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RELATIVE CHARACTER CORRESPONDENCES IN FINITE GROUPS

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ABSTRACT. In a coprime action, we study character correspondences between the invariant characters of the group and those of subgroups containing the fixed points subgroup. Some character degree consequences are obtained.

1. **Introduction.** The relationship between different Glauberman-Isaacs correspondents, when considering subgroups of the operator group or subgroups of the group acted on, is not an easy one.

If A acts coprimely on G, $^*\colon \mathrm{Irr}_A(G) \longrightarrow \mathrm{Irr}\big(\mathbf{C}_G(A)\big)$ is the Glauberman-Isaacs correspondence and $B \subseteq A$, it has not been possible to show, for instance, that the A-correspondent of $\chi \in \mathrm{Irr}_A(G)$ lies under its B-correspondent (even when G is a solvable group, [13]).

Here we do not vary the operator group but consider A-invariant subgroups containing the fixed points subgroup: if $\mathbf{C}_G(A) \subseteq H \subseteq G$ is an A-invariant subgroup of G, then the equation

$$\chi^* = \chi'^*$$

where $\chi \in \mathrm{Irr}_A(G)$ and $\chi' \in \mathrm{Irr}_A(H)$, defines a natural (relative) character correspondence between $\mathrm{Irr}_A(G)$ and $\mathrm{Irr}_A(H)$.

This correspondence was already studied by T. Wolf when H had a very particular form, *i.e.*, when $H = L\mathbf{C}_G(A)$ for some normal A-invariant subgroup L of G. In this case, and when A was solvable, it was proven in [13] that χ' is an irreducible constituent of χ_H.

The general fact is our Theorem A below announced in [10].

THEOREM A. *Suppose that A acts coprimely on G. If $\mathbf{C}_G(A) \subseteq H \subseteq G$ is an A-invariant subgroup of G and $\chi \in \mathrm{Irr}_A(G)$, then χ' is an irreducible constituent of χ_H.*

Although there are several reasons to study relative character correspondences (as happened in [13]), we have a very particular one here.

In [8], it was shown that there is a rich connection between the set $\mathrm{cd}_A(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_A(G)\}$ and the A-invariant group theoretical structure of G (as happens in the well known case $A = 1$). Concretely, we proved that if a prime p does not divide any member of the set $\mathrm{cd}_A(G)$ is because an A-invariant Sylow p-subgroup P of G satisfies

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some normal-abelian condition (as happens in the celebrated Ito-Michler Theorem). This normal condition was the most natural one: P is normalized by the fixed points subgroup $\mathbf{C}_G(A)$. This result suggests that some kind of A-invariant Clifford theory is going on between G and A-invariants subgroups H of G normalized by $\mathbf{C}_G(A)$. Since the A-invariant character theory between H and $H\mathbf{C}_G(A)$ is so tight, the problem reduces to study the connections between $\mathrm{Irr}_A(G)$ and $\mathrm{Irr}_A(H)$ for A-stabilized subgroups H of G containing $\mathbf{C}_G(A)$.

Perhaps the following result is surprising.

THEOREM B. *Suppose that A acts coprimely on a solvable group G and assume that H is an A-invariant odd order subgroup of G containing $\mathbf{C}_G(A)$. If $\chi \in \mathrm{Irr}_A(G)$, then $\chi'(1)$ divides $\chi(1)$. Moreover, $\frac{\chi(1)}{\chi'(1)}$ divides $|G:H|$.*

It should be remarked that it is not even obvious that χ_H has some irreducible constituent of degree dividing $\chi(1)$. Theorem B, thus, can be used as a key tool for obtaining A-information on G from $\mathrm{cd}_A(G)$, at least, as we see, for groups of odd order.

The obvious question at this point is: what happens if we only assume the subgroup H to contain the fixed points subgroup? When dealing with character correspondences and degrees, we have to make the safe hypothesis of assuming the group G to be solvable (one of the deepest problems in character theory is to show that $\chi^*(1)$ divides $\chi(1)$). So, assuming solvability, is Theorem B again true if we no longer assume H to have odd order?

Once we know that Theorem B is true with the odd order hypothesis (that is to say, for the Isaacs correspondence) it is perhaps more surprising to find out that it is false if this hypothesis is removed (that is to say, for the Glauberman correspondence). It is the first time, that we are aware of, that some result is true for just one half of the character bijection.

