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BESSEL- AND GRÜSS-TYPE INEQUALITIES IN INNER PRODUCT MODULES

SENKA BANIĆ[1], DIJANA ILIŠEVIĆ[2] AND SANJA VAROŠANEC[2]

[1] *Faculty of Civil Engineering and Architecture, University of Split, Matice hrvatske 15, 21000 Split, Croatia* (senka.banic@gradst.hr)

[2] *Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia* (ilisevic@math.hr; varosans@math.hr)

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Abstract In this paper we give Bessel- and Grüss-type inequalities in an inner product module over a proper H^*-algebra or a C^*-algebra.

Keywords: Bessel inequality; Grüss inequality; inner product H^*-module; inner product C^*-module

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1. Introduction and preliminaries

It is well known that in an inner product space $(\mathcal{H}, \langle \cdot\,, \cdot \rangle)$ over the real or complex number field $\mathbb{K}$ the Bessel inequality holds. Namely, if $\{e_i\}_{i\in I}$ is a family of orthonormal vectors in $\mathcal{H}$, then for any $x \in \mathcal{H}$ we have

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 \leqslant \langle x, x\rangle. \tag{1.1}$$

Furthermore, some results concerning upper bounds for the expression

$$\langle x, x\rangle - \sum_{i\in I} |\langle x, e_i\rangle|^2 \qquad (x \in \mathcal{H}) \tag{1.2}$$

and for the expression related to the Grüss-type inequality

$$\left|\langle x, y\rangle - \sum_{i\in I} \langle x, e_i\rangle\langle e_i, y\rangle\right| \quad (x, y \in \mathcal{H}) \tag{1.3}$$

have appeared in [**5**].

In this paper we give an analogue of the Bessel inequality (1.1) and we investigate expressions analogous to (1.2) and (1.3) in an inner product module over a proper H^*-algebra or a C^*-algebra.

Throughout this paper we denote by A a proper H^*-algebra or a C^*-algebra.

As we know, a proper H^*-algebra is a complex Banach algebra $(A, \|\cdot\|)$ whose underlying Banach space is a Hilbert space $(A, \langle\cdot\,,\cdot\rangle)$ equipped with an involution $a \mapsto a^*$ satisfying $\langle ab, c\rangle = \langle b, a^*c\rangle = \langle a, cb^*\rangle$ for all $a, b, c \in A$. An element a of a proper H^*-algebra A is called positive ($a \geqslant 0$) if $\langle ax, x\rangle \geqslant 0$ for every $x \in A$. Each $a \geqslant 0$ is self-adjoint ($a^* = a$). For a proper H^*-algebra A, the trace class associated with A is $\tau(A) = \{ab : a, b \in A\}$. For every positive $a \in \tau(A)$ there exists the square root of a, that is, a unique positive $a^{1/2} \in A$ such that $(a^{1/2})^2 = a$. There are a positive linear functional 'tr' on $\tau(A)$ and a norm τ on $\tau(A)$, related to the norm of A by the equality $\mathrm{tr}(a^*a) = \tau(a^*a) = \|a\|^2$ for every $a \in A$.

A C^*-algebra is a complex Banach $*$-algebra $(A, \|\cdot\|)$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$. An element a of a C^*-algebra A is called positive ($a \geqslant 0$) if it is self-adjoint and has positive spectrum. The square root of a exists for every positive $a \in A$.

In both structures, the relation '$\leqslant$' is given by $a \leqslant b$ if and only if $b - a \geqslant 0$.

An inner product module over A is a right module H over A together with a generalized inner product, i.e. with a mapping $[\cdot\,,\cdot]$ on $H \times H$ which is $\tau(A)$-valued if A is a proper H^*-algebra or A-valued if A is a C^*-algebra satisfying the following properties:

(H1) $[f, g + h] = [f, g] + [f, h]$ for all $f, g, h \in H$;

(H2) $[f, ga] = [f, g]a$ for all $f, g \in H$, $a \in A$;

(H3) $[f, g]^* = [g, f]$ for all $f, g \in H$;

(H4) $[f, f] \geqslant 0$ for every $f \in H$;

(H5) $[f, f] = 0$ implies $f = 0$.

We shall say that H is an inner product H^*-module if A is a proper H^*-algebra and that H is an inner product C^*-module if A is a C^*-algebra.

A mapping $[\cdot\,,\cdot]$ satisfying (H1)–(H4) is called a generalized semi-inner product, and H is called a semi-inner product H^*- or C^*-module.

The absolute value of $f \in H$ is defined as the square root of $[f, f]$ and it is denoted by $|f|$. Let us emphasize that $|f|$ is a positive element of A and is thus self-adjoint. It is said that $f, g \in H$ are orthogonal if and only if $[f, g] = 0$.

For some additional facts about H^*-algebras, C^*-algebras and (semi-)inner product modules over these structures we refer to the literature: see [**8**], [**10**], [**11**], [**12**], [**13**], [**14**] and the references therein.

2. Bessel-type inequalities

An element $p \in A$ is called a projection if p is non-zero and $p^* = p = p^2$. If A is a C^*-algebra, then $\|p\| = 1$, while if A is a proper H^*-algebra, then $\|p\|$ is not equal to 1 in general. An element h from a semi-inner product module over A is called a lifted projection if $|h|$ is a projection in A. According to [**7**, Lemma 2.4], h is a lifted projection in an inner product module over A if and only if h is non-zero and $h|h| = h$.

