

$$\left|\sum_{\Gamma_1} fV_2(g;\, T) - \sum_{\Gamma_2} fV_2(g;\, T)\right| \leqslant \sum_{i=1}^{2} \omega f V_2(g;\, [x_{i-1}, x_i]).$$

The extension of this result to Γ_1 subdivisions containing more than five points is straightforward and, as indicated earlier, the details will be omitted.

THEOREM 2. *A necessary and sufficient condition that* $(\sigma)\int_a^b f(d^2 g/dx)$ *exists is that*

$$\inf_{\Gamma} \sum_{\Gamma} \omega f V_2(g;\, I) \equiv \inf_{\Gamma} \sum_{i=1}^{n} \omega f V_2(g;\, [x_{i-1}, x_i]) = 0. \tag{3}$$

PROOF. We first show that the condition is sufficient. Accordingly, suppose that (3) holds. Then, for each $\varepsilon > 0$ there exists a Γ_ε subdivision such that

$$\sum_{\Gamma_\varepsilon} \omega f V_2(g;\, I) < \varepsilon. \tag{4}$$

Now suppose that $\Gamma_1 \geqslant \Gamma_\varepsilon$. Then, using Lemma 1, we obtain

$$\left|\sum_{\Gamma_\varepsilon} fV_2(g;\, T) - \sum_{\Gamma_1} fV_2(g;\, T)\right| \leqslant \sum_{\Gamma_\varepsilon} \omega f V_2(g;\, I) < \varepsilon.$$

The existence of the integral now follows from Theorem 1.

To prove the condition necessary we assume that for each $\varepsilon > 0$ there exists Γ_ε such that whenever $\Gamma_1 \geqslant \Gamma_\varepsilon$ and $\Gamma_2 \geqslant \Gamma_\varepsilon$,

$$\left|\sum_{\Gamma_1} fV_2(g;\, T) - \sum_{\Gamma_2} fV_2(g;\, T)\right| < \varepsilon. \tag{5}$$

Let $\Gamma \geqslant \Gamma_\varepsilon$ and let Γ consist of the points $x_{-1}, x_0, \dots, x_n, x_{n+1}$. For each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$, Definition 5 shows that we can find subdivisions Γ_i' and Γ_i'' of $[x_{i-1}, x_i]$ such that

$$\left|\sum_{\Gamma_i'} fV_2(g;\, T) - \sum_{\Gamma_i''} fV_2(g;\, T)\right| > \omega f V_2(g;\, [x_{i-1}, x_i]) - \frac{\varepsilon}{n}.$$

By interchanging Γ_i' and Γ_i'' if necessary we can also have

$$\sum_{\Gamma_i'} fV_2(g;\, T) - \sum_{\Gamma_i''} fV_2(g;\, T) \geqslant 0,$$

and the modulus signs in the previous inequality can be omitted. If we put $\Gamma' = \bigcup_{i=1}^{n} \Gamma_i'$ and $\Gamma'' = \bigcup_{i=1}^{n} \Gamma_i''$, then $\Gamma' \geqslant \Gamma_\varepsilon$, $\Gamma'' \geqslant \Gamma_\varepsilon$, and

$$0 \leqslant \sum_{i=1}^{n} \omega f V_2(g;\, [x_{i-1}, x_i]) \leqslant \sum_{i=1}^{n} \left[\sum_{\Gamma_i'} fV_2(g;\, T) - \sum_{\Gamma_i''} fV_2(g;\, T)\right] + \varepsilon$$

$$< \varepsilon + \varepsilon = 2\varepsilon \tag{6}$$

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provided that $\Gamma'_1, \ldots, \Gamma'_n$ are synchronized, and $\Gamma''_1, \ldots, \Gamma''_n$ are also synchronized. When the subdivisions are not synchronized we can make use of (5) and this will have the effect of introducing an extra ε in (6). The required result now follows.

COROLLARY. *If* $(\sigma)\int_a^b f(d^2g/dx)$ *exists, then*

$$\inf_{\Gamma} \sum_{i=1}^{n-1} \operatorname{osc}(f;\, [x_{i-1}, x_{i+1}])\,|V_2(g;\, T_i)| = 0.$$

PROOF. Let Γ_1 and Γ_2 be identical Γ subdivisions of an interval $[c, d]$. Denote their points by $x_{-1}, x_0, x_1, x_2, x_3$, where $x_{-1} < x_0 = c < x_1 < x_2 = d < x_3$. Since ξ_i in Definition 4 is arbitrary within the subinterval $[x_{i-1}, x_{i+1}]$, we choose $\xi_1 = \alpha$ and $\xi_1 = \beta$, respectively for the Γ_1 and Γ_2 subdivisions. It then follows from Definition 5 that

$$\omega f V_2(g;\, [c, d]) \geqslant |f(\alpha) - f(\beta)|\,|V_2(g;\, x_0, x_1, x_2)|$$

whenever α and β are in $[c, d]$. Hence, replacing $[c, d]$ by $[x_{i-1}, x_{i+1}]$ and making other obvious changes, we have

$$\omega f V_2(g;\, [x_{i-1}, x_{i+1}]) \geqslant \operatorname{osc}(f;\, [x_{i-1}, x_{i+1}])\,|V_2(g;\, x_{i-1}, x_i, x_{i+1})|.$$

The required result now follows readily from Theorem 2.