We will provide with an example (that we learned from M. Isaacs and for which we thank him) of a solvable group G of order $2^3 5^3$ acted by a group A of order 3 with an A-invariant subgroup H of order $2^3 5$ satisfying the hypotheses but not the conclusions of Theorem B.

We will prove Theorem B in two steps: first when the group G is (solvable) of odd order and then, as an application of Isaacs B_π-characters, in the more general case.

Theorem C below will be the key for doing this.

We recall that in a π-separable group G there exists a canonical subset $\mathrm{B}_\pi(G)$ of $\mathrm{Irr}(G)$ which lifts bijectively $\mathrm{I}_\pi(G)$, the set of Isaacs π-Brauer characters of G (for the reader not familiar with π-theory, we stress that when π is the complement of a single prime p, $\mathrm{I}_\pi(G)$ is just the set $\mathrm{IBr}(G)$ of irreducible Brauer characters of G).

If $\chi \in \mathrm{B}_\pi(G)$ and H is a Hall π-subgroup of G, the irreducible constituents α of minimal degree in χ_H are called the *Fong characters* of H associated with χ. It turns out that $\alpha(1) = \chi(1)_\pi$, and, therefore, that $\frac{\chi(1)}{\alpha(1)}$ divides $|G:H|$.

THEOREM C. *Suppose that A acts coprimely on a π-separable group G and assume that $\mathbf{C}_G(A)$ is a π-group. Then $\mathrm{Irr}_A(G) \subseteq \mathrm{B}_\pi(G)$. Also, if H is an A-invariant Hall π-subgroup of G containing $\mathbf{C}_G(A)$ and $\chi \in \mathrm{Irr}_A(G)$, then $\chi' \in \mathrm{Irr}_A(H)$ is the unique A-invariant Fong character associated with $\chi \in \mathrm{B}_\pi(G)$.*

As we see from this introduction, still one problem continues to be open: if H is an A-invariant subgroup of G, what is the right definition for H to be 'normal' in G from the A-point of view?

2. **Proof of Theorems A and B.** We begin with two easy lemmas. Recall that if A acts coprimely on G, it is a well known fact that A fixes some Sylow p-subgroup of G, that all of them are $\mathbf{C}_G(A)$-conjugate and that A-invariant p-subgroups of G are contained in A-invariant Sylow p-subgroups. Also, if G is π-separable the same happens for A-invariant π-subgroups and A-invariant Hall π-subgroups.

These first lemmas are immediate consequences of these facts.

Throughout this paper we will assume that A acts coprimely on G, and we will write $C = \mathbf{C}_G(A)$, for the fixed points subgroup and $\Gamma = GA$, for the semidirect product.

LEMMA 2.1. *Let H be an A-invariant subgroup of G containing C. If P is an A-invariant Sylow p-subgroup of G, then $P \cap H$ is a Sylow p-subgroup of H.*

PROOF. Let P_0 be an A-invariant Sylow p-subgroup of H containing $P \cap H$. Then, $P_0 \subseteq P_1$, where P_1 is an A-invariant Sylow p-subgroup of G. We know (this is an easy consequence of Glauberman's Lemma (13.8) and (13.9) of [2]), that $P_1 = P^c$, for some $c \in C$. Now

$$P \cap H \subseteq P_0 \subseteq P_1 \cap H = P^c \cap H = (P \cap H)^c$$

and the result follows by order considerations. ∎

LEMMA 2.2. *Let H be an A-invariant subgroup of G containing C and let T be any A-invariant subgroup of G. Then $|H : H \cap T|$ divides $|G : T|$.*

PROOF. It suffices to show that $|H : H \cap T|_p$ divides $|G : T|_p$, for all primes p. Let P_1 be an A-invariant Sylow p-subgroup of T and let P be an A-invariant Sylow p-subgroup of G containing P_1. Therefore, $P_1 = P \cap T$. By Lemma 2.1, we have that $P \cap H \in \mathrm{Syl}_p(H)$ and by the same reason $P \cap H \cap T \in \mathrm{Syl}_p(H \cap T)$. Now, $|H : H \cap T|_p = \frac{|H|_p}{|T \cap H|_p} = |P \cap H : P \cap H \cap T|$ and $|G : T|_p = \frac{|G|_p}{|T|_p} = |P : P \cap T|$. However, by elementary group theory, we know that $|P \cap H : P \cap H \cap T| \leq |P : P \cap T|$, and the lemma is proved. ∎