The following theorem gives us an A-valued Bessel-type inequality in an inner product module.

Theorem 2.1. *Let A be a proper H^*-algebra or a C^*-algebra. Let $(H, [\cdot\,,\cdot])$ be an inner product module over A and let $\{h_1, \dots, h_n\}$ be a set of mutually orthogonal lifted projections. Then, for all $f \in H$,*

$$\sum_{i=1}^{n} [f, h_i][h_i, f] \leqslant |f|^2 \tag{2.1}$$

holds.

Proof. Using the properties of the generalized inner product and the characterization of lifted projections we have

$$\begin{aligned}
|f|^2 - \sum_{i=1}^{n}[f, h_i][h_i, f] \\
&= [f, f] - \sum_{i=1}^{n}[f, h_i][h_i, f] \\
&= [f, f] - \sum_{i=1}^{n}[h_i, f]^*[h_i, f] - \sum_{i=1}^{n}[f, h_i][h_i, f] + \sum_{i=1}^{n}[f, h_i][h_i, f] \\
&= [f, f] - \sum_{i=1}^{n}[h_i, f]^*[h_i, f] - \sum_{i=1}^{n}[f, h_i][h_i, f] + \sum_{i=1}^{n}[f, h_i|h_i|][h_i|h_i|, f] \\
&= [f, f] - \sum_{i=1}^{n}[h_i, f]^*[h_i, f] - \sum_{i=1}^{n}[f, h_i][h_i, f] + \sum_{i=1}^{n}[f, h_i]|h_i|^2[h_i, f] \\
&= [f, f] - \sum_{i=1}^{n}[h_i, f]^*[h_i, f] - \sum_{i=1}^{n}[f, h_i][h_i, f] + \sum_{i,j=1}^{n}[h_i, f]^*[h_i, h_j][h_j, f] \\
&= \left[f - \sum_{i=1}^{n} h_i[h_i, f], f - \sum_{i=1}^{n} h_i[h_i, f]\right] \geqslant 0.
\end{aligned}$$

□

The following theorem gives an upper bound for the expression

$$|f|^2 - \sum_{i=1}^{n}[f, h_i][h_i, f],$$

that is, the A-valued best approximation formula.

Theorem 2.2. *Let A be a proper H^*-algebra or a C^*-algebra. Let $(H, [\cdot\,,\cdot])$ be an inner product module over A and let $\{h_1, \dots, h_n\}$ be a set of mutually orthogonal lifted projections. For all $f \in H$ and $a_1, \dots, a_n \in A$,*

$$|f|^2 - \sum_{i=1}^{n}[f, h_i][h_i, f] \leqslant \left|f - \sum_{i=1}^{n} h_i a_i\right|^2. \tag{2.2}$$

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Proof. We have

$$\begin{aligned}\left|f-\sum_{i=1}^{n}h_i a_i\right|^2 &\geqslant \left|f-\sum_{i=1}^{n}h_i a_i\right|^2 - \sum_{i=1}^{n}|[h_i,f]-|h_i|a_i|^2\\
&= \left[f-\sum_{i=1}^{n}h_i a_i, f-\sum_{i=1}^{n}h_i a_i\right] - \sum_{i=1}^{n}([f,h_i]-a_i^*|h_i|)([h_i,f]-|h_i|a_i)\\
&= |f|^2 - \sum_{i=1}^{n}a_i^*[h_i,f] - \sum_{i=1}^{n}[f,h_i]a_i + \sum_{i=1}^{n}a_i^*|h_i|^2 a_i\\
&\quad - \sum_{i=1}^{n}[f,h_i][h_i,f] + \sum_{i=1}^{n}a_i^*|h_i|[h_i,f] + \sum_{i=1}^{n}[f,h_i]|h_i|a_i - \sum_{i=1}^{n}a_i^*|h_i|^2 a_i\\
&= |f|^2 - \sum_{i=1}^{n}[f,h_i][h_i,f].\end{aligned}$$

□

Remark 2.3. Results such as these can be found in [**1**] and [**2**], where a concept of an orthonormal basis for Hilbert H^*-modules and Hilbert C^*-modules is discussed.

Let us emphasize that there are inner product H^*-modules and inner product C^*-modules that lack the property of being a complex vector space that is compatible with the structure of A (see, for example, [**6**, Example 4.1]). However, for $\lambda \in \mathbb{C}$, $h \in H$ and $a \in A$ we put $\lambda(ha) := h(\lambda a)$. In particular, if $h \in H$ is a lifted projection, then $h|h| = h$, so we are able to define $\lambda h := h(\lambda|h|)$ and we have $(\lambda h)a = h(\lambda a) = \lambda(ha)$ for every $\lambda \in \mathbb{C}$ and every $a \in A$.

Corollary 2.4. *Let A be a proper H^*-algebra or a C^*-algebra. Let $(H,[\cdot\,,\cdot])$ be an inner product module over A and let $\{h_1,\dots,h_n\}$ be a set of mutually orthogonal lifted projections. For all $f \in H$ and $\lambda_1,\dots,\lambda_n \in \mathbb{C}$,*

$$|f|^2 - \sum_{i=1}^{n}[f,h_i][h_i,f] \leqslant \left|f-\sum_{i=1}^{n}\lambda_i h_i\right|^2.$$

Proof. It is sufficient to define $a_i = \lambda_i|h_i|$ and apply Theorem 2.2. □

3. Grüss-type inequalities in inner product H^*-modules

Let us recall the original Grüss inequality [**9**]. Let f and g be real functions that are defined and integrable on $[a,b] \subset \mathbb{R}$. If there exist real constants φ, ϕ, γ, Γ such that

$$\varphi \leqslant f(x) \leqslant \phi, \quad \gamma \leqslant g(x) \leqslant \Gamma \quad (x \in [a,b]),$$

then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x \leqslant \tfrac{1}{4}(\phi-\varphi)(\Gamma-\gamma).$$

The constant $\frac{1}{4}$ is the best possible.

