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Bohr–Rogosinski radius for a certain class of close-to-convex harmonic mappings

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Abstract. Let $\mathcal{B}$ be the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}$. If $f \in \mathcal{B}$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ for $|z| = r \leq 1/3$ and $1/3$ cannot be improved. This inequality is called Bohr inequality and the quantity $1/3$ is called Bohr radius. If $f \in \mathcal{B}$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $|\sum_{n=0}^{N} a_n z^n| < 1$ for $|z| < 1/2$ and the radius $1/2$ is the best possible for the class $\mathcal{B}$. This inequality is called Bohr–Rogosinski inequality and the corresponding radius is called Bohr–Rogosinski radius. Let $\mathcal{H}$ be the class of all complex-valued harmonic functions $f = h + \bar{g}$ defined on the unit disk $\mathbb{D}$, where h and g are analytic in $\mathbb{D}$ with the normalization $h(0) = h'(0) - 1 = 0$ and $g(0) = 0$. Let $\mathcal{H}_0 = \{f = h + \bar{g} \in \mathcal{H} : g'(0) = 0\}$. For $\alpha \geq 0$ and $0 \leq \beta < 1$, let

$$\mathcal{W}^0_{\mathcal{H}}(\alpha, \beta) = \{f = h + \overline{g} \in \mathcal{H}_0 : \operatorname{Re}\left(h'(z) + \alpha z h''(z) - \beta\right) > |g'(z) + \alpha z g''(z)|, \ \ z \in \mathbb{D}\}$$

be a class of close-to-convex harmonic mappings in $\mathbb{D}$. In this paper, we prove the sharp Bohr–Rogosinski radius for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha, \beta)$.

1 Introduction

Harmonic mappings play the natural role in parameterizing minimal surfaces in the context of differential geometry. Planner harmonic mappings have application not only in the differential geometry but also in the various field of engineering, physics, operations research, and other intriguing aspects of applied mathematics. The theory of harmonic functions has been used to study and solve fluid flow problems (see [10]). The theory of univalent harmonic functions having prominent geometric properties like starlikeness, convexity, and close-to-convexity appear naturally while dealing with planner fluid dynamical problems. For instance, the fluid flow problem on a convex domain satisfying an interesting geometric property has been extensively studied by Aleman and Constantin [10]. With the help of geometric properties of harmonic mappings, Constantin and Martin [19] have obtained a complete solution of classifying all two-dimensional fluid flows.

In this paper, our purpose is to investigate several Bohr-type inequalities which will be harmonic analog of the inequalities for bounded analytic functions. Below,

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we first recall the classical result of Harald Bohr discovered in 1914, which generates intensive research activity, what is called Bohr phenomenon.

Let $\mathcal{B}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $|f(z)| < 1$ in $\mathbb{D}$. In the study of Dirichlet series, in 1914, Bohr [17] discovered the following interesting phenomenon that if $f \in \mathcal{B}$, then its associated majorant series:

$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \le 1 \ \text{ for } \ |z| = r \le \frac{1}{3}.$$

The constant $1/3$, known as the Bohr radius for the class $\mathcal{B}$, is the best possible. Furthermore, for $\psi_a(z) = (a - z)/(1 - az)$, $a \in [0, 1)$, it follows easily that $M_{\psi_a}(r) > 1$ if, and only if, $r > 1/(1 + 2a)$, and hence the radius $1/3$ is optimal as $a \to 1$. For the recent advancements of the Bohr-type inequalities, we refer to the articles [26, 35, 38, 44, 45] and references therein.

The Bohr inequality can be written in the following distance formulation:

$$d\left(\sum_{n=0}^{\infty} |a_n z^n|, |a_0|\right) = \sum_{n=1}^{\infty} |a_n z^n| \le 1 - |f(0)| = d(f(0), \partial\mathbb{D}), \tag{1.1}$$

for $|z| = r \le 1/3$ and the constant $1/3$ is independent on the coefficients of the Taylor series of f, where d is the Euclidean distance and $\partial f(\mathbb{D})$ is the boundary of $f(\mathbb{D})$. In view of the distance form (1.1), the notion of the Bohr phenomenon can be generalized to a class $\mathcal{G}$ consisting of analytic functions f in $\mathbb{D}$ which take values in a given domain $\Omega \subseteq \mathbb{C}$ such that $f(\mathbb{D}) \subset \Omega$ and the class $\mathcal{G}$ is said to satisfy the Bohr phenomenon if there exists largest radius $r_\Omega \in (0, 1)$ such that the inequality (1.1) holds for $|z| = r \le r_\Omega$ and for all functions $f \in \mathcal{G}$. The largest radius r_Ω is called the Bohr radius for the class $\mathcal{G}$.

A complex-valued function $f(z) = u(x, y) + iv(x, y)$ is called harmonic in $\mathbb{D}$ if both u and v satisfy the Laplace's equation $\bigtriangledown^2 u = 0$ and $\bigtriangledown^2 v = 0$, where

$$\bigtriangledown^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It is well known that under the assumption $g(0) = 0$, the harmonic function f has the unique canonical representation $f = h + \overline{g}$, where h and g are analytic functions in $\mathbb{D}$, respectively called, analytic and co-analytic parts of f. If, in addition, f is univalent, then we say that f is univalent harmonic on a domain Ω. A locally univalent harmonic mapping $f = h + \overline{g}$ is sense-preserving whenever its Jacobian $J_f(z) := |f_z(z)|^2 - |f_{\bar{z}}(z)| = |h'(z)|^2 - |g'(z)|^2 > 0$ for $z \in \mathbb{D}$.