When studying characters of solvable groups, the character theory associated with a fully ramified section is of fundamental importance. Recall that if L is a normal subgroup of G and $\varphi \in \mathrm{Irr}(L)$ and $\theta \in \mathrm{Irr}(G)$, φ (or θ) is said to be *fully ramified* with respect to G/L if $\theta_L = e\varphi$ with $e^2 = |G/L|$.

The deep properties of the character theory of fully ramified sections are contained in the following result of M. Isaacs.

We recall that (G,K,L,θ,φ) is a *character five* if L, K are normal subgroups of G with K/L abelian, and $\theta \in \mathrm{Irr}(K)$ and $\varphi \in \mathrm{Irr}(L)$ are G-invariant with $\theta_L = e\varphi$ and $e^2 = |K/L|$.

THEOREM 2.3. *Suppose that (G,K,L,θ,φ) is a character five and assume that $|G:K|$ or $|K:L|$ is odd. Then there exists a character $\psi \in \mathrm{Char}(G/K)$ and a subgroup $U \subseteq G$ satisfying:*

(a) $\psi^2(x) = \pm|C_{K/L}(x)|$ for all $x \in G$.

(b) $G = UK$ and $U \cap K = L$.

(c) For $\chi \in \mathrm{Irr}(G|\theta)$, $\chi_U = \psi_U\chi_0$ for a unique $\chi_0 \in \mathrm{Irr}(U|\varphi)$.

(d) The equation $\chi_U = \psi_U\chi_0$ defines a bijection between $\chi \in \mathrm{Irr}(G|\theta)$ and $\chi_0 \in \mathrm{Irr}(U|\varphi)$.

(e) If $|G:L|$ is odd and $\chi \in \mathrm{Irr}(G|\theta)$ and $\alpha \in \mathrm{Irr}(U)$, then $\chi_U = \psi_U\alpha$ if and only if $[\chi_U,\alpha]$ is odd.

(f) If $K \subseteq W \subseteq G$, then W satisfies (c), (d), (e) with respect to the character ψ_W and the subgroup $U \cap W$.

PROOF. Parts (a) to (e) are Theorem (9.1) of [1]. Observe that this theorem assures the existence of some complement U to K/L in G satisfying some 'good' conditions. However, it is not true that *any* complement to K/L in G satisfies these 'good' conditions. This is why part (f) is not completely obvious and needs some work (which was done in [11]). ∎

If U satisfies conditions (a) to (f) in Theorem 2.3, we will say that U is a *good complement* with respect to (G,K,L,θ,φ). Observe that if $K \subseteq W \subseteq G$, then $U \cap W$ is also a good complement with respect to (W,K,L,θ,φ).

The exact values of the character ψ are well known (see p. 619 of [1]) and are uniquely determined by (G,K,L,θ,φ). So if a is any automorphism of G fixing componentwise (G,K,L,θ,φ) it follows that ψ is fixed by a.

The following is a key result for proving our main theorems.

THEOREM 2.4. *Suppose that A acts coprimely on G, a group of odd order, and let (G,K,L,θ,φ) be an A-invariant character five. Assume that U is an A-invariant good complement of (G,K,L,θ,φ) containing $\mathbf{C}_G(A)$. If $\chi \in \mathrm{Irr}_A(G|\theta)$ and $\chi_U = \psi_U\chi_0$, as in Theorem 2.3, then $\chi_0^* = \chi^*$.*

PROOF. We argue by induction on $|G|$. We observe first that, by comments above, ψ is A-invariant. Also, since χ and χ_0 determine uniquely one each other, χ_0 is also A-invariant.

Suppose first that $|[G,A]| \subseteq K$. In this case, since K/L is abelian, we have that $[G,A]'C \subseteq U \subseteq G$, and, since $|G|$ is odd, that χ_0 is the unique irreducible constituent of χ_U with odd multiplicity (by Theorem 2.3(e)). Therefore, in the notation of Section 4 of [12], $\chi_0 = \chi\sigma(G,U,A)$. Now, by Theorem (4.6) of [12], we have that $\chi_0^* = \chi^*$ and in this case we are done.