Recently, some papers on the Grüss inequality in real and complex inner product spaces have appeared [**3**, **4**, **15**]. Here we reproduce the basic result from [**4**].

Theorem 3.1. *Let* $(\mathcal{H},\langle\cdot\,,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ *or* $\mathbb{C}$) *and let* $e\in\mathcal{H}$, $\|e\|=1$. *If* φ, ϕ, γ, Γ *are real or complex numbers and* x, y *are vectors in* $\mathcal{H}$ *such that the conditions*

$$\operatorname{Re}\langle\phi e - x, x-\varphi e\rangle \geqslant 0 \quad \textit{and} \quad \operatorname{Re}\langle\Gamma e - y, y-\gamma e\rangle \geqslant 0 \tag{3.1}$$

hold, then we have the inequality

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leqslant \tfrac{1}{4}|\phi-\varphi|\cdot|\Gamma-\gamma|.$$

It is easy to see that the assumptions (3.1) are equivalent to the following conditions:

$$\left\|x-\frac{\phi+\varphi}{2}e\right\| \leqslant \tfrac{1}{2}|\phi-\varphi| \quad \text{and} \quad \left\|y-\frac{\Gamma+\gamma}{2}e\right\| \leqslant \tfrac{1}{2}|\Gamma-\gamma|.$$

Generalizations of Theorem 3.1 for inner product modules are given in [**7**].

In the following text we give Grüss-type inequalities which are based on Bessel-type inequalities (2.1) and (2.2) described in the previous section.

Let $(H,[\cdot\,,\cdot])$ be an inner product H^*-module or an inner product C^*-module and let $\{h_1,\dots,h_n\}$ be a set of mutually orthogonal lifted projections in H. Below we will use the notation $[\![\cdot\,,\cdot]\!]$ for the mapping defined on $H\times H$ by

$$[\![f,g]\!] = [f,g] - \sum_{i=1}^{n}[f,h_i][h_i,g].$$

The mapping $[\![\cdot\,,\cdot]\!]$ is a generalized semi-inner product on H. The properties (H1)–(H3) can be verified by simple calculation and (H4) is, in fact, the inequality (2.1).

Theorem 3.2. *Let* $(A,\|\cdot\|)$ *be a proper* H^**-algebra. Let* $(H,[\cdot\,,\cdot])$ *be an inner product module over* A *and let* $\{h_1,\dots,h_n\}$ *be a set of mutually orthogonal lifted projections in* H. *If* $a_i,b_i,c_i,d_i\in A$ $(i=1,\dots,n)$ *and* $f,g\in H$ *are such that the assumptions*

$$\left.\begin{aligned} \left|\!\left|\!\left| f-\sum_{i=1}^{n}h_i\left(\frac{a_i+b_i}{2}\right)\right|\!\right|\!\right| &\leqslant \frac{1}{2}\left(\sum_{i=1}^{n}\|a_i-b_i\|^2\right)^{1/2}, \\ \left|\!\left|\!\left| g-\sum_{i=1}^{n}h_i\left(\frac{c_i+d_i}{2}\right)\right|\!\right|\!\right| &\leqslant \frac{1}{2}\left(\sum_{i=1}^{n}\|c_i-d_i\|^2\right)^{1/2} \end{aligned}\right\} \tag{3.2}$$

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hold, then we have the inequality

$$\tau\biggl([f,g]-\sum_{i=1}^{n}[f,h_i][h_i,g]\biggr) \leqslant \frac{1}{4}\biggl(\sum_{i=1}^{n}\|a_i-b_i\|^2\sum_{i=1}^{n}\|c_i-d_i\|^2\biggr)^{1/2} - \biggl(\frac{1}{4}\sum_{i=1}^{n}\|a_i-b_i\|^2-\biggl|\biggl|\biggl|f-\sum_{i=1}^{n}h_i\biggl(\frac{a_i+b_i}{2}\biggr)\biggr|\biggr|\biggr|^2\biggr)^{1/2} \times\biggl(\frac{1}{4}\sum_{i=1}^{n}\|c_i-d_i\|^2-\biggl|\biggl|\biggl|g-\sum_{i=1}^{n}h_i\biggl(\frac{c_i+d_i}{2}\biggr)\biggr|\biggr|\biggr|^2\biggr)^{1/2}. \quad (3.3)$$