Let $\mathcal{H}$ be the class of all complex-valued harmonic functions $f = h + \bar{g}$ defined on the unit disk $\mathbb{D}$, where h and g are analytic $\mathbb{D}$ with the normalization $h(0) = h'(0) - 1 = 0$ and $g(0) = 0$. Let $\mathcal{H}_0$ be defined by $\mathcal{H}_0 = \{f = h + \bar{g} \in \mathcal{H} : g'(0) = 0\}$. Then each $f = h + \bar{g} \in \mathcal{H}_0$ has the following form:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.2}$$

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In 2013, Ponnusamy *et al.* [46] considered the following class of harmonic mappings:

$$\mathcal{P}^0_{\mathcal{H}} = \{f = h + \bar{g} \in \mathcal{H} : \operatorname{Re} h'(z) > |g'(z)| \ \text{ with } \ g'(0) = 0 \ \text{ for } \ z \in \mathbb{D}\}$$

and motivated by the class $\mathcal{P}^0_{\mathcal{H}}$, Li and Ponnusamy [36] have studied the class $\mathcal{P}^0_{\mathcal{H}}(\alpha)$ defined by

$$\mathcal{P}^0_{\mathcal{H}}(\alpha) = \{f = h + \overline{g} \in \mathcal{H} : \operatorname{Re}(h'(z) - \alpha) > |g'(z)|, \ 0 \le \alpha < 1, \ g'(0) = 0 \text{ for } z \in \mathbb{D}\}.$$

It is easy to see that the class $\mathcal{H}$ reduces to the class $\mathcal{A}$ of normalized analytic functions if the co-analytic part of f is zero. A function $h \in \mathcal{A}$ is called close-to-convex in $\mathbb{D}$ if the complement of $h(\mathbb{D})$ can be written as the union of nonintersecting half lines. A function $h \in \mathcal{A}$ is said to be close-to-convex function of order β $(0 \le \beta < 1)$ if $\operatorname{Re}(h'(z)) > \beta$ for $z \in \mathbb{D}$ (see [47]). For $\alpha \ge 0$ and $0 \le \beta < 1$, let

$$\mathcal{W}(\alpha, \beta) = \{h \in \mathcal{A} : \operatorname{Re}(h'(z) + \alpha z h''(z)) > \beta \ \text{ in } \mathbb{D}\}.$$

The class $\mathcal{W}(\alpha, \beta)$ has been studied by Gao and Zhou [23] for $\beta < 1$ and $\alpha > 1$.

In 1977, Chichra [18] introduced the following class $\mathcal{W}(\alpha)$ for $\alpha \ge 0$:

$$\mathcal{W}(\alpha) = \{h \in \mathcal{A} : \operatorname{Re}(h'(z) + \alpha z h''(z) > 0) \ \text{ in } \mathbb{D}\}.$$

Moreover, Chichra [18] has proved that functions in the class $\mathcal{W}(\alpha)$ constitute a subclass of close-to-convex functions in $\mathbb{D}$. In 2014, Nagpal and Ravichandran [42] studied a new subclass $\mathcal{W}^0_{\mathcal{H}}$ of univalent harmonic mappings and obtained the coefficient bounds for functions in the class $\mathcal{W}^0_{\mathcal{H}}$, where

$$\mathcal{W}^0_{\mathcal{H}} = \{f = h + \bar{g} \in \mathcal{H} : \operatorname{Re}(h'(z) + z h''(z)) > |g'(z) + z g''(z)| \ \text{ for } z \in \mathbb{D}\}.$$

In 2019, Ghosh and Allu [24] studied the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$, where

$$\mathcal{W}^0_{\mathcal{H}}(\alpha) = \{f = h + \bar{g} \in \mathcal{H} : \operatorname{Re}(h'(z) + \alpha z h''(z)) > |g'(z) + \alpha z g''(z)| \ \text{ for } z \in \mathbb{D}\}.$$

Chichra [18] has shown that if $0 \le \beta < \alpha$, then $\mathcal{W}(\alpha) \subset \mathcal{W}(\beta)$. Hence, it is easy to see that $\mathcal{W}^0_{\mathcal{H}}(\alpha) \subset \mathcal{W}^0_{\mathcal{H}}(\beta)$ if $0 \le \beta < \alpha$. Therefore, for $\alpha \ge 1$, $\mathcal{W}^0_{\mathcal{H}}(\alpha) \subseteq \mathcal{W}^0_{\mathcal{H}}(1)$. Since functions in $\mathcal{W}^0_{\mathcal{H}}(1)$ are starlike, it follows that functions in $\mathcal{W}^0_{\mathcal{H}}(\alpha)$ are starlike for $\alpha \ge 1$. For $\alpha \ge 0$ and $0 \le \beta < 1$, let

$$\mathcal{W}^0_{\mathcal{H}}(\alpha, \beta) = \{f = h + \overline{g} \in \mathcal{H}^0 : \operatorname{Re}\left(h'(z) + \alpha z h''(z) - \beta\right) > |g'(z) + \alpha z g''(z)| \ \ z \in \mathbb{D}\}.$$

It is known that functions in $\mathcal{W}^0_{\mathcal{H}}(\alpha, \beta)$ are close-to-convex harmonic mappings (see [47]).