So we may assume that $1 < [G/K,A]$. Since G is solvable, we have that $[G/K,A]' < [G/K,A]$ and by general properties of coprime action, we have that the subgroup $W/K = [G/K,A]'CK/K$ is proper in G/K. Therefore, we have produced an A-invariant proper subgroup W of G containing K and, at the same time, containing $[G,A]'C$.

By Corollary (4.3) of [12], we may write

$$\chi_W = \alpha + 2\Delta + \Theta$$

where $\alpha \in \mathrm{Irr}_A(W)$, each irreducible constituent of Δ is A-invariant (or Δ is zero) and each irreducible constituent of Θ is not A-invariant (if Θ is not zero). Also, $\alpha^* = \chi^*$, by Theorem (4.6) of [12]. By the same reasons, we may write

$$(\chi_0)_{W\cap U} = \beta + 2\Xi + \Phi$$

where $\beta \in \mathrm{Irr}_A(W \cap U)$, each irreducible constituent of Ξ is A-invariant (or Ξ is zero) and each irreducible constituent of Φ is not A-invariant (if Φ is not zero). Also, $\beta^* = \chi_0^*$.

Now, by parts (f) and (e) of Theorem 2.3, we may write

$$\alpha_{W\cap U} = \alpha_0 + 2\Upsilon$$

and by induction we have that $\alpha_0^* = \alpha^*$. Notice that to prove the theorem it suffices to show that $\alpha_0 = \beta$. By Theorem 2.3, we may write

$$\chi_U = \chi_0 + 2\Omega$$

and hence, we have that

$$\chi_{U\cap W} = (\chi_U)_{U\cap W} = \beta + 2\Lambda + \Phi.$$

Also

$$\chi_{U\cap W} = (\chi_W)_{U\cap W} = \alpha_o + 2\Psi + \Theta_{U\cap W}.$$

Suppose now that ξ is an irreducible constituent of Θ. Then ξ is not A-invariant, and by Theorem 2.3(e)(f), we may write $\xi_{U\cap W} = \xi_0 + 2\Lambda_\xi$, where ξ_0 is not A-invariant (because ξ is not). Since β is A-invariant, it follows that

$$[\Theta_{U\cap W}, \beta] \equiv 0 \bmod 2.$$

Now

$$1 \equiv [\chi_{U\cap W}, \beta] \equiv [\alpha_o, \beta] \bmod 2$$

and consequently, $\alpha_0 = \beta$, as wanted. ∎

Before proceeding to prove Theorems A and B, for the reader's convenience, we state a useful technical lemma proved by M. Isaacs in [6] and some of the main results of [5] and [9].

First the technical result.

LEMMA 2.5. *Let G be p-solvable and suppose that M/N is a p-chief factor of G. Let $C = \mathbf{C}_G(M/N)$ and suppose that $\varphi \in \mathrm{Irr}(N)$ is G-invariant. Then one of the following occurs.*

(a) C acts transitively on the set $\mathrm{Irr}(M|\varphi)$.

(b) Some member $\theta \in \mathrm{Irr}(M|\varphi)$ is G-invariant.

Furthermore, if $C < G$ and situation (b) occurs, then θ is unique.

PROOF. See Lemma (6.1) of [6]. ∎

The following is the main result of [5].

THEOREM 2.6. *Suppose that A acts coprimely on G and let H be an A-invariant subgroup of G. Let $\chi \in \mathrm{Irr}_A(G)$ and $\theta \in \mathrm{Irr}_A(H)$.*

(a) If $\theta^G = \chi$, then $(\theta^*)^C = \chi^*$.

(b) If $\chi_H = \theta$, then $\chi^*_{H\cap C} = \theta^*$.

PROOF. See Theorem A of [5]. ∎

Notice that when A is solvable, Theorem 2.6 is a consequence of the following result.