The constant $\frac{1}{4}$ *is the best possible.*

Proof. From the strong Cauchy–Schwarz inequality (see [**10**]) for a semi-inner product H^*-module $(H,[\![\cdot\,,\cdot]\!])$, we get

$$\begin{aligned}\tau\biggl([f,g]-\sum_{i=1}^{n}[f,h_i][h_i,g]\biggr)^2 &= \tau([\![f,g]\!])^2\leqslant \operatorname{tr}([\![f,f]\!])\operatorname{tr}([\![g,g]\!])\\ &=\operatorname{tr}\biggl([f,f]-\sum_{i=1}^{n}[f,h_i][h_i,f]\biggr)\operatorname{tr}\biggl([g,g]-\sum_{i=1}^{n}[g,h_i][h_i,g]\biggr)\\ &=\biggl(\operatorname{tr}(|f|^2)-\sum_{i=1}^{n}\operatorname{tr}([h_i,f]^*[h_i,f])\biggr)\biggl(\operatorname{tr}(|g|^2)-\sum_{i=1}^{n}\operatorname{tr}([h_i,g]^*[h_i,g])\biggr)\\ &=\biggl(|||f|||^2-\sum_{i=1}^{n}|||[h_i,f]|||^2\biggr)\biggl(|||g|||^2-\sum_{i=1}^{n}|||[h_i,g]|||^2\biggr). \quad (3.4)\end{aligned}$$

From Theorem 2.2 we have

$$|f|^2-\sum_{i=1}^{n}[f,h_i][h_i,f]\leqslant\biggl|f-\sum_{i=1}^{n}h_i\biggl(\frac{a_i+b_i}{2}\biggr)\biggr|^2,$$

which implies that

$$\operatorname{tr}\biggl(|f|^2-\sum_{i=1}^{n}[f,h_i][h_i,f]\biggr)\leqslant\operatorname{tr}\biggl(\biggl|f-\sum_{i=1}^{n}h_i\biggl(\frac{a_i+b_i}{2}\biggr)\biggr|^2\biggr).$$

This yields

$$|||f|||^2-\sum_{i=1}^{n}|||[h_i,f]|||^2\leqslant\biggl|\biggl|\biggl|f-\sum_{i=1}^{n}h_i\biggl(\frac{a_i+b_i}{2}\biggr)\biggr|\biggr|\biggr|^2. \quad (3.5)$$

In the same way we obtain

$$|||g|||^2 - \sum_{i=1}^n \|[h_i, g]\|^2 \leqslant \left|\left|\left| g - \sum_{i=1}^n h_i \left(\frac{c_i + d_i}{2}\right) \right|\right|\right|^2. \tag{3.6}$$

Multiplying (3.5) by (3.6) we get

$$\left(|||f|||^2 - \sum_{i=1}^n \|[h_i, f]\|^2\right) \cdot \left(|||g|||^2 - \sum_{i=1}^n \|[h_i, g]\|^2\right) \\ \leqslant \left|\left|\left| f - \sum_{i=1}^n h_i \left(\frac{a_i + b_i}{2}\right) \right|\right|\right|^2 \cdot \left|\left|\left| g - \sum_{i=1}^n h_i \left(\frac{c_i + d_i}{2}\right) \right|\right|\right|^2. \tag{3.7}$$

Let us define

$$m = \frac{1}{2}\left(\sum_{i=1}^n \|a_i - b_i\|^2\right)^{1/2}, \qquad p = \frac{1}{2}\left(\sum_{i=1}^n \|c_i - d_i\|^2\right)^{1/2}.$$

Taking into account the assumptions (3.2), we can also define

$$n = \left(\frac{1}{4}\sum_{i=1}^n \|a_i - b_i\|^2 - \left|\left|\left| f - \sum_{i=1}^n h_i \left(\frac{a_i + b_i}{2}\right) \right|\right|\right|^2\right)^{1/2},$$
$$q = \left(\frac{1}{4}\sum_{i=1}^n \|c_i - d_i\|^2 - \left|\left|\left| g - \sum_{i=1}^n h_i \left(\frac{c_i + d_i}{2}\right) \right|\right|\right|^2\right)^{1/2}.$$

Applying the inequality $(m^2 - n^2)(p^2 - q^2) \leqslant (mp - nq)^2$, we get

$$\left|\left|\left| f - \sum_{i=1}^n h_i \left(\frac{a_i + b_i}{2}\right) \right|\right|\right|^2 \cdot \left|\left|\left| g - \sum_{i=1}^n h_i \left(\frac{c_i + d_i}{2}\right) \right|\right|\right|^2 \\ \leqslant \left(\frac{1}{4}\left(\sum_{i=1}^n \|a_i - b_i\|^2 \sum_{i=1}^n \|c_i - d_i\|^2\right)^{1/2} \right. \\ - \left(\frac{1}{4}\sum_{i=1}^n \|a_i - b_i\|^2 - \left|\left|\left| f - \sum_{i=1}^n h_i \left(\frac{a_i + b_i}{2}\right) \right|\right|\right|^2\right)^{1/2} \\ \left. \times \left(\frac{1}{4}\sum_{i=1}^n \|c_i - d_i\|^2 - \left|\left|\left| g - \sum_{i=1}^n h_i \left(\frac{c_i + d_i}{2}\right) \right|\right|\right|^2\right)^{1/2}\right)^2. \tag{3.8}$$

The inequality (3.3) is obtained after comparing the inequalities (3.4), (3.7) and (3.8).