Remark 1.1 The class $\mathcal{W}^0_{\mathcal{H}}(\alpha, \beta)$ generalizes some of the well-known classes of harmonic mappings. For example, $\mathcal{W}^0_{\mathcal{H}}(\alpha, 0) = \mathcal{W}^0_{\mathcal{H}}(\alpha)$, $\mathcal{W}^0_{\mathcal{H}}(0, \beta) = \mathcal{P}^0_{\mathcal{H}}(\beta)$, $\mathcal{W}^0_{\mathcal{H}}(1, 0) = \mathcal{W}^0_{\mathcal{H}}$, and $\mathcal{W}^0_{\mathcal{H}}(0, 0) = \mathcal{P}^0_{\mathcal{H}}$.

From the following results, it is easy to see that functions in the class $\mathcal{W}^0_{\mathcal{H}}(\alpha, \beta)$ are univalent for $\alpha \ge 0$ and $0 \le \beta < 1$, and they are closely related to functions in $\mathcal{W}(\alpha, \beta)$.

Lemma 1.1 [47] *The harmonic mapping $f = h + \bar{g}$ belongs to $\mathcal{W}^0_{\mathcal{H}}(\alpha, \beta)$ if, and only if, the analytic function $F_\varepsilon = h + \varepsilon g$ belongs to $\mathcal{W}(\alpha, \beta)$ for each $|\varepsilon| = 1$.*

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The sharp coefficient bounds and the sharp growth estimates for functions in the class $\mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$ have been studied in [47].

Lemma 1.2 [47] *Let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$ for $\alpha \geq 0$, $0 \leq \beta < 1$ and be of the form (1.2). Then, for any $n \geq 2$,*

(i) $|a_n| + |b_n| \leq \dfrac{2(1-\beta)}{n\left(1+\alpha(n-1)\right)}$;

(ii) $||a_n| - |b_n|| \leq \dfrac{2(1-\beta)}{n\left(1+\alpha(n-1)\right)}$;

(iii) $|a_n| \leq \dfrac{2(1-\beta)}{n\left(1+\alpha(n-1)\right)}$.

All these inequalities are sharp for the function $f = f_{\alpha,\beta}$ given by

$$f_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\beta)\bar{z}^n}{n\left(1+\alpha(n-1)\right)}. \tag{1.3}$$

Lemma 1.3 [47] *Let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$ and be of the form (1.2) with $0 < \alpha \leq 1$ and $0 \leq \beta < 1$. Then*

$$|z| + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}|z|^n}{n\left(1+\alpha(n-1)\right)} \leq |f(z)| \leq |z| + \sum_{n=2}^{\infty} \frac{2(1-\beta)|z|^n}{n\left(1+\alpha(n-1)\right)}. \tag{1.4}$$

Both the inequalities are sharp for the function $f = f_{\alpha,\beta}$ given by (1.3).

Bohr–Rogosinski radius is an analogous to the Bohr radius which has been defined (see [48]) as follows: If $f \in \mathcal{B}$ is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then for an integer $N \geq 1$, we have $|S_N(z)| < 1$ in the disk $\{z \in \mathbb{C} : |z| < 1/2\}$ and the radius $1/2$ is the best possible, where $S_N(z) = \sum_{n=0}^{N} a_n z^n$ denotes the Nth partial sum of f. The radius $r = 1/2$ is called the Rogosinski radius. For $f \in \mathcal{B}$ is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the Bohr–Rogosinski sum $R_N^f(z)$ of f defined by

$$R_N^f(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n| r^n, \quad |z| = r. \tag{1.5}$$

It is worth to notice that for $N = 1$, the quantity (1.5) is related to the classical Bohr sum in which $|f(0)|$ is replaced by $|f(z)|$. The inequality $R_N^f(z) \leq 1$ is called Bohr–Rogosinski inequality. If B and R denote the Bohr radius and Bohr–Rogosinski radius, respectively, then $B \leq R$ because $|S_N(z)| \leq \sum_{n=0}^{N} |a_n||z|^n \leq \sum_{n=0}^{\infty} |a_n||z|^n$.

In 2005, Aizenberg *et al.* [8] generalized the Bohr–Rogosinski inequality for the holomorphic mappings of the open unit ball into an arbitrary convex domain as well as studied the multidimensional analog of Rogosinski's theorem with some applications. In 2009, Aizenberg [9] proved that the abscissas of Bohr and Rogosinski for ordinary Dirichlet series, mapping the right half-plane into the bounded convex domain $G \subset \mathbb{C}$ are independent of the domain. In 2012, Aizenberg [7] studied Bohr and Rogosinski radii for Hardy classes H^p of holomorphic functions in the unit disk $\mathbb{D}$ as well as the Bohr and Rogosinski radii for the mappings of Reinhradt domains in $\mathbb{C}^n$ into Reinhardt domains in $\mathbb{C}^n$. In 2020, Alkhaleefah *et al.* [12] obtained Bohr–Rogosinski radius for analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in unit disk $\mathbb{D}$ replacing the coefficient a_n of the power series by the derivatives $f^{(n)}(z)/n!$. Recently, Huang *et al.* [25] have generalized and improved some refined versions of Bohr–Rogosinski inequalities which have been studied by Liu *et al.* [39]. In 2021, Kayumov *et al.* [29] investigated Bohr–Rogosinski phenomenon for analytic functions defined on $\mathbb{D}$ in a general setting and derived Bohr–Rogosinski radii for Cesáro operators on the space of bounded analytic functions. For the extensive study of Bohr phenomenon including recent developments, we refer the reader to the articles [1–5, 11, 14–16, 20–22, 27, 28, 30–32, 34, 37, 40, 41, 43].