THEOREM 2.7. *Suppose that a solvable group A acts coprimely on G and let $H \subseteq G$ be an A-stabilized subgroup of G. If $\chi \in \mathrm{Irr}_A(G)$ and $\theta \in \mathrm{Irr}_A(H)$, then $[\chi_H, \theta] \geq [\chi^*_{C\cap H}, \theta^*]$.*

PROOF. See [9]. ∎

At this point, we observe that if $C \subseteq H \subseteq G$ and $\chi^* = (\chi')^*$, for $\chi \in \mathrm{Irr}_A(G)$ and $\chi' \in \mathrm{Irr}_A(H)$, we will have that $[\chi_H, \chi']| \geq 1$ when A is solvable. Then, to prove Theorem A and Theorem B (in the case $|G|$ odd) it suffices to show the following result.

THEOREM 2.8. *Suppose that A acts coprimely on a group G of odd order and let $C \subseteq H \subseteq G$ be an A-invariant subgroup of G. Let $\chi \in \mathrm{Irr}_A(G)$ and $\chi' \in \mathrm{Irr}_A(H)$ be such that $\chi^* = (\chi')^*$. Then χ' is an irreducible constituent of χ_H, $\chi'(1)$ divides $\chi(1)$ and $\frac{\chi(1)}{\chi'(1)}$ divides $|G : H|$.*

PROOF. We argue by double induction, first on $|G|$ and second on $|G : H|$.

Suppose first that $H < J < G$ where J is an A-invariant subgroup of G and let $\eta \in \mathrm{Irr}_A(J)$ be such that $\eta^* = \chi^*$. By induction on $|G : H|$, we have that η is an irreducible constituent of χ_J, that $\eta(1)$ divides $\chi(1)$ and that $\frac{\chi(1)}{\eta(1)}$ divides $|G : J|$. Now, observe that by definition, $\chi' = \eta'$ (relative to H), and therefore, by induction on $|G|$, we have that χ' is an irreducible constituent of η_J, that $\chi'(1)$ divides $\eta(1)$ and that $\frac{\eta(1)}{\chi'(1)}$ divides $|J : H|$. In this case, we see that the theorem is proved.

So we may assume that H is A-maximal (*i.e.*, that G is the unique A-invariant subgroup containing properly H).

If H is normal in G, by Lemma (2.5) of [13], we have that χ' is an irreducible constituent of χ_H. But in this case, $\chi'(1)$ divides $\chi(1)$, by Clifford's Theorem and $\frac{\chi(1)}{\chi'(1)}$ divides $|G : H|$ by (11.29) of [2].

So we assume that H is not normal in G. Let $L = \mathrm{core}_G H$ and let K/L be a chief factor of Γ contained in G. By a general group theoretical argument, we know that H is a complement of K/L in G, that $\mathbf{C}_G(K/L) = K$ and that all A-invariant complements of K/L in G are C-conjugate to H, and therefore, equal to H.

By Theorem (13.27) of [2], we may choose $\varphi \in \mathrm{Irr}_A(L)$ under χ. Now, let $T = \mathrm{I}_G(\varphi)$ be the stabilizer in G of φ (an A-invariant subgroup) and let $\mu \in \mathrm{Irr}_A(T|\varphi)$ be the Clifford correspondent of χ over φ (μ is A-invariant by the uniqueness of the Clifford correspondence).

Suppose first that T is proper in G. Since $T\cap C \subseteq T\cap H \subseteq T$, we let $\mu' \in \mathrm{Irr}_A(T\cap H)$ be such that $\mu^* = (\mu')^*$. By induction, we have that μ' is an irreducible constituent of μ that (since $L \subseteq T\cap H$) necessarily lies over φ. Hence, $\mu' \in \mathrm{Irr}(T\cap H|\varphi)$ and thus, by the Clifford correspondence, $(\mu')^H \in \mathrm{Irr}(H)$.

By Lemma (2.5) of [13] (or Theorem 2.6(a), if we wish), we have that

$$(\mu^*)^C = \chi^*$$

and

$$\big((\mu')^H\big)^* = \big((\mu')^*\big)^C = (\mu^*)^C = \chi^*.$$

Thus,

$$(\mu')^H = \chi'$$

and since μ' is an irreducible constituent of $\chi_{T\cap H}$, it follows that $\chi' = (\mu')^H$ is an irreducible constituent of χ_H. So in the case that T is proper in G we just have to check the divisibility conditions.

