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On the differential properties of algebraic morphisms into Grassmannians

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In the thesis [Pe1] it was introduced, studied and applied a general theory of Weierstrass loci for vector bundles on a smooth curve. The results, proofs, background, examples and motivations of this thesis are contained in [Pe2]. We believe that this theory, at least in characteristic 0, is the 'right' one. The aim of this paper is to introduce and study an extension of [Pe1] to the case of higher dimensional varieties. At least two possible theories seem to be useful and natural; see the discussion just after Remark 1.1 and Section 4. We strongly prefer the 'symmetric' one (see Definition 1.5). In the first section we introduce the general theory and give the main general results. In the second section we study in details the case of $\mathbf{P}^2$ for three reasons: it is nice; it shows how to use the general theory and what could be expected in more general situations and (last but not least) to convince the reader that it is technically easier and often more interesting to work in the 'symmetric' set up. Then in the third section we apply the method of Section 2 to a much more general situation (essentially, any variety X as base of the vector bundle). In the fourth section we give the set up and start the analysis of specific examples of what happens near a specific point P of the base variety X (even when X is singular at P). Here, except at the first step we are able to work only with the 'symmetric' definition.

Many of the main statements in the first section of this paper are the same as in the case of curves considered in [Pe1] and [Pe2]. Sometimes, also the proofs are the same; in this case we will just quote [Pe2]. Sometimes the proofs are different and we give the details. Sometimes the proofs are just a reduction to the corresponding statement for curves proved in [Pe1] and [Pe2]. In all 3 situations, the main point of this paper is that the general set-up introduced here is the right one to make life easy.

We assume that the algebraically closed base field has characteristic 0. A sheaf F on a normal scheme is called *reflexive* if the natural map from F to its bidual F^{**} is an isomorphism. Let X be a normal scheme and U an open subscheme with

$\operatorname{codim}(X\backslash U) \geqslant 2$. For every reflexive sheaf F on U there is a unique reflexive sheaf G on X with $G|U = F$; even if F is locally free, G need not be locally free. Vice versa, every reflexive sheaf on X is uniquely determined by its restriction to any open subscheme of X whose complement has codimension at least 2. In our opinion the natural category will turn out to be the one of 'normal algebraic schemes up to codimension 2'. This will be our working category in Sections 1 and 3. We will sometimes call any reflexive sheaf on X a *bundle* and say that a map $u : A \to B$ between reflexive sheaves on X is *surjective* if $\operatorname{Codim}(\operatorname{Supp}(\operatorname{coker}(u))) \geqslant 2$. Furthermore, exact sequences of bundles in this category will be just complexes of reflexive sheaves whose cohomology has support of codimension $\geqslant 2$. Often (and in particular in most of the main statements) we will write the words 'up to codimension $\geqslant 2$', just as a warning for the reader. A very important technical motivation for working in th is category is that for the duality theorem and the 'symmetric' definition we had to take duals.

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1.

Let X be a normal connected variety, E a rank r reflexive sheaf (i.e. a 'bundle') on X and $V \subseteq H^0(X, E)$ a vector space of sections spanning E outside codimension $\geqslant 2$. We stress again that we are working 'up to codimension 2'. Let $f : X \to G := G(V, r)$ (the Grassmannian of r dimensional quotient spaces of V) be the associated morphism (i.e. the rational map of schemes defined outside codimension $\geqslant 2$). On the Grassmannian G there is a 'tautological' exact sequence:

$$0 \to S \to V_G \to Q \to 0, \tag{1}$$

with V_G the trivial vector bundle of rank $\dim(V)$, Q the universal rank r quotient bundle and S the universal rank $(\dim(V) - r)$ subbundle. By definition of f we have $f^*(Q) = E$. Let

$$0 \to S_E \to V_X \to E \to 0 \tag{2}$$

be the pull-back of (1) by f. Since $TG \cong \operatorname{Hom}(S, Q)$, the differential $TX \to f^*(TG)$ induces a morphism $\partial' : S_E \to \Omega_X \otimes E$ or equivalently a morphism $\partial\colon S_E \otimes TX \to E$. Set $E_1 := (\operatorname{Coker}(\partial)/\operatorname{Tors}(\operatorname{Coker}(\partial)))^{**}$ and $r_1 := \operatorname{rank}(E_1)$; E_1 is a reflexive sheaf on X. The codimension 1 torsion part of $\operatorname{Coker}(\partial)$ is called the *first ramification locus*. The surjection $V_X \to E$ factors through the natural projection from the bundle of first order principal parts $P^1(E)$ of E to E. Let

$$0 \to \Omega_X \otimes E \to P^1(E) \to E \to 0 \tag{3}$$

be the exact sequence associated to $P^1(E)$ and $u\colon V_X \to P^1(E)$ 'the first order Taylor map' (see for instance [Pe2], Sect. 2). As remarked by the referee, the following commutative diagram with exact rows

$$\begin{tikzcd} & & V_X \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega_X \otimes E \arrow[r] & P^1(E) \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

induces a map $\partial'' : S_E \to \Omega_X \otimes E$. It was checked in [Pe2], Proposition 2.1, when X is a smooth curve that, up to a sign, $\partial' = \partial''$, giving a geometric reason for the definition of E_1. The same proof works for any normal X because his proof is based essentially on the fact that the corresponding assertion is true for the identity morphism of the Grassmannian G. Note that E_1, as quotient (outside codimension 2) of E, is a quotient of V (outside codimension 2). Hence, if $r_1 \neq 0$, we may iterate the construction obtaining another bundle E_2 as a quotient of E_1 and another ramification locus corresponding to the torsion part. Hence we get a chain of surjections (outside codimension 2)

$$V \to E \to E_1 \to E_2 \to E_3 \to \cdots . \tag{4}$$

Set $E_0 := E$. Set $r_i := \mathrm{rank}(E_i)$. As in the case of smooth curves, the bundle E_i is called the *ith derived bundle* of (X, E, V). The integer $r_i - r_{i+1}$ is called the *ith differential rank* of (X, E, V). The non increasing sequence of integers r_i stabilizes to a certain value $\alpha \geqslant 0$.