If the submodule of H generated by $\{h_1, \dots, h_n\}$ is not equal to H, then there exists $h \in H$ such that $h \neq \sum_{i=1}^n h_i[h_i, h]$. If we put $k = h - \sum_{i=1}^n h_i[h_i, h]$, then $k \in H$ is non-zero and for any $j \in \{1, 2, \dots, n\}$ we have

$$[k, h_j] = [h, h_j] - \sum_{i=1}^n [h, h_i][h_i, h_j] = [h, h_j] - [h, h_j]|h_j|^2 = 0.$$

Let us define

$$f = k(\lambda|k|) + \sum_{i=1}^{n} h_i\left(\frac{a_i + b_i}{2}\right), \qquad g = k(\mu|k|) + \sum_{i=1}^{n} h_i\left(\frac{c_i + d_i}{2}\right),$$

where

$$\lambda = \frac{1}{2\||k|^2\|}\left(\sum_{i=1}^{n} \|a_i - b_i\|^2\right)^{1/2}, \qquad \mu = \frac{1}{2\||k|^2\|}\left(\sum_{i=1}^{n} \|c_i - d_i\|^2\right)^{1/2}.$$

For these $f, g \in H$, (3.2) and (3.3) become equalities. More precisely,

$$\left|\!\left|\!\left| f - \sum_{i=1}^{n} h_i\left(\frac{a_i + b_i}{2}\right)\right|\!\right|\!\right| = \frac{1}{2}\left(\sum_{i=1}^{n} \|a_i - b_i\|^2\right)^{1/2},$$
$$\left|\!\left|\!\left| g - \sum_{i=1}^{n} h_i\left(\frac{c_i + d_i}{2}\right)\right|\!\right|\!\right| = \frac{1}{2}\left(\sum_{i=1}^{n} \|c_i - d_i\|^2\right)^{1/2}$$

(so the second summand on the right-hand side in (3.3) vanishes), and

$$\tau\left([f, g] - \sum_{i=1}^{n} [f, h_i][h_i, g]\right) = \frac{1}{4}\left(\sum_{i=1}^{n} \|a_i - b_i\|^2 \sum_{i=1}^{n} \|c_i - d_i\|^2\right)^{1/2}.$$

Hence, the constant $\frac{1}{4}$ is the best possible. □

Corollary 3.3. *Let $(A, \|\cdot\|)$ be a proper H^*-algebra. Let $(H, [\cdot\,,\cdot])$ be an inner product module over A and let $\{h_1, \dots, h_n\}$ be a set of mutually orthogonal lifted projections in H. If $a_i, b_i, c_i, d_i \in A$ $(i = 1, \dots, n)$, and $f, g \in H$ are such that the assumptions (3.2) hold, then*

$$\tau\left([f, g] - \sum_{i=1}^{n} [f, h_i][h_i, g]\right) \leqslant \frac{1}{4}\left(\sum_{i=1}^{n} \|a_i - b_i\|^2 \sum_{i=1}^{n} \|c_i - d_i\|^2\right)^{1/2}.$$

The constant $\frac{1}{4}$ is the best possible.

For $\varphi_i, \phi_i, \gamma_i, \Gamma_i \in \mathbb{C}$ $(i = 1, \dots, n)$ we set $a_i = \phi_i|h_i|$, $b_i = \varphi_i|h_i|$, $c_i = \Gamma_i|h_i|$, $d_i = \gamma_i|h_i|$ $(i = 1, \dots, n)$ in Theorem 3.2 and Corollary 3.3 to obtain the following corollary.

Corollary 3.4. *Let $(A, \|\cdot\|)$ be a proper H^*-algebra. Let $(H, [\cdot\,,\cdot])$ be an inner product module over A and let $\{h_1, \dots, h_n\}$ be a set of mutually orthogonal lifted projections in H. If $\varphi_i, \phi_i, \gamma_i, \Gamma_i \in \mathbb{C}$ $(i = 1, \dots, n)$ and $f, g \in H$ are such that the assumptions*

$$\left.\begin{aligned} \left|\!\left|\!\left| f - \sum_{i=1}^{n} \frac{\phi_i + \varphi_i}{2} h_i\right|\!\right|\!\right| &\leqslant \frac{1}{2}\left(\sum_{i=1}^{n} |\phi_i - \varphi_i|^2 \||h_i|\|^2\right)^{1/2}, \\ \left|\!\left|\!\left| g - \sum_{i=1}^{n} \frac{\Gamma_i + \gamma_i}{2} h_i\right|\!\right|\!\right| &\leqslant \frac{1}{2}\left(\sum_{i=1}^{n} |\Gamma_i - \gamma_i|^2 \||h_i|\|^2\right)^{1/2} \end{aligned}\right\} \tag{3.9}$$

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hold, then we have the following inequality:

$$\begin{aligned}
\tau\biggl([f,g]-\sum_{i=1}^{n}[f,h_i][h_i,g]\biggr)
&\leqslant \frac{1}{4}\biggl(\sum_{i=1}^{n}|\phi_i-\varphi_i|^2|||h_i|||^2\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2|||h_i|||^2\biggr)^{1/2}\\
&\qquad -\biggl(\frac{1}{4}\sum_{i=1}^{n}|\phi_i-\varphi_i|^2|||h_i|||^2-\biggl|\biggl|\biggl|f-\sum_{i=1}^{n}\frac{\phi_i+\varphi_i}{2}h_i\biggr|\biggr|\biggr|^2\biggr)^{1/2}\\
&\qquad\quad \times\biggl(\frac{1}{4}\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2|||h_i|||^2-\biggl|\biggl|\biggl|g-\sum_{i=1}^{n}\frac{\Gamma_i+\gamma_i}{2}h_i\biggr|\biggr|\biggr|^2\biggr)^{1/2}\\
&\leqslant \frac{1}{4}\biggl(\sum_{i=1}^{n}|\phi_i-\varphi_i|^2|||h_i|||^2\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2|||h_i|||^2\biggr)^{1/2}.
\end{aligned}$$

The constant $\frac{1}{4}$ is the best possible.