In 2017, Kayumov and Ponnusamy [33] obtained the following result on the Bohr–Rogosinski radius for the analytic functions in the unit disk $\mathbb{D}$.

Theorem 1.1 [33] *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disk $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 \text{ for } r \leq R_N,$$

where R_N is the positive root of the equation $2(1+r)r^N - (1-r^2) = 0$. The radius R_N is the best possible. Moreover,

$$|f(z)|^2 + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 \text{ for } r \leq R'_N,$$

where R'_N is the positive root of the equation $(1+r)r^N - (1-r^2) = 0$. The radius R'_N is the best possible.

Recently, Kayumov and Ponnusamy [33] have obtained another interesting result concerning Bohr–Rogosinski radius considering some power of z in $f(z)$ as the following.

Theorem 1.2 [33] *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ such that $|f(z)| \leq 1$ in $\mathbb{D}$. Then, for each $m, N \in \mathbb{N}$,*

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 \text{ for } r \leq R_{m,N},$$

where $R_{m,N}$ is the positive root of of the equation $2r^N(1+r^m) - (1-r)(1-r^m) = 0$, and the number $R_{m,N}$ cannot be improved. Moreover, $\lim_{N\to\infty} R_{m,N} = 1$ and $\lim_{m\to\infty} R_{m,N} = A_N$, where A_N is the positive root of the equation $2r^N = 1 - r$. Also, $A_1 = 1/3$ and $A_2 = 1/2$.

Study of sharp refine Bohr–Rogosinski inequalities for functions in the class $\mathcal{B}$ is an important topic in the geometric function theory. The refine Bohr–Rogosinski inequality for the class $\mathcal{B}$ is extensively studied by Liu *et al.* [39] and Ponnusamy *et al.* [44]. For $N = 1$, it is easy to see that $R_1 = \sqrt{5} - 2$ and $R'_1 = 1/3$. However, recently, Liu *et al.* [39] proved the following interesting result showing that the two constants R_1

and R_1' can be improved for any individual function in $\mathcal{B}$ (in the context of Theorem 1.1 with $N = 1$).

Theorem 1.3 [39] *Suppose that $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$A(z) := |f(z)| + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_1|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 1$$

for $|z| = r \le r_{a_0} = 2/\left(3 + |a_0| + \sqrt{5}\,(1+|a_0|)\right)$. *The radius* r_{a_0} *is best possible and* $r_{a_0} > \sqrt{5} - 2$. *Moreover,*

$$B(z) := |f(z)|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_1|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 1$$

for $|z| = r \le r'_{a_0}$, *where* r'_{a_0} *is the unique positive root of the equation*

$$\left(1 - |a_0|^3\right) r^3 - (1 + 2|a_0|)\, r^2 - 2r + 1 = 0.$$

The radius r'_{a_0} *is best possible and* $1/3 < r'_{a_0} < 1/(2 + |a_0|)$.

Motivated from the papers [6, 13], the main objective of this paper is to find the sharp Bohr–Rogosinski radius which are the harmonic analog of Theorems 1.1–1.3 for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$ for $\alpha \ge 0$ and $0 \le \beta < 1$. In Section 2, we state the main results of the paper and the proof of the main results will be discussed in Section 3.

2 Main results

Using Lemmas 1.2 and 1.3, we obtain the following sharp Bohr–Rogosinski inequality and sharp Bohr–Rogosinski radius for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$.

Theorem 2.1 *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$ for $\alpha \ge 0$ and $0 \le \beta < 1$, be of the form (1.2) with $b_1 = 0$. Then, for $N \ge 2$,*

(i) $|f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \le d(f(0), \partial f(\mathbb{D}))$ *for* $|z| = r \le R_N(\alpha,\beta)$, *where* $R_N(\alpha,\beta)$ *is the unique root of*

$$r + \sum_{n=2}^{\infty} \frac{2(1-\beta) r^n}{n\,(1+\alpha(n-1))} + \sum_{n=N}^{\infty} \frac{2(1-\beta) r^n}{n\,(1+\alpha(n-1))} - 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\,(1+\alpha(n-1))} = 0.$$

Here, $R_N(\alpha,\beta)$ is the best possible.

(ii) $|f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \le d(f(0), \partial f(\mathbb{D}))$ *for* $|z| = r \le R'_N(\alpha,\beta)$, *where* $R'_N(\alpha,\beta)$ *is the unique root of*

$$\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta) r^n}{n\,(1+\alpha(n-1))} \right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\beta) r^n}{n\,(1+\alpha(n-1))} - 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\,(1+\alpha(n-1))} = 0.$$

Here, $R'_N(\alpha,\beta)$ is best possible.