Remark 1.1. By [Pe2], Remark 6.1.2, if X is a smooth curve (in characteristic 0, as always) the integer α is exactly the rank of a trivial factor W of E (hence the geometric situation is determined by $(V/W, E/W)$); the same proof works for every X. Alternatively, the result for reflexive sheaves follows formally from the result for bundles on curves applied to a family of smooth curves covering a Zariski open subset of X and whose general member is contained in the open subset of X_{reg} on which the reflexive sheaf is locally free.

Let $P^t(E)$ be the bundle of t-order principal parts of E. We have a chain of maps

$$V_X \to \cdots \to P^t(E) \to P^{t-1}(E) \to \cdots \to E.$$

These maps, except the first one, are surjections. The image $G^t(E)$ of V_X in $P^t(E)$ is called ([Pi]) the *osculating bundle of order t*.

In our opinion the main drawback of the iterative definition of the bundles E_j comes from the fact that if $\dim(X) > 1$ in this way one cannot recover from the theory the integrability condition (i.e. the symmetry condition $\partial^2_{xy} = \partial^2_{yx}$). This is the reason (both technical and conceptual) why in this paper most of the results on derived bundles concern just the first derived bundle.

In [Pe2], Proposition 4.4, it was proved that when $\dim(X) = 1$ for every $i > 0$ the surjection $E \to E_i$ factors through the surjection $G^i(E_i) \to E_i$. We

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leave to the reader to check that (thanks to our set up) for $i = 1$ the proof of [Pe2], Proposition 4.4, works verbatim for arbitrary $\dim(X)$, i.e. that we have the following result.

THEOREM 1.2. *The surjection $E \to E_1$ factors through $G^1(E_1)$.*

Major results contained in [Pe1] and [Pe2] are two duality theorems ([Pe2], 7.1 and 7.1.2, for derived bundles, and [Pe2], 8.4.1, for torsion sheaves) for $\dim(X) = 1$. Now we will prove that the statement of [Pe2], 7.1, and the part with $i = 1$ of [Pe2], 8.4.1, hold for arbitrary X, i.e. that (up to codimension 2, as always) we have the following duality theorems.

THEOREM 1.3 (Duality Theorem for the first derived bundle). *The kernel of the map $f_1 : V_X \to E_1$ is the dual of the first osculating bundle of $V_X^* \to S_E^*$. The kernel of the map $V \to G^1(E)$ is the dual of the first derived bundle of $V_X^* \to S_E^*$.*

Proof. As in [Pe2], 7.1, the second statement follows in a formal way from the first one. Let K be the double dual of the cokernel of the natural inclusion $G^1(S_E^*)^* \to V_X$. We have to check that $K \cong E_1$ as quotients of V_X. Fix a smooth curve $C \subseteq X$ such that the map $V_C \to E|C$ is surjective; denote with $E_{1;C}$ the corresponding first derived bundle on C. The natural surjection $G^1(S_E^*) \to S_E^*$ induces a commutative diagram (5):

$$\begin{tikzcd}
0 \arrow[r] & S_E \arrow[r] \arrow[d] & V_X \arrow[r] \arrow[d, "\mathrm{Id}"] & E \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & G^1(S_E^*)^* \arrow[r] & V_X \arrow[r] & K &
\end{tikzcd} \tag{5}$$

We have to check that the map $\alpha : E \to K$ factors through E_1 and that the induced map $\beta : E_1 \to K$ is an isomorphism. Since E is reflexive, to obtain β we have to check that Ker(α) contains Im(∂). Fix a general $P \in X$; since a is linear it is sufficient to show that for a general tangent vector $\mathbf{t}$ to X at P, we have $\alpha(E_P \otimes \mathbf{t}) = 0$. We may find a smooth curve C as above with $P \in C$ and $\mathbf{t}$ as tangent space to C at P. The natural surjection $P_X^1(E)|C \to P_C^1(E|C)$ induces an inclusion $G_C^1(S_{E|C}^*)^* \to G^1(S_E^*)^*|C$ and this inclusion and the Duality Theorem for C give that $\alpha(E_P \otimes \mathbf{t}) = 0$. Hence we have $\beta : E_1 \to K$ which by construction is surjective (outside codimension 2). To show that β is injective we will use the following diagram chasing. Take a general $P \in X$ and an element a of the fiber $E_1|P$ of E_1 at P with $\beta(a) = 0$. Lift a to $b \in V_X|P$. Since $\beta(a) = 0$, we have $b \in G^1(S_E^*)^*|P$. There is a smooth curve C with $P \in C$ and such that the natural inclusion $G_C^1(S_{E|C}^*)^* \to G^1(S_E^*)^*|C$ has b in its image at C; call b'

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the corresponding element of $G^1_C(S^*_{E|C})^*|P$. By the result for C, i.e. by the exact sequence

$$0 \to G^1_C(S^*_{E|C})^* \to V_C \to E_{1;C} \to 0,$$

b' goes to 0 into $E_{1;C}$. By the definitions and the generality of P, the vector space $E_1|P$ is a quotient of $E_{1;C}|P$. Hence $a = 0$, as wanted. □

THEOREM 1.4. (Duality Theorem for the first torsion sheaf). *The first torsion sheaf of (E, V) is the first torsion sheaf of the pair $(G^1(S^*_E), V^*)$.*

Proof. It is sufficient to check that the two sheaves on X in the statements have the same restriction to any general member of any large family of curves. This follows from the corresponding result for curves ([Pe2], Prop. 8.4.1) and the way we proved the Duality Theorem for the first derived bundle. □

Motivated by Theorem 1.3, the discussion just after Remark 1.1 and the duality theorems for curves ([Pe2], Cor. 7.2.1 and Prop. 8.4.1), we introduce the following definition.

DEFINITION 1.5. The *ith symmetric differential bundle* E^i of E is the double dual of the cokernel of the natural inclusion of $G^i(S^*_E)^*$ into V_X; the *ith symmetric torsion sheaf* is the first torsion sheaf of the surjection $V_X \to G^i(S^*_E)$.