The following discussion motivates the next theorem. Consider the function g defined by

$$g(x) = \beta x, \quad x \geqslant 0,$$

$$g(x) = \alpha x, \quad x \leqslant 0,$$

where α and β are constants. Consider a Γ subdivision of $[-1, 1]$, and let $0 = x_p \in \Gamma$. Then, if $f(x) \equiv 1$,

$$\sum_{\Gamma} f V_2(g;\, T) = (\beta - \alpha) f(\xi_p),$$

where $x_{p-1} \leqslant \xi_p \leqslant x_{p+1}$. Consequently, if $(\sigma)\int_{-1}^{1} f(d^2g/dx)$ exists, it must have the value $(\beta - \alpha) f(0)$. Hence, if f is discontinuous at 0, we must have $\beta = \alpha$, in which case g is differentiable at 0. On the other hand, if $\alpha = g'_-(0) \neq \beta = g'_+(0)$, then f must be continuous at 0.

THEOREM 3. *If* $(\sigma)\int_a^b f(d^2g/dx)$ *exists, and* $a < c < b$, *then the conditions* f *discontinuous at* c, *and* g *non-differentiable at* c *cannot occur simultaneously.*

PROOF. If $(\sigma)\int_a^b f(d^2g/dx)$ exists, then it follows from Theorem 2, Corollary, that for each $\varepsilon>0$ there exists Γ_ε such that whenever $\Gamma \geqslant \Gamma_\varepsilon$,

$$\sum_{i=1}^{n-1} \sup_{\xi_i,\eta_i \in I_i} |f(\xi_i)-f(\eta_i)|\,|V_2(g;\, x_{i-1}, x_i, x_{i+1})| < \varepsilon,$$

where $I_i = [x_{i-1}, x_{i+1}]$.

If c is a point of discontinuity of f, then by including c in Γ it follows that if $c = x_p$,

$$\sup_{x,y \in I_p} |f(x)-f(y)|\,|Q_1(g;\, x_{p+1}, c)-Q_1(c, x_{p-1})| < \varepsilon,$$

where

$$Q_1(g;\, x, y) = \frac{g(y)-g(x)}{y-x}.$$

Since c is a point of discontinuity of f, there exists a positive number k such that

$$\sup_{x,y \in I_p} |f(x)-f(y)| > k$$

no matter how small $x_{p+1}-x_{p-1}$. Consequently, no matter how small $x_{p+1}-x_{p-1}$,

$$|Q_1(g;\, x_{p+1}, c)-Q_1(g;\, c, x_{p-1})| < \varepsilon/k.$$

Since x_{p-1} and x_{p+1} are independent, it follows from Cauchy's principle of convergence that $g'_-(c)$ and $g'_+(c)$ both exist, and are equal. Thus, we have shown that if f is discontinuous at c, then g must be differentiable at that point. It now follows that if g is not differentiable at c, then f must be continuous there. This completes the proof of the theorem.

COROLLARY. *If $(\sigma)\int_a^b f(d^2g/dx)$ exists, then the conditions f discontinuous and g non-differentiable on the right at a cannot occur simultaneously. Similarly f discontinuous and g non-differentiable on the left at b cannot occur simultaneously.*

3. An application

THEOREM 4. *If $a<c<b$, and any two of the integrals*

$$(\sigma)\int_a^c f\frac{d^2g}{dx}, \quad (\sigma)\int_c^b f\frac{d^2g}{dx} \quad \text{and} \quad (\sigma)\int_a^b f\frac{d^2g}{dx}$$

exist, then so does the other, and

$$(\sigma)\int_a^b f\frac{d^2g}{dx} = (\sigma)\int_a^c f\frac{d^2g}{dx} + (\sigma)\int_c^b f\frac{d^2g}{dx}. \tag{7}$$

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PROOF. We shall only prove one case, the proofs of other cases being similar to the one given.