By the inductive hypothesis, we have that $\mu'(1)$ divides $\mu(1)$ and that $\frac{\mu(1)}{\mu'(1)}$ divides $|T : T\cap H|$. By Lemma 2.2, we know that $|H : T\cap H|$ divides $|G : T|$. Now, we have that $\mu'(1)|H : T\cap H|$ divides $\mu(1)|G : T|$ and thus $\chi'(1)$ divides $\chi(1)$, as wanted. Also,

$$\frac{\chi(1)}{\chi'(1)} = \frac{|G : T|}{|H : T\cap H|}\frac{\mu(1)}{\mu'(1)}$$

divides

$$\frac{|G : T|}{|H : T\cap H|}|T : T\cap H| = |G : H|$$

and this proves the theorem in the case that T is proper in G.

So we may assume that φ is Γ-invariant. Now we want to apply Lemma 2.5 and we check its hypotheses. First of all, since H is A-maximal, observe that K/L is a chief factor of Γ and since G is solvable, it is an abelian p-group for some prime p dividing $|G|$ (and hence not dividing $|A|$). Then Γ is p-solvable. Now, since $\mathbf{C}_\Gamma(K/L)$ is a normal subgroup of Γ with normal Hall subgroup $K = \mathbf{C}_G(K/L)$, by the Schur-Zassenhaus Theorem, we have that $\mathbf{C}_\Gamma(K/L) \subseteq KA < \Gamma$. By Theorem (11.27) of [2], let $\theta \in \mathrm{Irr}_A(K)$ under χ which necessarily lies over φ (and, therefore, observe that θ is $\mathbf{C}_\Gamma(K/L)$-invariant). Since $C\cap K \subseteq H\cap K = L$, by Lemma (1.5) of [13], θ is the unique A-invariant member of $\mathrm{Irr}(K|\varphi)$. Therefore, by Lemma 2.5, in both cases (a) and (b), we have that θ is Γ-invariant.

Now, since K/L is an abelian chief factor of Γ, by the Going Down Theorem (6.18) of [2], we have two possibilities: $\theta_L = \varphi$ or $\theta_L = e\varphi$, with $e^2 = |K : L|$.

In the first case, by Lemma (10.5) of [1], we have that $\chi_H \in \mathrm{Irr}_A(H)$. By Theorem 2.6(b), $(\chi_H)^* = \chi^*$ and hence $\chi_H = \chi'$. In this case, everything follows.

So suppose that $\theta_L = e\varphi$, with $e^2 = |K/L|$. Since $|K/L|$ is odd and $(\Gamma, K, L, \theta, \varphi)$ is a character five, we may choose X to be a good complement for this character five and, since conjugates of good complements are good complements, there is no loss if we assume, by the Schur-Zassenhaus Theorem, that X contains A. In this case, $X \cap G$ is an A-invariant good complement of $(G, K, L, \theta, \varphi)$ and by comments above we deduce that H is an A-invariant good complement for $(G, K, L, \theta, \varphi)$. In the notation of Theorem 2.3, we may write $\chi_H = \psi_H \chi_0$. Since G has odd order, we know that χ_0 is an irreducible constituent of χ_H and also that $\chi(1) = \psi(1)\chi_0(1) = e\chi_0(1)$. Therefore, $\chi_0(1)$ divides $\chi(1)$ and $\frac{\chi(1)}{\chi_0(1)} = e$ divides $|G : H|$. Since $\chi_0 = \chi'$, by Theorem 2.4, the proof is complete. ∎

3. **An example.** Let K be an extraspecial group of order 5^3 with exponent 5 and write $L = \mathbf{Z}(K)$. If we consider SL(2, 5), it is well known that SL(2, 5) acts on K as automorphisms centralizing L and acting in the natural way on the 2-dimensional $\mathbf{Z}_5$-vectorial space K/L. It is well known that $S = \mathrm{SL}(2,3)$ lives in SL(2, 5) and acts Frobenius on K/L. We write $\Gamma = KS$, for the semidirect product.