Thus we took the duality theorems as the definition of ith symmetric differential bundle and ith torsion sheaf. We hope to convince the reader that in this way we will obtain a very powerful notion. In this way we loose any hope to obtain also a positive characteristic theory, because $S^t(E)^* \neq S^t(E^*)$ in characteristic $< t$.

If X, E and V are clear from the context, s_i will denote the rank of the ith symmetric differential bundle of (E, V) (with $s_0 := r := \text{rank}(E)$).

PROPOSITION 1.6. *Assume the existence of a dense open subset U of X such that the Taylor series map $V \to P^1_X(E)$ is surjective on U. Then $E_1 = 0$ and the first torsion sheaf is supported on $X \backslash U$.*

Proof. By the discussion just after equation (3) the map ∂ is the composition of $(\partial', \text{Id}_{TX}) : S_E \otimes TX \to E \otimes \Omega \otimes TX$ with the surjective map $(\text{Id}_E, c) : E \otimes \Omega \otimes TX \to E$ induced by the contraction morphism $\Omega \otimes TX \to \boldsymbol{O}$. The result follows from the surjectivity of the map $\partial' : S_E \to E \otimes \Omega$. □

Using [L], Theorem 1.3, it would be easy to apply 1.6 for the case $X = \mathbf{P}^2$ and $r = 2$. However, in the next section we will obtain stronger results.

2.

In this section we will consider in detail the case $X = \mathbf{P}^2$. The aim is to convince the reader that it is technically easy to work with the notions of ith symmetric

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differential bundles and torsion sheaves. In this section X will denote $\mathbf{P}^2$. We will look for stable vector bundles on $\mathbf{P}^2$. If we drop the requirement of stability (either for E or for S_E) it is much easier to obtain the corresponding results (in a much wider range) for every X. This will be the content of Section 3. For all integers r, c_1 and c_2 let $M(r; c_1, c_2)$ be the moduli scheme of rank r stable vector bundles on $\mathbf{P}^2$ with Chern classes c_1 and c_2. The triples (r, c_1, c_2) such that $M(r; c_1, c_2) \neq \emptyset$ are completely described in [DL] and [HL]. It is known ([El]) that $M(r; c_1, c_2)$ is always irreducible and smooth (if not empty). For any $P \in X$, let $P(k+1)$ be the kth infinitesimal neighborhood of P in X, i.e. define $P(k+1)$ by the relation $\boldsymbol{I}_{P(k+1)}Y := (\boldsymbol{I}_P)^{k+1}$. By [Br], Theorem 5.1, there is a Zariski open dense subset U of $M(2; c_1, c_2)$ such that for every $F \in U$ and every integer t we have either $h^0(F(t)) = 0$ or $h^1F(t)) = 0$; this open set U will be called the *open stratum*. Any bundle U satisfying the cohomology condition 'for each integer t at most 1 of the integers $h^i(X, U(t)), i = 0$, 1 and 2, is $\neq 0$' is said to have the natural cohomology. A stable bundle U with $\mathrm{rank}(U) = 2$ has the natural cohomology if and only if it is in the open stratum because $V^* \cong V(-c_1(V))$ for every rank 2 bundle V. We stress that Remarks 2.14, 2.16, 2.17.1 and Theorem 2.15 are just examples and that with more effort other related statements could be proved. We will need the following computations of Chern classes.

Fix integers r and k with $r \geqslant 2$, $k > 0$. Note that for every rank s vector bundle U and every integer t we have $c(U) = s + c_1(U)(c_1(U)+3)/2 - c_2(U)$ (Riemann–Roch), $c_1(U(t)) = c_1(U) + st$ and $c_2(U(t)) = c_2(U) + (s-1)c_1(U) \cdot t + (s(s-1)/2)t^2$. Consider the following exact sequence of vector bundles:

$$0 \to F \to (r+2)\boldsymbol{O} \to E \to 0 \tag{6}$$

with $\mathrm{rank}(E) = r$ and $\mathrm{rank}(F) = 2$. Set $c_i := c_i(E)$. We have $F^* \cong F(c_1)$, $c_1(F) = -c_1$, $c_2(F) = c_1^2 - c_2$, $c_i(F)^*) = (-1)^i c_i(F)$, $c_1(F^*(-k)) = c_1 - 2k$, $c_2(F^*(-k)) = c_1^2 - c_2 + kc_1 + k^2$ and $c(F^*(-k)) = 2 + (c_1 - 2k)(c_1 - 2k + 3)/2 - c_1^2 + c_2 - kc_1 - k^2$.

Let A be a set of vector bundles on X; we will say that A is 'generic' at $M \in A$ if A contains a neighborhood of a versal deformation space of E. This is roughly consistent with the notion of generic used in [HL]. Let A, B be two sets of vector bundles and $\mathbf{b}\colon A \to B$ a map (of sets); we will say that $\mathbf{b}$ is 'generic' at $M \in A$ if A is 'generic' at M, B is 'generic' at $\mathbf{b}(M)$ and $\mathbf{b}$ sends a small (versal) neighborhood of M in A onto a versal neighborhood of $\mathbf{b}(M)$. We will use the following map $\mathbf{a}$ from two sets of vector bundles respectively of rank s and rank r. Set $v := s + r$. Let M be a rank s vector bundle such that $h^0(M) = 0$ and M is embedded in a rank v trivial vector bundle, $V \otimes \boldsymbol{O}$, i.e. such that $h^0(M^*) \geqslant v$ and M^* is spanned by its global sections. Then we have an exact sequence

$$0 \to M \to V \otimes \boldsymbol{O} \to N \to 0 \tag{7}$$

with N rank r vector bundle; set $N := \mathbf{a}(M)$. If $h^0(M^*) = v$, then N is uniquely determined by M; if $h^0(M^*) > v$, we make this construction $\mathbf{a}$ for all such pairs

(M, V). Vice versa, starting with a spanned rank r vector bundle with $h^0(N) \geqslant v$ and N spanned, we have an exact sequence (7) and we set $M := \mathbf{a}'(N)$. Now assume that the inclusion $M \to V \otimes \boldsymbol{O}$ is not an embedding as vector bundles, but only an inclusion of sheaves with torsion free quotient N (i.e. it is an embedding in the category of sheaves up to codimension two). Note that $h^2(N) = h^2(N^{**})$ and $h^1(N) \geqslant h^1(N^{**})$ because $\mathrm{Supp}(N^{**}/N)$ is finite. Thus Lemmas 2.1 and 2.2 below hold even in this more general set up. However, $c_2(N^{**})$ is not uniquely determined by the Chern classes of M.