For $n=1$, Theorem 3.2 and Corollary 3.3 become the generalized Grüss inequality and its refinement.

Corollary 3.5. *Let $(A,\|\cdot\|)$ be a proper H^*-algebra. Let $(H,[\cdot,\cdot])$ be an inner product module over A and let $h\in H$ be a lifted projection. If $a,b,c,d\in A$ and $f,g\in H$ are such that the assumptions*

$$\biggl|\biggl|\biggl|f-h\biggl(\frac{a+b}{2}\biggr)\biggr|\biggr|\biggr|\leqslant\tfrac{1}{2}\|a-b\|,\qquad \biggl|\biggl|\biggl|g-h\biggl(\frac{c+d}{2}\biggr)\biggr|\biggr|\biggr|\leqslant\tfrac{1}{2}\|c-d\| \tag{3.10}$$

hold, then we have the following inequality:

$$\begin{aligned}
\tau([f,g]-[f,h][h,g])
&\leqslant \tfrac{1}{4}\|a-b\|\|c-d\|\\
&\quad -\biggl(\tfrac{1}{4}\|a-b\|^2-\biggl|\biggl|\biggl|f-h\biggl(\frac{a+b}{2}\biggr)\biggr|\biggr|\biggr|^2\biggr)^{1/2}\biggl(\tfrac{1}{4}\|c-d\|^2-\biggl|\biggl|\biggl|g-h\biggl(\frac{c+d}{2}\biggr)\biggr|\biggr|\biggr|^2\biggr)^{1/2}\\
&\leqslant \tfrac{1}{4}\|a-b\|\|c-d\|.
\end{aligned}\tag{3.11}$$

The constant $\frac{1}{4}$ is the best possible.

Furthermore, from Corollary 3.4 in the case in which $n=1$ we obtain [**7**, Theorem 4.1] and [**7**, Corollary 4.2].

Remark 3.6. For $f=g$, Corollary 3.3 becomes the Bessel-type inequality. Namely, we have the following result.

If the assumptions of Corollary 3.3 are satisfied and if

$$\biggl|\biggl|\biggl|f-\sum_{i=1}^{n}h_i\biggl(\frac{a_i+b_i}{2}\biggr)\biggr|\biggr|\biggr|\leqslant\frac{1}{2}\biggl(\sum_{i=1}^{n}\|a_i-b_i\|^2\biggr)^{1/2} \tag{3.12}$$

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holds, then we have the following inequality:

$$\tau\biggl(|f|^2-\sum_{i=1}^{n}[f,h_i][h_i,f]\biggr)\leqslant\frac{1}{4}\sum_{i=1}^{n}\|a_i-b_i\|^2.$$

4. Grüss-type inequalities in inner product C^*-modules

Theorem 4.1. *Let $(A,\|\cdot\|)$ be a C^*-algebra. Let $(H,[\cdot,\cdot])$ be an inner product module over A and let $\{h_1,\dots,h_n\}$ be a set of mutually orthogonal lifted projections in H. If $a_i,b_i,c_i,d_i\in A$ $(i=1,\dots,n)$ and $f,g\in H$ are such that the assumptions (3.2) hold, then we have the following inequality:*

$$\begin{aligned}\biggl\|[f,g]-\sum_{i=1}^{n}[f,h_i][h_i,g]\biggr\|&\\ \leqslant\frac{1}{4}\biggl(\sum_{i=1}^{n}\|a_i-b_i\|^2\sum_{i=1}^{n}\|c_i-d_i\|^2\biggr)^{1/2}&\\ -\biggl(\frac{1}{4}\sum_{i=1}^{n}\|a_i-b_i\|^2-\biggl\|\biggl|f-\sum_{i=1}^{n}h_i\biggl(\frac{a_i+b_i}{2}\biggr)\biggr|\biggr\|^2\biggr)^{1/2}&\\ \times\biggl(\frac{1}{4}\sum_{i=1}^{n}\|c_i-d_i\|^2-\biggl\|\biggl|g-\sum_{i=1}^{n}h_i\biggl(\frac{c_i+d_i}{2}\biggr)\biggr|\biggr\|^2\biggr)^{1/2}&.\end{aligned}\tag{4.1}$$

The constant $\frac{1}{4}$ is the best possible.

Proof. First we have the Cauchy–Schwarz inequality for a semi-inner product C^*-module $(H,[\![\cdot,\cdot]\!])$ (see, for example, [**8**, Proposition 1.1] together with [**11**, Theorem 2.2.5.(3)]):

$$\biggl\|[f,g]-\sum_{i=1}^{n}[f,h_i][h_i,g]\biggr\|^2\leqslant\biggl\||f|^2-\sum_{i=1}^{n}[f,h_i][h_i,f]\biggr\|\cdot\biggl\||g|^2-\sum_{i=1}^{n}[g,h_i][h_i,g]\biggr\|.\tag{4.2}$$

Applying [**11**, Theorem 2.2.5.(3)] on (2.2), taking into account (2.1), we get

$$\biggl\||f|^2-\sum_{i=1}^{n}[f,h_i][h_i,f]\biggr\|\leqslant\biggl\|\biggl|f-\sum_{i=1}^{n}h_i\biggl(\frac{a_i+b_i}{2}\biggr)\biggr|^2\biggr\|,$$

that is,

$$\biggl\||f|^2-\sum_{i=1}^{n}[f,h_i][h_i,f]\biggr\|\leqslant\biggl\|\biggl|f-\sum_{i=1}^{n}h_i\biggl(\frac{a_i+b_i}{2}\biggr)\biggr|\biggr\|^2.\tag{4.3}$$