Now, let Q be the quaternion Sylow 2-subgroup of S and let A be a Sylow 3-subgroup of S. If $G = KQ$, A acts coprimely on G. We let $H = LQ$, an A-invariant subgroup of G. Since $\mathbf{C}_{K/L}(A) = 1$, by Lemma (3.5) of [7], we have that $C = \mathbf{C}_G(A) \subseteq H$. Also, if $Z = \mathbf{Z}(Q)$, we have that $LZ \subseteq C \subseteq H$. Since $\mathbf{C}_{Q/Z}(A) = 1$, it follows that $\mathbf{C}_{H/LZ}(A) = 1$ and hence $C = LZ$.

Now, let $\varphi \in \mathrm{Irr}(L)$ be a faithful linear character of L and write $\varphi^K = 5\theta$, where $\theta \in \mathrm{Irr}(K)$ has degree 5. Then, θ is Γ-invariant (because φ is) and since $(|K|, |\Gamma : K|) = 1$, θ extends to some character $\hat{\chi} \in \mathrm{Irr}(\Gamma)$.

Now let $\chi = \hat{\chi}_G \in \mathrm{Irr}_A(G)$. To know how χ restricts to H we use Theorem 2.3 and the values of the character ψ given in p. 619 of [1]. By Theorem 2.3, since $|K : L|$ is odd, we have that $\chi_H = \psi_H \chi_0$, where $\chi_0 \in \mathrm{Irr}_A(H|\varphi)$.

Since $\mathbf{C}_{K/L}(x) = 1$ for all $1 \neq x \in Q$, the values of $\psi \in \mathrm{Char}(Q)$ are $\psi(x) = -1$ if $o(x) = 4$, $\psi(z) = 1$ if $o(z) = 2$ and $\psi(1) = 5$. That is to say

$$\psi = \beta + \lambda_2 + \lambda_3 + \lambda_4$$

where β is the unique character of Q of degree 2 (and therefore A-invariant) and $\mathrm{Irr}(Q/Q') = \{1_Q, \lambda_2, \lambda_3, \lambda_4\}$. Since $\mathbf{C}_{Q/Q'}(A) = 1$, 1_Q is the unique A-invariant character of Q/Q'. Also, observe that $\chi_0(1) = 1$. Now

$$\chi_H = \beta\chi_0 + \lambda_2\chi_o + \lambda_3\chi_0 + \lambda_4\chi_0.$$

By Gallagher's Theorem (6.17) of [2], it follows that $\beta\chi_0$ is the unique A-invariant constituent of χ_H which necessarily (by Theorem A) has to be χ'. Since $\chi'(1) = 2$ and $\chi(1) = 5$, this example shows that Theorem B is not true for solvable groups if H is of even order.

4. **Proof of Theorem C.** The proof of Theorem C requires the main results of [7]. If χ is a character of a group G, we denote by χ^0 the restriction of χ to the set of π-elements of G.

THEOREM 4.1. *Suppose that A acts coprimely on a π-separable group G. Write $C = \mathbf{C}_G(A)$ and let $\chi \in \mathrm{B}_\pi(G)$ be A-invariant. Then $(\chi^*)^0 \in \mathrm{I}_\pi(C)$. Also, if H is an A-invariant Hall π-subgroup of G, then χ_H has an A-invariant Fong constituent. If α is any such constituent, then α^* is a Fong character belonging to $(\chi^*)^0$.*

PROOF. See Theorem (4.4) of [14], and Theorems A and (4.1) of [7]. ∎

Now we are ready to give the proof of Theorem C.

THEOREM 4.2. *Suppose that A acts coprimely on a π-separable group G and assume that $\mathbf{C}_G(A)$ is a π-group. Then $\mathrm{Irr}_A(G) \subseteq \mathrm{B}_\pi(G)$. Also, if H is an A-invariant Hall π-subgroup of G containing $\mathbf{C}_G(A)$ and $\chi \in \mathrm{Irr}_A(G)$, then $\chi' \in \mathrm{Irr}_A(H)$ is the unique A-invariant Fong character associated with $\chi \in \mathrm{B}_\pi(G)$.*

PROOF. We prove that $\mathrm{Irr}_A(G) \subseteq \mathrm{B}_\pi(G)$ by induction on $|G|$. Let $\chi \in \mathrm{Irr}_A(G)$ and let N be a maximal normal A-invariant subgroup of G. By (11.27) of [2], we may choose $\theta \in \mathrm{Irr}_A(N)$ under χ and by induction we know that $\theta \in \mathrm{B}_\pi(N)$. Since G is π-separable, it follows that G/N is a π-group or a π'-group.