LEMMA 2.1. *Assume that the construction* **a** *is defined at* M. *If* $h^1(M^*) = 0$, *then* **a** *is defined over a 'generic' bundle near* M.

Proof. Since $h^1(M^*) = 0$, by semicontinuity we have $h^0(G^*) = h^0(M^*)$ for all bundles G near M. Since the spannedness condition is an open condition in a family of bundles with constant h^0, we conclude. □

LEMMA 2.2. *Assume that the construction* **a** *is defined at* M. *Assume* $h^1(M^*) = 0$ *and* $h^1(\mathbf{a}(M)) = 0$. *Then* **a** *is 'generic' at* M. *Furthermore,* $h^1(\mathbf{a}(M)) = 0$ *if* $h^2(M) = 0$, *i.e. by Serre duality if* $h^0(M^*(-3)) = 0$.

Proof. By semicontinuity we see that $h^0(A) = h^0(\mathbf{a}(M))$ for all bundles A near $\mathbf{a}(M)$. Hence the reverse construction $\mathbf{a}'$ is defined on a neighborhood of $\mathbf{a}(M)$ in its versal deformation space. Hence **a** is 'generic'. The last assertion follows from the cohomology of the exact sequence (7). □

Now we will give some criteria to check that the kth principal part bundle $P^k(A)$ of a bundle A is spanned or is generically spanned by the global sections of A. First, we have three obvious remarks and an easy lemma. We fix the integer $k > 0$. Note that for a point $P \in X$ we have $\mathrm{length}(P(k+1)) = (k+2)(k+1)/2$.

First, the obvious rank 1 case (e.g. use the homogeneity of every line bundle on X).

Remark 2.3. The kth principal part bundle of a line bundle $\boldsymbol{O}(x)$ is generically spanned if and only if it is spanned and this is the case if and only if $x \geqslant k$.

Remark 2.4. Let A be a vector bundle. Then $P^k(A(t))$ is spanned by $H^0(A(t))$ for large t. If A is spanned, then $P^k(A(t))$ is spanned for $t \geqslant k$.

Remark 2.5. Set $s := \mathrm{rank}(A)$. Assume that $P^k(A)$ is generically spanned by $H^0(A)$. Then for a general vector space $V \subseteq H^0(A)$ with $\dim(V) \geqslant s(k+2)(k+1)/2$, $P^k(A)$ is generically spanned by V.

LEMMA 2.6. *Set* $s := \mathrm{rank}(A)$. *Assume that* $P^k(A)$ *is spanned by* $H^0(A)$. *Then for a general vector space* $V \subseteq H^0(A)$ *with* $v := \dim(V) \geqslant 2 + s(k+2)(k+1)/2$, $P^k(A)$ *is spanned by* V.

Proof. This is a dimensional count similar to a proof of Bertini theorem. Fix $P \in X$. Note that length$(P^k(A)|P) = s(k+2)(k+1)/2$. Hence the set of all subspaces W of $H^0(A)$ with dim$(W) = v$ and such that W does not span $P^k(A)$ at P has codimension $v - s(k+2)(k+1)/2 - 1$ in the Grassmannian $G(v, H^0(A))$. Then use dim$(X) = 2$. □

(2.7) In this subsection we will consider the case rank$(A) = 2$ and set $d := c_1(A)$, $d'' := c_2(A)$. Note that if $M = A^*$ and M and N are related by (7), we have $c_1(M) = -d$, $c_2(M) = d''$ and $d'' = c_1(N)^2 - c_2(N)$. We fix an integer $k > 0$ and we assume that A fits in an exact sequence

$$0 \to \boldsymbol{O}(k) \to A \to \boldsymbol{I}_W(d-k) \to 0 \tag{8}$$

with W general finite subset of X with $w :=$ card$(W) = d'' - k(d-k)$. Hence

$$0 \leqslant w \leqslant d'' - k(d-k). \tag{9}$$

Assume $2k < d$, i.e. that (8) is not a destabilizing sequence for A. The Cayley–Bacharach condition for (8) which gives the existence of such locally free A with W general is $w \geqslant 1 + h^0(\boldsymbol{O}(d-2k-3))$, i.e.

$$w \geqslant 1 + (d-2k-1)(d-2k-2)/2. \tag{10}$$

Hence (9) and (10) may be simultaneously satisfied if and only if

$$d^2 - (2k+3)d + 2k^2 + 6k + 4 \leqslant 2d''. \tag{11}$$

Now we study the conditions of Lemmas 2.1 and 2.2 and the local freeness of the corresponding sheaf N (when $A = M^*$).

LEMMA 2.8. *We have* $\chi(A) = 2 + d(d-3)/2 - d''$, $\chi(A(-3)) = 2 + (d-6)(d-9)/2 - d'' + 3d - 9 = \chi(A) - 3d + 18$, $\chi(A(-2)) = \chi(A) - 2d + 10$.