Analogously,

$$\biggl\||g|^2-\sum_{i=1}^{n}[g,h_i][h_i,g]\biggr\|\leqslant\biggl\|\biggl|g-\sum_{i=1}^{n}h_i\biggl(\frac{c_i+d_i}{2}\biggr)\biggr|\biggr\|^2.\tag{4.4}$$

Multiplying (4.3) by (4.4) yields

$$\begin{aligned}\Bigg\| |f|^2 - \sum_{i=1}^{n}[f,h_i][h_i,f]\Bigg\| \cdot \Bigg\| |g|^2 - \sum_{i=1}^{n}[g,h_i][h_i,g]\Bigg\| \\ \leqslant \Bigg|\Bigg\|f - \sum_{i=1}^{n} h_i\bigg(\frac{a_i+b_i}{2}\bigg)\Bigg\|\Bigg|^2 \cdot \Bigg|\Bigg\|g - \sum_{i=1}^{n} h_i\bigg(\frac{c_i+d_i}{2}\bigg)\Bigg\|\Bigg|^2. \quad (4.5)\end{aligned}$$

The inequality (3.8) is then obtained as in the proof of Theorem 3.2. It remains to compare the inequalities (4.2), (4.5) and (3.8).

The equality in (4.1) holds for the same $f, g \in H$ that give the equality in (3.3). Note that the second summand on the right-hand side in (4.1) vanishes for such f and g as well. □

Corollary 4.2. *Let $(A, \|\cdot\|)$ be a C^*-algebra. Let $(H, [\cdot\,,\cdot])$ be an inner product module over A and let $\{h_1, \dots, h_n\}$ be a set of mutually orthogonal lifted projections in H. If $a_i, b_i, c_i, d_i \in A$ $(i = 1, \dots, n)$ and $f, g \in H$ are such that the assumptions (3.2) hold, then*

$$\Bigg\|[f,g] - \sum_{i=1}^{n}[f,h_i][h_i,g]\Bigg\| \leqslant \frac{1}{4}\bigg(\sum_{i=1}^{n}\|a_i - b_i\|^2 \sum_{i=1}^{n}\|c_i - d_i\|^2\bigg)^{1/2}.$$

The constant $\frac{1}{4}$ is the best possible.

As in the case of inner product H^*-modules, Theorem 4.1 and Corollary 4.2 for

$$\left.\begin{aligned} a_i &= \phi_i|h_i|,\\ b_i &= \varphi_i|h_i|,\\ c_i &= \Gamma_i|h_i|,\\ d_i &= \gamma_i|h_i|, \end{aligned}\right\} \quad i = 1, \dots, n,$$

give the following corollary. Note that, in contrast with the H^*-case, here we have $\||h_i|\| = 1$.

Corollary 4.3. *Let $(A, \|\cdot\|)$ be a C^*-algebra. Let $(H, [\cdot\,,\cdot])$ be an inner product module over A and let $\{h_1, \dots, h_n\}$ be a set of mutually orthogonal lifted projections in H. If $\varphi_i, \phi_i, \gamma_i, \Gamma_i \in \mathbb{C}$ $(i = 1, \dots, n)$ and $f, g \in H$ are such that the assumptions*

$$\left.\begin{aligned} \Bigg|\Bigg\|f - \sum_{i=1}^{n}\frac{\phi_i+\varphi_i}{2}h_i\Bigg\|\Bigg| &\leqslant \frac{1}{2}\bigg(\sum_{i=1}^{n}|\phi_i - \varphi_i|^2\bigg)^{1/2},\\ \Bigg|\Bigg\|g - \sum_{i=1}^{n}\frac{\Gamma_i+\gamma_i}{2}h_i\Bigg\|\Bigg| &\leqslant \frac{1}{2}\bigg(\sum_{i=1}^{n}|\Gamma_i - \gamma_i|^2\bigg)^{1/2} \end{aligned}\right\} \quad (4.6)$$

hold, then we have the following inequality:

$$\begin{aligned}\left\|[f,g]-\sum_{i=1}^{n}[f,h_i][h_i,g]\right\| &\leqslant \frac{1}{4}\left(\sum_{i=1}^{n}|\phi_i-\varphi_i|^2\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2\right)^{1/2}\\ &\qquad -\left(\frac{1}{4}\sum_{i=1}^{n}|\phi_i-\varphi_i|^2-\left\|\left|f-\sum_{i=1}^{n}\frac{\phi_i+\varphi_i}{2}h_i\right|\right\|^2\right)^{1/2}\\ &\qquad\quad \times\left(\frac{1}{4}\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2-\left\|\left|g-\sum_{i=1}^{n}\frac{\Gamma_i+\gamma_i}{2}h_i\right|\right\|^2\right)^{1/2}\\ &\leqslant \frac{1}{4}\left(\sum_{i=1}^{n}|\phi_i-\varphi_i|^2\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2\right)^{1/2}.\end{aligned}$$

The constant $\frac{1}{4}$ is the best possible.

For $n=1$ we get [**7**, Theorem 5.1] and [**7**, Corollary 5.2].

The following result is the generalized Grüss inequality and its refinement and it is obtained from Theorem 4.1 and Corollary 4.2 for $n=1$.