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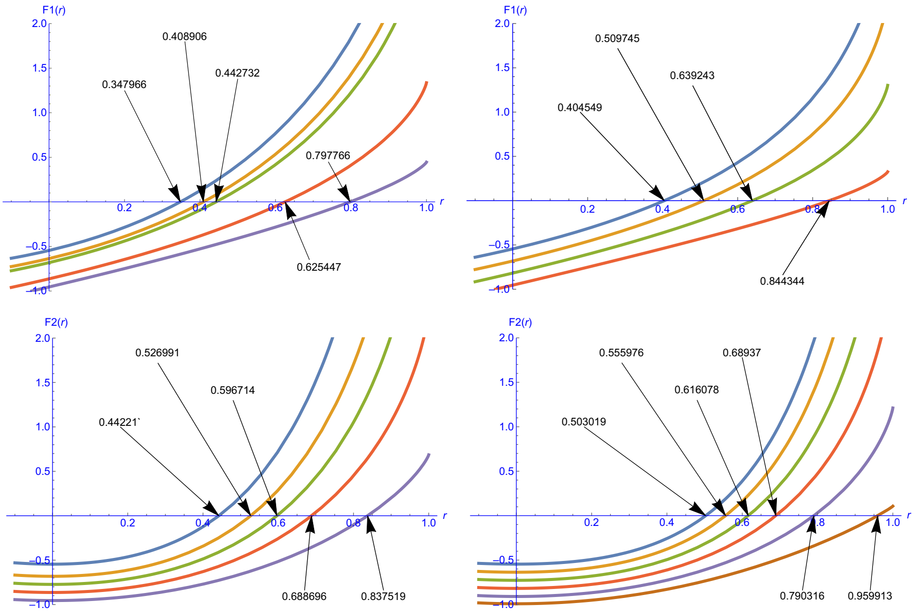

Figure 1: The pictorial representation of the radii in Remark 2.1.

Remark 2.1 For some particular values of α and β, a simple computation gives the following Bohr–Rogosinski radii (Figure 1):

(i) $R_2(0.5, 0) \approx 0.347966$, $R_2(0.5, 0.2) \approx 0.408906$, $R_2(0.5, 0.3) \approx 0.442732$, $R_2(0.5, 0.7) \approx 0.625477$, $R_2(0.5, 0.9) \approx 0.797766$, $R_5(0.5, 0) \approx 0.404549$, $R_5(0.5, 0.3) \approx 0.509745$, $R_5(0.5, 0.6) \approx 0.609243$, $R_5(0.5, 0.9) \approx 0.844344$.

(ii) $R_2'(0.5, 0) \approx 0.44221$, $R_2'(0.5, 0.3) \approx 0.526991$, $R_2'(0.5, 0.5) \approx 0.596714$, $R_2'(0.5, 0.7) \approx 0.688696$, $R_2'(0.5, 0.9) \approx 0.837519$, $R_5'(0.5, 0) \approx 0.503019$, $R_5'(0.5, 0.2) \approx 0.555796$, $R_5'(0.5, 0.4) \approx 0.616078$, $R_5'(0.5, 0.6) \approx 0.68937$, $R_5'(0.5, 0.98) \approx 0.959913$.

We obtain the next result which is the harmonic analog of Theorem 1.2 for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha, \beta)$.

Theorem 2.2 *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(\alpha, \beta)$ for $\alpha \geq 0$ and $0 \leq \beta < 1$, be of the form (1.2) with $b_1 = 0$. Then, for $N \geq 2$, we have*

$$|f(z^m)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \leq d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \leq R_{m,N}(\alpha, \beta)$, where $R_{m,N}(\alpha, \beta)$ is the unique root of

$$r^m + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^{mn}}{n(1+\alpha(n-1))} + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))} - 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))} = 0.$$

Here, $R_{m,N}(\alpha, \beta)$ is best possible.

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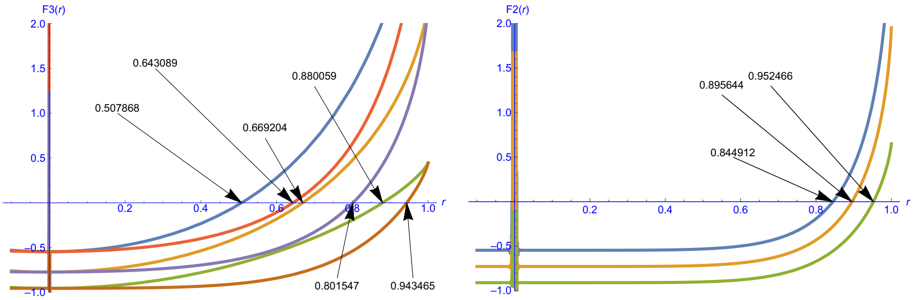

Figure 2: The pictorial representation of the radii in Remark 2.2.

Remark 2.2 For particular values of α and β, a simple computation gives the Bohr–Rogosinski radii $R_{2,2}(0.5,0) \approx 0.507868$, $R_{2,2}(0.5,5) \approx 0.669204$, $R_{2,2}(0.5,0.9) \approx 0.880059$, $R_{5,2}(0.5,0) \approx 0.643089$, $R_{5,2}(0.5,0,5) \approx 0.801547$, $R_{5,2}(0.5,0.9) \approx 0.943465$, $R_{7,5}(0.5,0) \approx 0.844912$, $R_{7,5}(0.5,0.4) \approx 0.895644$, $R_{7,5}(0.5,0.8) \approx 0.952466$ (Figure 2).

In order to establish a harmonic analog of Theorem 1.3 for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$, we obtain the following result.