Hence we have $\chi(A(-3)) \leqslant 0$ if and only if

$$d^2 - 9d + 40 \leqslant 2d'' \tag{12}$$

and we have simultaneously $\chi(A(-3)) \leqslant 0$ and $\chi(A(-2)) \geqslant 0$ if and only if

$$d^2 - 9d + 40 \leqslant 2d'' \leqslant d^2 - 7d + 24. \tag{13}$$

However, by the Cayley–Bacharach type condition (11), the inequality (13) may be satisfied only if

$$k^2 + 3k - 10 \leqslant (k-2)d. \tag{14}$$

Since $2k < d$, this inequality is often satisfied. However, even for the values of (d, k) satisfying (14) the condition (13) for large k gives strong restrictions on the allowable pairs (d, d''). The condition '$\chi(A(-2)) \geqslant 0$' would give easily in our setting the spannedness of a suitable A and hence very explicit examples for the construction **a** with $\mathbf{a}(A)$ locally free and good integers $s_i(\mathbf{a}(M))$. However, this condition is too strong and we prefer to give a condition (see Lemma 2.12) for the spannedness of a general $M \in M(2, d, d'')$. An easy condition (see the first part of the proof of Theorem 2.15 or, alternatively, use the exact sequence (8) for $k = 1$) would be '$\chi(A(-1)) \geqslant 0$', i.e. $2d'' \leqslant d^2 - 2$.

LEMMA 2.9. *Assume $d \geqslant 2k$ (resp. $d > 2k$) and $w \geqslant 1 + (d - 2k - 1)(d - 2k - 2)/2$. Then for a general W every locally free A fitting in the exact sequence (8) is semistable (resp. stable).*

Proof. Since by the generality of W and the Cayley–Bacharach condition (10) we have $h^0(\boldsymbol{I}_W(d - 2k - 3)) = 0$, we see that there is no inclusion of $\boldsymbol{O}(x)$ into A with $x > k$ and $2x \geqslant d$. □

LEMMA 2.10. *Fix a locally free A given by (8) for a general W. Assume $d \geqslant 2k > 0$ and*

$$d^2 - (2k - 3)d - 4k + 2 \geqslant 2w. \tag{15}$$

Then $P^{k-1}(A)$ is generically spanned by $H^0(A)$.

Proof. Fix $P \in X$ and tensor (8) by $\boldsymbol{I}_{P(k)}$. If $P \notin W$ we have $\boldsymbol{I}_W \otimes \boldsymbol{I}_{P(k)} = \boldsymbol{I}_{W \cup P(k)}$. Note that (15) is equivalent to $(d-k+2)(d-k+1)/2 \geqslant w + k(k+1)/2$. Hence, moving W, by the generality of W and the assumptions on d, k and w we have $h^1(\boldsymbol{I}_{W \cup P(k)}(d - k)) = 0$. Hence we have $h^0(A \otimes \boldsymbol{I}_{P(k)}(d - k)) = h^0(A) - 2 \cdot \text{length}(P(k))$, as wanted. □

By Riemann–Roch and [Br], Theorem 5.1 (i.e. the fact that a general $M \in M(2, d, d'')$ has natural cohomology) we have the following lemma.

LEMMA 2.11. *A general $M \in M(2, d, d'')$ has $h^0(M) = h^0(A)$ if and only if*

$$2w = (d - k + 2)(d - k + 1). \tag{16}$$

LEMMA 2.12. *Assume $d \geqslant 4, 4d'' > d^2$ and $d(d - 3)/2 \geqslant 2 + d''$. Then a general $M \in M(2, d, d'')$ is spanned by its global sections.*

Proof. By assumption $M(2, d, d'') \neq \vee$. Since by [Br], Theorem 5.1, a general $M \in M(2, d, d'')$ has natural cohomology, by Riemann–Roch the last condition of the lemma is equivalent to the condition '$h^0(M) \geqslant 4$ for general M'. Fix a general $M \in M(2, d, d'')$. By [Br] it is given as an extension (8) with $k = 0$ and W general. By (8) it is sufficient to show that $\boldsymbol{I}_W(d)$ is spanned. By assumption we have $d \geqslant 4$ and $\chi(\boldsymbol{I}_W(d)) \geqslant 3$. If $\chi(\boldsymbol{I}_W(d)) \geqslant 6$, this is a particular case of [AH], Theorem 2.3. If we assume only $\chi(\boldsymbol{I}_W(d)) \geqslant 3$ the proof of [AH], Theorem 2.3, (i.e. essentially the quotation [BH], q.1) works taking $N + 1$ instead of N (hence with fibers of the first projection of dimension 2). □

Now we will consider the case of unstable rank 2 kernel bundles.

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Remark 2.13. Fix integers x, a, w with $0 \leqslant w$, $0 < a \leqslant x$. Let $W \subseteq X$ be a general subset with $\mathrm{card}(W) = w$. Then for a general $P \in X$ the restriction map $H^0(X, \boldsymbol{I}_W(x)) \to H^0(P(a), \boldsymbol{I}_W(x)|P(a))$ is surjective (i.e. rank $G^{a-1}(\boldsymbol{I}_W(x)) = a(a+1)$) if and only if $2w + a(a+1) \leqslant (x+2)(x+1)$.

Since $H^1(X, \boldsymbol{O}(t)) = 0$ for every t, the previous remark gives the following remark.

Remark 2.14. Fix a rank 2 vector bundle A fitting in the exact sequence (8) with W general. Assume $2k \geqslant d$ (i.e. A not stable). If $2k = d$ assume $w := \mathrm{card}(W) > 0$, i.e. $A \neq 2\boldsymbol{O}(d/2)$. Let e be the maximal integer with

$$0 \leqslant e \leqslant d - k \quad \text{and} \quad 2w + (e+2)(e+1) \leqslant (d-k+2)(d-k+1). \tag{17}$$

Then $\mathrm{rank}(G^b(A)) = (b+2)(b+1)$ for every $b < e$ and $\mathrm{rank}(G^b(A)) = (e+2)(e+1) + (b-e)(b-e+1)/2$ for every $e < b \leqslant k$.