Corollary 4.4. *Let $(A,\|\cdot\|)$ be a C^*-algebra. Let $(H,[\cdot,\cdot])$ be an inner product module over A and let $h\in H$ be a lifted projection. If $a,b,c,d\in A$ and $f,g\in H$ are such that the assumptions (3.10) hold, then we have the following inequality:*

$$\begin{aligned}&\|[f,g]-[f,h][h,g]\|\\ &\quad\leqslant \tfrac{1}{4}\|a-b\|\|c-d\|\\ &\qquad -\left(\tfrac{1}{4}\|a-b\|^2-\left\|\left|f-h\left(\frac{a+b}{2}\right)\right|\right\|^2\right)^{1/2}\left(\tfrac{1}{4}\|c-d\|^2-\left\|\left|g-h\left(\frac{c+d}{2}\right)\right|\right\|^2\right)^{1/2}\\ &\quad\leqslant \tfrac{1}{4}\|a-b\|\|c-d\|. \end{aligned}\tag{4.7}$$

The constant $\frac{1}{4}$ is the best possible.

Remark 4.5. If we put $f=g$ in Corollary 4.2, then we obtain the Bessel-type inequality. More precisely, if the assumptions of Corollary 4.2 and (3.12) hold, then

$$\left\||f|^2-\sum_{i=1}^{n}[f,h_i][h_i,f]\right\|\leqslant\frac{1}{4}\sum_{i=1}^{n}\|a_i-b_i\|^2.$$

5. Grüss-type inequalities in complex inner product spaces

A complex inner product space is an inner product module over the complex numbers. If $(\mathcal{H},\langle\cdot,\cdot\rangle)$ is a complex inner product space, then the absolute value in $\mathcal{H}$ coincides with the norm in $\mathcal{H}$ induced by $\langle\cdot,\cdot\rangle$, that is, with the mapping $\|\cdot\|$ defined by $\|f\|=\sqrt{\langle f,f\rangle}$ for every $f\in\mathcal{H}$. Therefore, $e\in\mathcal{H}$ is a lifted projection if and only if $\|e\|=1$, and $\{e_1,\dots,e_n\}$ is a set of mutually orthogonal lifted projections in $\mathcal{H}$ if and only if it is an orthonormal set in $\mathcal{H}$ (that is, $\|e_i\|=1$ $(i=1,\dots,n)$ and $\langle e_i,e_j\rangle=0$ for $i\neq j$). Then the previous results give us the following theorem.

Theorem 5.1. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex inner product space and let $\|\cdot\|$ be the norm in $\mathcal{H}$ induced by $\langle\cdot,\cdot\rangle$. Let $\{e_1,\dots,e_n\}$ be an orthonormal set in $\mathcal{H}$. If $\varphi_i,\phi_i,\gamma_i,\Gamma_i\in\mathbb{C}$ $(i=1,\dots,n)$ and $f,g\in\mathcal{H}$ are such that the assumptions*

$$\left.\begin{aligned} \left\|f-\sum_{i=1}^{n}\frac{\phi_i+\varphi_i}{2}e_i\right\| &\leqslant \frac{1}{2}\left(\sum_{i=1}^{n}|\phi_i-\varphi_i|^2\right)^{1/2}, \\ \left\|g-\sum_{i=1}^{n}\frac{\Gamma_i+\gamma_i}{2}e_i\right\| &\leqslant \frac{1}{2}\left(\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2\right)^{1/2} \end{aligned}\right\} \tag{5.1}$$

hold, then we have the following inequality:

$$\begin{aligned} \left|\langle f,g\rangle-\sum_{i=1}^{n}\langle f,e_i\rangle\langle e_i,g\rangle\right| &\leqslant \frac{1}{4}\left(\sum_{i=1}^{n}|\phi_i-\varphi_i|^2\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2\right)^{1/2} \\ &\quad -\left(\frac{1}{4}\sum_{i=1}^{n}|\phi_i-\varphi_i|^2-\left\|f-\sum_{i=1}^{n}\frac{\phi_i+\varphi_i}{2}e_i\right\|^2\right)^{1/2} \\ &\quad \times\left(\frac{1}{4}\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2-\left\|g-\sum_{i=1}^{n}\frac{\Gamma_i+\gamma_i}{2}e_i\right\|^2\right)^{1/2} \\ &\leqslant \frac{1}{4}\left(\sum_{i=1}^{n}|\phi_i-\varphi_i|^2\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2\right)^{1/2}. \end{aligned} \tag{5.2}$$

The constant $\frac{1}{4}$ is the best possible.

Using the fact that

$$\operatorname{Re}\left\langle\sum_{i=1}^{n}\phi_ie_i-f, f-\sum_{i=1}^{n}\varphi_ie_i\right\rangle=\frac{1}{4}\sum_{i=1}^{n}|\phi_i-\varphi_i|^2-\left\|f-\sum_{i=1}^{n}\frac{\phi_i+\varphi_i}{2}e_i\right\|^2$$

and

$$\operatorname{Re}\left\langle\sum_{i=1}^{n}\Gamma_ie_i-g, g-\sum_{i=1}^{n}\gamma_ie_i\right\rangle=\frac{1}{4}\sum_{i=1}^{n}|\Gamma_i-\gamma_i|^2-\left\|g-\sum_{i=1}^{n}\frac{\Gamma_i+\gamma_i}{2}e_i\right\|^2,$$

we get [**5**, Theorem 5]. An analogue of inequality (5.2) in the case of a real inner product space is [**15**, Theorem 2].

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