Accordingly, assume that $(\sigma)\int_a^b f(d^2g/dx)$ exists. Then it follows from Theorem 2 that the other two integrals in (7) also exist. Consequently, given $\varepsilon>0$ there exist subdivisions Γ'_ε and Γ''_ε of $[a,c]$ and $[c,b]$ such that

(8) $$\Big|\sum_{\Gamma'} fV_2(g;\,T)-L'\Big|<\tfrac{1}{2}\varepsilon \quad \text{whenever } \Gamma'\geqq\Gamma'_\varepsilon$$

and

(9) $$\Big|\sum_{\Gamma''} fV_2(g;\,T)-L''\Big|<\tfrac{1}{2}\varepsilon \quad \text{whenever } \Gamma''\geqq\Gamma''_\varepsilon,$$

where

$$L'=(\sigma)\int_a^c f\frac{d^2g}{dx} \quad \text{and} \quad L''=(\sigma)\int_c^b f\frac{d^2g}{dx}.$$

Let Γ'_ε consist of the points $x_{-1}, x_0=a, x_1,\dots,x_m=c, x_{m+1}$, and let Γ''_ε consist of the points $y_{-1}, y_0=c, y_1,\dots,y_n=b, y_{n+1}$. If the subdivisions Γ'_ε and Γ''_ε are not synchronized, several cases can arise. One of these will be considered; others can be dealt with in a similar way. Consequently, suppose that

$$y_{-1}<x_{m-1}<x_m=c=y_0<x_{m+1}<y_1.$$

Let Γ^*_ε be the refinement of Γ''_ε obtained by choosing $y^*_{-1}=x_{m-1}$ and introducing an additional point $y^*_1=x_{m+1}$. Then Γ^*_ε consists of the points $y^*_{-1}=x_{m-1}$, $y_0=x_m$, $y^*_1=x_{m+1}, y_1,\dots,y_n=b, y_{n+1}$. Then

$$\Big|\sum_{\Gamma''} fV_2(g;\,T)-L''\Big|<\tfrac{1}{2}\varepsilon \quad \text{whenever } \Gamma''>\Gamma^*_\varepsilon,$$

and from (8)

$$\Big|\sum_{\Gamma'} fV_2(g;\,T)-L'\Big|<\tfrac{1}{2}\varepsilon \quad \text{whenever } \Gamma'\geqq\Gamma'_\varepsilon.$$

We observe that Γ'_ε and Γ^*_ε are now synchronized. Consequently there is no loss of generality in assuming that (8) and (9) are valid for synchronized subdivisions Γ'_ε and Γ''_ε.

Let $\Gamma_\varepsilon=\Gamma'_\varepsilon\cup\Gamma''_\varepsilon$, and let Γ be any subdivision of $[a,b]$ such that $\Gamma\geqq\Gamma_\varepsilon$. Then we can write $\Gamma=\Gamma'\cup\Gamma''$, where by the above discussion, Γ' and Γ'' are synchronized subdivisions of $[a,c]$ and $[c,b]$ respectively. If Γ' and Γ'' consist, respectively, of the points

$$x_{-1},\quad x_0=a,\dots,x_m=c,\quad x_{m+1}$$

and

$$x_{m-1},\quad x_m=c,\quad x_{m+1},\dots,x_{m+n}=b,\quad x_{m+n+1},$$

then

$$\sum_{\Gamma} f V_2(g;\, T) = \left\{\tfrac{1}{2} f(a)\, V_2(g;\, T_0) + \sum_{i=1}^{m-1} f(\xi_i)\, V_2(g;\, T_i) + \tfrac{1}{2} f(c)\, V_2(g;\, T_m)\right\}$$

$$+ \left\{\tfrac{1}{2} f(c)\, V_2(g;\, T_m) + \sum_{i=m+1}^{m+n-1} f(\xi_i)\, V_2(g;\, T_i) + \tfrac{1}{2} f(b)\, V_2(g;\, T_{m+n})\right\}$$

$$+ \{f(\xi_m) - f(c)\}\, V_2(g;\, T_m). \tag{10}$$

It now follows from the proof of Theorem 3 that the last term in (10) tends to zero under refinement irrespective of whether f is continuous or discontinuous at c. Hence, from (10), the limit of $\sum_{\Gamma} f V_2(g;\, T)$ under refinement exists by assumption and equals $L' + L''$, as required.

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References

T. H. Hildebrandt (1963), *Introduction to the theory of integration* (Pure and Appl. Math. Vol. 13, Academic Press, New York).

A. M. Russell (1973), 'Functions of bounded kth variation, and Stieltjes-type integrals' (*Doctoral dissertation*, University of Melbourne).

A. M. Russell (1975), 'Stieltjes-type integrals', *J. Austral. Math. Soc.* (*Ser A*) **20**, 431–448.

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