THEOREM 2.15. *Fix integers r, d, c_2 with $d > r > 0$, $M(r, d, c_2) \neq \emptyset$, $d(d+3)/2 \geqslant 2 + c_2$. Then a general $N \in M(r, d, c_2)$ is spanned. Hence we may consider a general exact sequence* (7) *with $V \subseteq H^0(X, N)$, $\dim(V) = r + 2$ and $N \in M(r, d, c_2)$. Assume $5d^2 - 6d - 16 > 8c_2$. Then M is stable and general in $M(2, -d, d'')$ with $d'' := d^2 - c_2$.*

Proof. By [HL], Remarque 1 at p. 94 (i.e. by [HL], Cor. 3.2 at p. 93 which is based on the proof of [HL], Prop. 1.5 at p. 89), a general $N \in M(r, d, c_2)$ has the natural minimal free resolution and the natural cohomology. Since $c_1(N(-1)) = d - r > 0$, this implies that if $\chi(N(-1)) \geqslant 0$ we have $h^0(X, N(-1)) \neq 0$ and N, being a quotient of a direct sum of line bundles of non negative degree, is spanned. Hence we may assume $\chi(N(-1)) < 0$. Look at the proof of [HL], Proposition 1.5, which gives the minimal free resolution of N. In the first case of that proof, N is spanned. Hence we may assume to be in the second case, i.e. to have a minimal free resolution

$$0 \to (b+2+n)\boldsymbol{O}(-2) \xrightarrow{f,g} b\boldsymbol{O}(-1) \oplus (r+2+n)\boldsymbol{O} \to N \to 0, \tag{18}$$

with $n \geqslant 0$. Now we follow the referee. The morphism of vector bundles $f: (b+2+n)\boldsymbol{O}(-2) \to b\boldsymbol{O}(-1)$ is injective if and only if N is spanned. We claim the injectivity of f for general N, i.e. for general (f, g). This is proved by computing the dimension of the linear subspaces of $H^0(N)$ consisting of the sections vanishing at a given point x of $\mathbf{P}^2$. This subspace is exactly $g_x(\ker(f_x))$. Since g is injective on $\ker(f)$ by the injectivity (as map of bundles) of the map (f, g), this linear subspace is of dimension $n + 2$. By the claim a general N given by (18) is spanned, as wanted.

Since N^* is stable and $d > 0$, we have $h^2(X, N(-3)) = h^0(X, N^*) = 0$. Hence by (7) $h^1(X, M(-3)) = 0$, i.e. $h^1(X, M^*) = 0$. Hence M is 'generic' by Lemma 2.1. Since N is general, by [HL], Remarque 1 at p. 94, N has the natural cohomology, i.e. $h^1 X, N) = 0$, $h^0(X, N) = \chi(N) = d(d+3)/2 + r \geqslant r + 2$.

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Assume M not stable and hence $A := M^*$ not stable. Fix a destabilizing exact sequence (8) for A with $2k \geqslant d > 0$. Since $d - 2k - 3 < 0$, the Cayley–Bacharach condition is trivially satisfied and any locally complete intersection W with

$$w := \text{length}(W) = d'' - k(d - k) \tag{19}$$

corresponds to a locally free sheaf. Since A is 'generic' we may assume W reduced and W general. Since A is spanned we have $h^0(X, \boldsymbol{I}_W(d - k)) \neq 0$. Since W is general this implies

$$w \leqslant 1 + (d - k + 2)(d - k + 1)/2. \tag{20}$$

Since $2k \geqslant d$, by (19) and (20) we find $d^2 - c_2 := d'' \leqslant (\frac{3}{8})d^2 + (\frac{3}{4})d + 2$, contradiction. □

Remark 2.16. The union of 2.15 and 2.10 gives examples of stable spanned E with stable kernel bundle and with a few symmetric derived bundles with the smallest possible rank.

(2.17) In this subsection we will consider briefly the case of homogeneous vector bundles. Since we are in characteristic 0 by a theorem of Matsushima they are in principle known from the representation theory of $SL(3)$. If a homogeneous bundle is spanned and the map to the Grassmannian is given by a complete linear system, the kernel bundle is homogeneous. Since all the associated maps are equivariant, they have everywhere constant rank. In particular all torsion sheaves and all symmetric torsion sheaves vanish. Here are the examples with a line bundle as kernel rank.

EXAMPLE 2.17.1. Fix an integer $k > 0$ and set $v := (k+2)(k+1)/2$, $4r := v - 1$. Let $i(k)$ be the embedding of $\boldsymbol{O}(-k)$ into $\boldsymbol{O}^{\oplus v}$ induced by $H^0(X, \boldsymbol{O}(k))$. Set $J\{k\} := \text{Coker}(i(k))$. The bundles on $\mathbf{P}^n$ corresponding to $J\{k\}$ were studied in recent years for many different reasons (see e.g. [Pa] and references therein). $J\{k\}$ is a homogeneous rank r vector bundle. Since $TX(-1)$ is spanned and $\text{rank}(G^m(\boldsymbol{O}(k)) = (m + 1)(m + 2)/2$ for every $m \geqslant 0$, by Remarks 2.3 and 2.4 we have $r - s_m = (m + 1)(m + 2)/2$ for every $m < k$ and $s_k = r$.

3.

In this section we will consider the same problem on an arbitrary X but taking as kernel bundle an unstable vector bundle built using direct sums of line bundles. In this way we will cover a very large range of possible Chern classes for E, a range containing an aerea for which no stable bundle exists (see Remark 3.4). As in Section 2 it is sufficient to find a bundle E such that S_E^* has good cohomological properties. This will be done in the proof of Theorem 3.3 for $\dim(X) = 2$ and in

3.1 for $\dim(X) > 2$. Recall that in this section we work in the category of sheaves on X 'up to codimension 2'.