Theorem 2.3 *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$ for $\alpha \geq 0$ and $0 \leq \beta < 1$, be of the form (1.2). Then, for any integer $p \geq 1$,*

$$|f(z)|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^n + \left(\frac{1}{2} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\, r^{2n} \leq d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \leq R^*_p(\alpha,\beta)$, where $R^*_p(\alpha,\beta)$ is the unique root of

$$\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))}\right)^p + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))} + \frac{1+r}{2(1-r)} \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 r^{2n}}{(n\,(1+\alpha(n-1)))^2}$$
$$= 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\,(1+\alpha(n-1))}$$

in $(0,1)$. Here, $R^*_p(\alpha,\beta)$ is best possible.

We have the following immediate corollary of Theorem 2.3 which is the harmonic analog of Theorem 1.3 for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$.

Corollary 2.1 *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$ for $\alpha \geq 0$ and $0 \leq \beta < 1$, be of the form (1.2). Then:*

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(i) $|f(z)| + \sum_{n=2}^{\infty} (|a_n| + |b_n|) |z|^n + \left(\frac{1}{2} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 r^{2n} \le d(f(0), \partial f(\mathbb{D}))$ *for* $|z| = r \le R_1^*(\alpha, \beta)$*, where* $R_1^*(\alpha, \beta)$ *is the unique root of*

$$r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))} + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))} + \frac{1+r}{2(1-r)} \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 r^{2n}}{(n(1+\alpha(n-1)))^2} = 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))}$$

in $(0,1)$*. Here,* $R_1^*(\alpha, \beta)$ *is best possible.*

(ii) $|f(z)|^2 + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + \left(\frac{1}{2} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 r^{2n} \le d(f(0), \partial f(\mathbb{D}))$ *for* $|z| = r \le R_2^*(\alpha, \beta)$*, where* $R_2^*(\alpha, \beta)$ *is the unique root of*

$$\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))}\right)^2 + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))} + \frac{1+r}{2(1-r)} \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 r^{2n}}{(n(1+\alpha(n-1)))^2} = 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))}$$

in $(0,1)$*. Here,* $R_2^*(\alpha, \beta)$ *is best possible.*

3 Proof of the main results

Proof of Theorem 2.1 Let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha, \beta)$ be given by (1.2). Then in view of Lemmas 1.2 and 1.3, it is easy to see that the Euclidean distance $d(f(0), \partial f(\mathbb{D}))$ between $f(0)$ and the boundary of $f(\mathbb{D})$ is

$$d(f(0), \partial f(\mathbb{D})) = \liminf_{|z|\to 1} |f(z) - f(0)| \ge 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))}. \tag{3.1}$$

(i) Let $F_1 : [0,1] \to \mathbb{R}$ be defined by

$$F_1(r) := r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))} + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))} - 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))}. \tag{3.2}$$

It is easy to see that $F_1(r)$ is continuous on $[0,1]$ and differentiable on $(0,1)$. Since

$$\left|\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n(1+\alpha(n-1))}\right| \le \frac{1}{2(1-\beta)} \quad \text{for} \quad n \ge 2,$$

it follows from (3.2) that

$$F_1(0) = -1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))} < 0.$$

On the other hand, we see that

$$\sum_{n=2}^{\infty}\frac{1}{\alpha n^2+(1-\alpha)n}\geq\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\alpha n^2+(1-\alpha)n}.$$

Therefore, a simple computation shows that

$$\begin{aligned}F_1(1)&=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)}{n\left(1+\alpha(n-1)\right)}+\sum_{n=N}^{\infty}\frac{2(1-\beta)}{n\left(1+\alpha(n-1)\right)}\\&\quad-1-\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{n\left(1+\alpha(n-1)\right)}\\&\geq\sum_{n=N}^{\infty}\frac{2(1-\beta)}{n\left(1+\alpha(n-1)\right)}>0.\end{aligned}$$

Clearly, we have $F_1(0)F_1(1)<0$, and hence, by the intermediate value theorem, $F_1(r)$ has at least one root in $(0,1)$. To show that $F_1(r)$ has the unique root in $(0,1)$, it is sufficient to show that F_1 is strictly monotone in $(0,1)$. Now a simple computation shows that

$$\frac{d}{dr}\left(F_1(r)\right)=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)nr^{n-1}}{n\left(1+\alpha(n-1)\right)}+\sum_{n=N}^{\infty}\frac{2(1-\beta)nr^{n-1}}{n\left(1+\alpha(n-1)\right)}>0$$

for all $r\in(0,1)$, and hence the function F_1 is strictly increasing. Therefore, $F_1(r)$ has the unique root in $(0,1)$, say $R_N(\alpha,\beta)$. That is, $F_1(R_N(\alpha,\beta))=0$ and hence, from (3.2), we obtain

$$\begin{aligned}R_N(\alpha,\beta)&+\sum_{n=2}^{\infty}\frac{2(1-\beta)(R_N(\alpha,\beta))^n}{n\left(1+\alpha(n-1)\right)}+\sum_{n=N}^{\infty}\frac{2(1-\beta)(R_N(\alpha,\beta))^n}{n\left(1+\alpha(n-1)\right)}\\&=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{n\left(1+\alpha(n-1)\right)}.\end{aligned}\tag{3.3}$$

In order to show that $R_N(\alpha,\beta)$ is the best possible, we consider the function $f=f_{\alpha,\beta}$ given by (1.3). It is easy to see that $f_{\alpha,\beta}\in\mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$. For $f=f_{\alpha,\beta}$, a straightforward computation shows that

$$d(f_{\alpha,\beta}(0),\partial f_{\alpha,\beta}(\mathbb{D}))=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{n\left(1+\alpha(n-1)\right)}.\tag{3.4}$$