If G/N is a π-group, by Corollary (7.6) of [3], $\chi \in \mathrm{B}_\pi(G)$ and we are done in this case. So we may assume that G/N is a π'-group. In this case, observe that $C \subseteq N$ and, therefore, by Lemma (1.5) of [13], we know that χ is the unique member of the set $\mathrm{Irr}_A(G|\theta)$. Now, by Theorem (6.2.b) of [3], we know that $\mathrm{B}_\pi(G|\theta) = \{\xi\}$. Since θ is A-invariant, by uniqueness ξ is A-invariant and hence $\chi = \xi \in \mathrm{B}_\pi(G)$, as wanted.

Now, let $C \subseteq H \subseteq G$ be an A-invariant Hall π-subgroup of G (in fact, H is the only one) and let $\chi \in \mathrm{Irr}_A(G)$. Since $\chi \in \mathrm{B}_\pi(G)$, by Theorem 4.1, we may find $\alpha \in \mathrm{Irr}_A(H)$ an A-invariant Fong character belonging to χ^0. Then, by Theorem 4.1, we have that α^* is a Fong character belonging to $(\chi^*)^0$. Since $C \subseteq H$, what we have is that $\chi^* = \alpha^* = (\chi')^*$. Hence, $\alpha = \chi'$ by the Glauberman-Isaacs bijection. Therefore, χ' is the unique A-invariant Fong character of H associated with $\chi \in B_\pi(G)$. ∎

As it was observed by the referee the following is also true. (We take this opportunity for thanking him for many useful comments).

THEOREM 4.3. *Suppose that A acts coprimely on a π-separable group G. If $\mathrm{Irr}_A(G) \subseteq \mathrm{B}_\pi(G)$, then $\mathbf{C}_G(A)$ is a π-group.*

PROOF. We prove that $C = \mathbf{C}_G(A)$ is a π-group by induction on $|G|$. If N is a nontrivial normal A-invariant proper subgroup of G, since $\mathrm{Irr}_A(G/N)$ and $\mathrm{Irr}_A(N)$ are B_π-characters (because any A-invariant irreducible character of N lies under some A-invariant irreducible character of G, by Theorem (13.31) of [2] and its preceding comments, and because normal irreducible constituents of B_π-characters are B_π-characters [3]), by induction, we have that $C \cap N$ and $\mathbf{C}_{G/N}(A) = CN/N$ are π-groups. So we may assume that G has no normal A-invariant subgroups. Since G is π-separable, by considering $\mathbf{O}_\pi(G)$ and $\mathbf{O}_{\pi'}(G)$, it follows that G is a π-group or a π'-group. In the first case, we are done. In the second, since, by definition, B_π-characters are induced from π-special characters and the unique π-special character in a π'-group is the trivial character, it follows that $\mathrm{Irr}_A(G) = \{1_G\}$. Hence, $\mathbf{C}_G(A) = 1$ (by the Glauberman-Isaacs bijection) and the result follows. ∎

Finally, we prove Theorem B of the introduction.

THEOREM 4.4. *Suppose that A acts coprimely on a solvable group G and assume that H is an A-invariant odd order subgroup of G containing $\mathbf{C}_G(A)$. If $\chi \in \mathrm{Irr}_A(G)$, then $\chi'(1)$ divides $\chi(1)$. Moreover, $\frac{\chi(1)}{\chi'(1)}$ divides $|G:H|$.*

PROOF. Since G is solvable, we may find M an A-invariant 2-complement of G containing H. By applying Theorem C, we have that $\chi \in \mathrm{B}_{2'}(G)$. Now, let $\alpha \in \mathrm{Irr}_A(M)$ be such that $\alpha^* = \chi^*$, and we know that α is a Fong character associated with χ and hence that $\alpha(1)$ divides $\chi(1)$ and that $\frac{\chi(1)}{\alpha(1)}$ divides $|G:M|$. Now, observe that $\alpha' = \chi'$ and since M is of odd order, by Theorem 2.8, it follows that $\chi'(1)$ divides $\alpha(1)$ and $\frac{\alpha(1)}{\chi'(1)}$ divides $|M:H|$. Now the result is proved. ∎

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