(3.1) We describe here the construction of vector bundles made in [BC]. Assume (to be on the safe side) X smooth. Fix an integer $s := v - r$ and a rank $(s-1)$ vector bundle F on X with $H^2(X, F) = 0$. Let Y be a locally complete intersection subscheme of codimension 2 of X. Let $N_{Y,X}$ be its normal bundle and assume that $\det(N_{Y,X}) \otimes F$ (considered as a rank $(s-1)$ vector bundle on Y) has a nowhere vanishing global section on Y. Then there exists an algebraic vector bundle S of rank s on X given as an extension

$$0 \to F \to S \to \boldsymbol{I}_Y \to 0. \tag{21}$$

For every $M \in \mathrm{Pic}(X)$ call $\{Y; M\}$ the contribution in the Chow ring of X made by the total Chern character of the sheaf $\boldsymbol{I}_Y \otimes M$ (taking a resolution of it by locally free sheaves). Then the Chern class $c_i(S)$ of S is computed in terms of the Chern classes of F and of $\{Y; \boldsymbol{O}_X\}$. For instance if either $\dim(X) = 3$ or $s \leqslant 3$ we have $c_1(S) = c_1(F)$, $c_2(S) = c_2(F) + [Y]$ and $c_3(S) = (c_1(F) + c_1(TX))[Y] - 2c(\boldsymbol{O}_Y) + c_3(F)$.

We will state 'explicitely' only the case in which F in 3.1 is a direct sum of line bundles.

THEOREM 3.2. *Assume X smooth. Fix integers $m > 0$, $s = v - r \geqslant 2$ and s line bundles L_j, $1 \leqslant j \leqslant s$ on X. Assume $H^2(L_i \otimes L_s^*) = 0$ for every i with $1 \leqslant i < s$. Let Y be a locally complete intersection subscheme of codimension 2 of X. Assume that* $\det(N_{Y,X}) \otimes (\oplus_{1 \leqslant i < s} L_i \otimes L_s^*)$ *has a nowhere vanishing section on Y. Assume that every line bundle $L_i \otimes L_s^*$, $1 \leqslant i < s$, is spanned by its global sections. Assume that the map $H^0(\boldsymbol{I}_Z \otimes L_s) \to P_X^m(L_s)$ is surjective.Then there is a* rank *s vector bundle A on X whose total Chern character is the product of $\{Y; L_s\}$ and the product of the total Chern characters of the line bundles L_i, $1 \leqslant i < s$, and such that the map $H^0(A) \otimes \boldsymbol{O}_X \to P_X^m(A)$ is surjective. Hence, taking A^* as kernel bundle we have $s_m = \max\{0, r - s \cdot ((x+m)!/(x!m!))\}$, where $x := \dim(X)$, and there is no mth symmetric torsion.*

Since very often in interesting ranges of Chern classes the assumptions of 3.1 and 3.2 are not satisfied if $\dim(X) = 2$, we will consider now the case $\dim(X) = 2$.

THEOREM 3.3. *Let X be a Gorenstein projective surface. Fix integers $m > 0$, $s := v - r > 1$ and s line bundles L_j, $1 \leqslant j \leqslant s$, with $H^1(K_X \otimes L_i \otimes L_s^*) = H^1(L_s) = 0$ for every i with $1 \leqslant i < s$. Fix an integer $z \geqslant 0$ and a 0-dimensional locally complete intersection subscheme Z of X with length z. Assume that every line bundle $L_i \otimes L_s^*$, $1 \leqslant i < s$, is spanned by its global sections. Assume that the map $H^0(\boldsymbol{I}_Z \otimes L_s) \to P_X^m(L_s)$ is surjective. Then there is a* rank *s vector bundle A on X with $c_1(A) = \Sigma_{1 \leqslant j \leqslant s} L_j$, $c_2(A) = \Sigma_{1 < j < k \leqslant s} L_j \cdot L_k + z$, and such that the map $H^0(A) \otimes \boldsymbol{O}_X \to P_X^m(A)$ is surjective. Hence, taking A^* as kernel bundle we have $s_m = \max 0, r - s(m+2)(m+1)/2$ and there is no mth symmetric torsion.*

Proof. We will find A as an extension of the following type:

$$0 \to \oplus_{1 \leqslant i < s} L_i \to A \to \boldsymbol{I}_Z \otimes L_s \to 0. \tag{22}$$

To prove the existence of a locally trivial sheaf A fitting in (22) just note that the Cayley–Bacharach condition is trivially satisfied (with no restriction on Z). Then, as in Section 2, any such bundle A proves 3.3. □

Remark 3.4. Even in the case $X = \mathbf{P}^2$ the range of Chern classes covered by Theorem 3.3 is much wider than the one covered by Theorem 2.15 and Remark 2.16. For instance taking z very small we obtain often $(r-1)c_1(S_E)^2 > 2rc_2(S_E)$ (hence E cannot be stable), while in Theorem 2.15 and Remark 2.16 we have always $c_1(S_E)^2 \leqslant 4c_2(S_E)$. As an example we discuss in detail the case $X = \mathbf{P}^2$. Fix the integers r, s, m and $z \geqslant 0$. Fix any line bundle L_s with $\deg(L_s) > z + m$. Take as L_i, $1 \leqslant i < s$, any line bundle with $\deg(L_i) \geqslant \deg(L_s)$. The cohomological conditions $H^1(K_X \otimes L_i \otimes L_s^{-1}) = H^1(L_s) = 0$ are always satisfied, while the other conditions are satisfied by the assumptions on the integers $\deg(L_s)$ and $\deg(L_i)$.

Remark 3.5. Note that the proof of Theorem 3.3 gives families of bundles E with large dimension.

We stress again that (as clear from 3.1) we may iterate the construction used in the proof of Theorems 3.2 and 3.3 to cover larger ranges of bundles with maximal drops of ranks.

4.

In this section X may be singular. We fix $P \in X$; in this section we are interested in the situation near P (or even at P); hence we do not work ‘up to codimension 2’. We allow that P is a singular point of X. Of course, for the first step (from E to E_1) we do not need to work up to codimension 2. For the other steps the non-inductive ‘symmetric’ Definition 1.5 allows us to consider the situation also for higher order invariants. In this paper we give the general set up and consider it in a few examples. A joint paper is planned (not for the near future, however) with a finer analysis for low-dimensional singular X.

Let X be an integral variety, $P \in X$, E a rank r vector bundle on X spanned by a v-dimensional vector space of global sections and $f\colon X \to G := G(r, v)$ the associated morphism. Since the map $TX \to f^*TG$ is defined under no assumption on X, we have again a morphism $\partial\colon TX \otimes S_E \to E$.