Furthermore, it is easy to see that

$$r+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{n\left(1+\alpha(n-1)\right)}+\sum_{n=N}^{\infty}\frac{2(1-\beta)r^n}{n\left(1+\alpha(n-1)\right)}\leq1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\alpha n^2+(1-\alpha)n}$$

for $r\leq R_N(\alpha,\beta)$. A simple computation using (3.3) and (3.4) for the function $f=f_{\alpha,\beta}$ and $r>R_N(\alpha,\beta)$ shows that

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$$
\begin{aligned}
&|f_{\alpha,\beta}(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \\
&= r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))} + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))} \\
&> R_N(\alpha,\beta) + \sum_{n=2}^{\infty} \frac{2(1-\beta)(R_N(\alpha,\beta))^n}{n\,(1+\alpha(n-1))} + \sum_{n=N}^{\infty} \frac{2(1-\beta)(R_N(\alpha,\beta))^n}{n\,(1+\alpha(n-1))} \\
&= 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\,(1+\alpha(n-1))} \\
&= d(f_{\alpha,\beta}(0), \partial f_{\alpha,\beta}(\mathbb{D})).
\end{aligned}
$$

Therefore, the constant $R_N(\alpha,\beta)$ is best possible. This completes the proof of (i).
(ii). In view of Lemmas 1.2 and 1.3, for $|z| = r$, we easily obtain

$$
\begin{aligned}
(3.5) \qquad &|f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \\
&\le \left(|z| + \sum_{n=2}^{\infty} \frac{2(1-\beta)|z|^n}{n\,(1+\alpha(n-1))}\right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\beta)|z|^n}{n\,(1+\alpha(n-1))} \\
&= \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))}\right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))}.
\end{aligned}
$$

It is easy to see that

$$
\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))}\right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))} \le 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\,(1+\alpha(n-1))}
$$

for $r \le R'_N(f)$, where $R'_N(f)$ is a root of $F_2(r) = 0$, where $F_2 : [0,1] \to \mathbb{R}$ is defined by

$$
\begin{aligned}
F_2(r) := &\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))}\right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))} \\
&- 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\,(1+\alpha(n-1))}.
\end{aligned}
$$

By the similar argument being used in the proof of (i), it is easy to see that $F_2(0) < 0$ and $F_2(1) > 0$. Since F_2 is continuous on $[0,1]$ and differentiable on $(0,1)$, by the intermediate value theorem there exists a root, say $R'_N(\alpha,\beta)$ of F_2 in $(0,1)$. In fact, we show that $R'_N(\alpha,\beta)$ is the unique root of F_2 in $(0,1)$. By a simple computation, we obtain

$$
\begin{aligned}
\frac{d}{dr}(F_2(r)) = &\,2\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n\,(1+\alpha(n-1))}\right)\left(1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)nr^{n-1}}{n\,(1+\alpha(n-1))}\right) \\
&+ \sum_{n=N}^{\infty} \frac{2(1-\beta)nr^{n-1}}{n\,(1+\alpha(n-1))} > 0
\end{aligned}
$$

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for $r \in (0,1)$ and this shows that $F_2(r)$ is strictly increasing in $(0,1)$. Therefore, $R'_N(\alpha,\beta)$ is the unique root of F_2 in $(0,1)$. Thus, we have

$$\left(R'_N(\alpha,\beta) + \sum_{n=2}^{\infty} \frac{2(1-\beta)\left(R'_N(\alpha,\beta)\right)^n}{n\left(1+\alpha(n-1)\right)}\right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\beta)\left(R'_N(\alpha,\beta)\right)^n}{n\left(1+\alpha(n-1)\right)} = 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\left(1+\alpha(n-1)\right)}. \tag{3.6}$$

To show that $R'_N(\alpha,\beta)$ is the best possible, we consider the function $f = f_{\alpha,\beta}$ defined by (1.3). Using (3.4)–(3.6), for $f = f_{\alpha,\beta}$ and $r > R'_N(\alpha,\beta)$, it is easy to see that

$$\begin{aligned} &|f_{\alpha,\beta}(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \\ &= \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n\left(1+\alpha(n-1)\right)}\right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n\left(1+\alpha(n-1)\right)} \\ &> \left(R'_N(\alpha,\beta) + \sum_{n=2}^{\infty} \frac{2(1-\beta)\left(R'_N(\alpha,\beta)\right)^n}{n\left(1+\alpha(n-1)\right)}\right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\beta)\left(R'_N(\alpha,\beta)\right)^n}{n\left(1+\alpha(n-1)\right)} \\ &= 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\left(1+\alpha(n-1)\right)} \\ &= d\left(f_{\alpha,\beta}(0), \partial f_{\alpha,\beta}(\mathbb{D})\right), \end{aligned}$$

which shows that $R'_N(\alpha,\beta)$ is best possible. This completes the proof of (ii). ∎

Proof of Theorem 2.2 In view of Lemmas 1.2 and 1.3, for $|z| = r$, we obtain

$$\begin{aligned} &|f(z^m)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \\ &\le r^m + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^{mn}}{n\left(1+\alpha(n-1)\right)} + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n\left(1+\alpha(n-1)\right)}. \end{aligned} \tag{3.7}$$

It is easy to see that

$$\begin{aligned} &r^m + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^{mn}}{n\left(1+\alpha(n-1)\right)} + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n\left(1+\alpha(n-1)\right)} \\ &\le 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\left(1+\alpha(n-1)\right)} \end{aligned}$$

for $r \le R_{m,N}(\alpha,\beta)$, where $R_{m,N}(\alpha,\beta)$ is a root of $F_3(r) = 0$, where $F_3 : [0,1] \to \mathbb{R}$ is defined by

$$\begin{aligned} F_3(r) := r^m &+ \sum_{n=2}^{\infty} \frac{2(1-\beta)r^{mn}}{n\left(1+\alpha(n-1)\right)} + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{n\left(1+\alpha(n-1)\right)} \\ &- 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\left(1+\alpha(n-1)\right)}. \end{aligned} \tag{3.8}$$

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Using the similar argument being used in the proof of Theorem 2.1, it can be shows that $R_{m,N}(\alpha,\beta)$ is the unique root of F_3 in $(0,1)$. Hence, we have

(3.9)
$$(R_{m,N}(\alpha,\beta))^m + \sum_{n=2}^{\infty} \frac{2(1-\beta)(R_{m,N}(\alpha,\beta))^{mn}}{n(1+\alpha(n-1))} + \sum_{n=N}^{\infty} \frac{2(1-\beta)(R_{m,N}(\alpha,\beta))^{n}}{n(1+\alpha(n-1))} = 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))}.$$

Let $f = f_{\alpha,\beta}$ be defined by (1.3). In view of (3.4), (3.8), and (3.9), for $r > R_{m,N}(\alpha,\beta)$, a simple computation shows that

$$\begin{aligned}
&|f_{\alpha,\beta}(z^m)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \\
&= r^m + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^{mn}}{n(1+\alpha(n-1))} + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^{n}}{n(1+\alpha(n-1))} \\
&> (R_{m,N}(\alpha,\beta))^m + \sum_{n=2}^{\infty} \frac{2(1-\beta)(R_{m,N}(\alpha,\beta))^{mn}}{n(1+\alpha(n-1))} + \sum_{n=N}^{\infty} \frac{2(1-\beta)(R_{m,N}(\alpha,\beta))^{n}}{n(1+\alpha(n-1))} \\
&= 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))} \\
&= d(f_{\alpha,\beta}(0), \partial f_{\alpha,\beta}(\mathbb{D})),
\end{aligned}$$

and hence the radius $R_{m,N}(\alpha,\beta)$ is best possible. This completes the proof. ■

Proof of Theorem 2.3 Let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha,\beta)$ and $p \geq 1$ be an integer. Then, in view of the Lemmas 1.2 and 1.3, we see that

$$\begin{aligned}
|f(z)|^p &+ \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \left(\frac{1}{2} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 r^{2n} \\
&\leq \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))}\right)^p + \frac{1+r}{2(1-r)} \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 r^{2n}}{(n(1+\alpha(n-1)))^2} \\
&\quad + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))} \\
&\leq 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))}
\end{aligned}$$

for $|z| = r \leq R^*_p(\alpha,\beta)$, where $R^*_p(\alpha,\beta)$ is a root in $(0,1)$ of the equation $F_4(r) = 0$, where

$$\begin{aligned}
F_4(r) := &\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))}\right)^p + \frac{1+r}{2(1-r)} \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 r^{2n}}{(n(1+\alpha(n-1)))^2} \\
&+ \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{n(1+\alpha(n-1))} - 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n(1+\alpha(n-1))}.
\end{aligned}$$

By the similar argument being used in the proof of the previous theorems, it is easy to show that $F_4(0)F_4(1) < 0$ and $\frac{d}{dr}(F_4(r)) > 0$ in $(0,1)$. Hence, F_4 being continuous and monotone increasing function, $R_p^*(\alpha,\beta)$ is the unique root of the equation $F_4(r) = 0$ in $(0,1)$.

Thus, we have

(3.10)
$$\left(R_p^*(\alpha,\beta) + \sum_{n=2}^{\infty} \frac{2(1-\beta)\left(R_p^*(\alpha,\beta)\right)^n}{n\,(1+\alpha(n-1))}\right)^p + \frac{1+R_p^*(\alpha,\beta)}{2(1-rR_p^*(\alpha,\beta))} \sum_{n=2}^{\infty} \frac{4(1-\beta)^2\left(R_p^*(\alpha,\beta)\right)^{2n}}{\left(n\,(1+\alpha(n-1))\right)^2}$$
$$+ \sum_{n=2}^{\infty} \frac{2(1-\beta)\left(R_p^*(\alpha,\beta)\right)^n}{n\,(1+\alpha(n-1))} = 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{n\,(1+\alpha(n-1))}.$$

Therefore, in view of (3.1), we have

$$|f(z)|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^n + \left(\frac{1}{2} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\, r^{2n} \le d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \le R_p^*(\alpha,\beta)$.

Considering the function $f = f_{\alpha,\beta}$ defined by (1.3). Using (3.4) and (3.10), for $f = f_{\alpha,\beta}$ and $r > R_p^*(\alpha,\beta)$, it can be shown that

$$|f_{\alpha,\beta}(z)|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, |z|^n + \left(\frac{1}{2} + \frac{r}{1-r}\right) \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\, r^{2n}$$
$$> d(f_{\alpha,\beta}(0), \partial f_{\alpha,\beta}(\mathbb{D}))$$

which shows that $R_p^*(\alpha,\beta)$ is best possible. This completes the proof. ∎

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