DEFINITION 4.1. Fix X, E, V and a morphism $f\colon C \to X$ with C smooth curve. Let $r_{i;f}$ or $r_{i;C}$ be the rank of the ith derived bundle on C for $(f^*(E), f^*V)$; set $r_{i;c} := \min\{r_{i;f}\}$, where the minimum is taken among all such f (with the curve C not fixed); $r - r_{i;c}$ is called the ith curvilinear differential rank. Fix a point $P \in X$ such that E is spanned by V at P; set $r_{i;c,P} := \min\{r_{i,f}\}$, where the minimum is

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taken among all such f (C is not fixed) with $P \in f(C)$ and E spanned by V at every point of $f(C)$; $r - r_{i;c,P}$ is called the ith curvilinear differential rank at P.

The same invariants are obtained if we take as f only the embeddings. Note that $r_{i;c} = \min\{r_{i;c,}\}$ and that $r_{i;c,P} = r_{i;c}$ for P in a Zariski dense open subset of X.

Note that for a smooth curve C and every $P \in C$ [Pe2] gives vector bundles, $E_i, i \geqslant 1$, spanned at P, plus the part of the ith torsion sheaf supported at P. Using the symmetric definition, it is possible to do the same essentially on any X.

DEFINITION 4.2. The ith symmetric derived datum at P of (E, V) is a triple (F_i, M_i, t_i) where: (a) F_i is the germ at P of the cokernel of the natural inclusion of $G^i(S_E^*)^{**}$ into V; (b) $M_i := (F_i)^{**}/F_i = E^i/F_i$; (c) t_i is the germ at P of the ith symmetric torsion sheaf.

We will consider this definition in a few easy examples (see Examples 4.3, 4.4 and 4.5) related to the following general theme. Fix a morphism $f : Y \to X$. For us the most interesting cases are if f is the normalization map or a desingularization map. We would like to know the relations between the pull-back of the derived data of (X, E, V) and the derived data of $(Y, f^*(E), f^*V)$. By [Pi], Theorem 6.2(iii), we have a natural morphism $u : f^*P_X^m(S_E^*) \to P_Y^m(f^*(S_E^*))$. If S is locally free (e.g. if E is locally free at P and V spans E at P) we have $f^*(S^*) = (f^*(S))^*$, where the last dual is taken with respect to Hom$(, \boldsymbol{O}_X)$. Composing the map u with this isomorphism we obtain (if E is a vector bundle spanned by V at P) the comparison map $f^*P_X^m(S_E^*) \to P_Y^m(f^*(S_E^*))$ we were looking for.

EXAMPLE 4.3. If $P \in f(Y)$ and f is etale at every point of $f^{-1}(P)$, then the comparison map is an isomorphism at every point of $f^{-1}(P)$.

EXAMPLE 4.4. Let f be the normalization map and X be smooth at P. Here any single explicit example may be analyzed using Taylor expansions.

EXAMPLE 4.5. Assume $\dim(X) = 2$, X smooth at P and f the blowing up of X at P. We leave the details to the interested reader with the following hints. Note that TY is the elementary transformation of $f^*(TX)$ along the exceptional divisor D with respect to $\boldsymbol{O}_D$. Hence if q is a function on Y coming from a function on X, then we have the vanishing of all the derivatives of q with respect to the variable in the direction of $f^{-1}(P)$.

Now we will show the existence in another situation of the comparison map for the first step.

Remark 4.6. Assume either X smooth at P or S locally free at P. Then we have an isomorphism between $f^*(TX \otimes S_E^*)$ and $f^*(TX) \otimes f^*(S_E^*)$. Composing this isomorphism with the map $TY \to f^*(TX)$ we obtain a comparison map.

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References

[AH] d'Almeida, J. and Hirschowitz, A.: Quelques plongements projectifs non speciaux de surfaces rationnelles, *Math. Z.* 211 (1992) 479–483.

[BC] Banica, C. and Coanda, J.: Existence of rank 3 vector bundles with given Chern classes on homogeneous rational 3-folds, *Manuscripta Math.* 51 (1985) 121–143.

[Br] Brun, J.: Les fibrés de rang deux sur $\mathbf{P}_2$ et leurs sectiones, *Bull. Soc. Math. France* 108 (1980) 457–473.

[BH] Brun, J. and Hirschowitz, A.: Le problème de Brill–Noether pour les idéaux de $\mathbf{P}^2$, *Ann. Sci. Ec. Norm. Super. (IV)* 20 (1987) 171–200.

[DL] Drezet, J.-M. and Le Potier, J.: Fibrés stables et fibrés exceptionnels sur $\mathbf{P}_2(\mathbf{C})$, *Ann. Ec. Norm. Sup.* 18 (1985) 193–244.

[El] Ellingsrud, G.: Sur l'irréducibilite du module des fibrés stables sur $\mathbf{P}^2$, *Math. Z.* 182 (1983) 189–192.

[HL] Hirschowitz, A. and Laszlo, Y.: Fibrés génériques sur le plan projectif, *Math. Ann.* 297 (1993) 85–102.

[L] Le Potier, J.: Stabilité et amplitude sur $\mathbf{P}_2(\mathbf{C})$, in: Vector Bundles and Differential Equations, Proc. Nice 1979, pp. 146–181, *Progress in Math.* 7, Birkhäuser, 1980.

[Pa] Paoletti, R.: Stability of a class of homogeneous vector bundles on $\mathbf{P}^n$, *Boll. U.M.I. (7)* 9-A (1995) 329–343.

[Pi] Piene, R.: Numerical characters of a curve in projective n-space, in: Real and Complex Singularities, Oslo 1976, pp. 475–495, Sijthoff and Noordhoff, 1977.

[Pe1] Perkinson, D.: *Jet bundles and curves in Grassmannians*, Ph.D. Thesis, Chicago 1990.

[Pe2] Perkinson, D.: Curves in Grassmannians, *Trans. Amer. Math. Soc.* 347 (1995), 3179–3246.