CORRESPONDENCE.

ON THE COMPONENT PARTS OF A TERMINABLE ANNUITY.

To the Editor of the Assurance Magazine.

SIR,—Circumstances have led me to attend to a problem which happens to be of some interest at the present time. I cannot say of course that it has not been investigated before. Very likely it has, although I have nowhere met with it.

It is known to the youngest of your readers, although our legislators and financial administrators have not yet realized the fact that the successive payments of a term annuity consist partly of a return of a portion of the capital producing it, and partly of interest accruing on the portion remaining in the hands of the borrower (*i. e.*, the payer of the annuity) since the last payment. Now, the payments of the annuity being uniform, while the portion of capital bearing interest diminishes from year to year, the ratio of the components of the annuity is in a state of annual variation; and it is the law that regulates this variation that I propose to investigate.

Let a be the number of pounds in an annuity for n years, and r the interest of one pound for a year. Then the consideration—the sum advanced—the present value of that annuity will be

$$\frac{a(1-v^n)}{r},$$

which, therefore, also is the amount of capital unpaid at the beginning of the term. Now, if to the amount unpaid at the end of any one year we add one year's interest (*i. e.*, multiply by $1+r$), and deduct a, the remainder will obviously be the amount of capital remaining unpaid at the end of the next year. We thus get for the amount unpaid at the end of the

1st year, $\dfrac{a(1+r)(1-v^n)}{r}-a$;

2nd ,, $\dfrac{a(1+r)^2(1-v^n)}{r}-a(1+r)-a$;

3rd ,, $\dfrac{a(1+r)^3(1-v^n)}{r}-a(1+r)^2-a(1+r)-a$;

⋮ ⋮

mth ,, $\dfrac{a(1+r)^m(1-v^n)}{r}-a(1+r)^{m-1}-a(1+r)^{m-2}-\dots-a(1+r)-a$;

or $\dfrac{a(1+r)^m(1-v^n)}{r}-a\{1+(1+r)+(1+r)^2+\dots+(1+r)^{m-1}\}$;

or $\dfrac{a(1+r)^m(1-v^n)}{r}-\dfrac{a\{(1+r)^m-1\}}{r}$; . . . . (A)

and this, on simplification, becomes

$$\frac{a(1-v^{n-m})}{r} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(B)}$$

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This expression visibly denotes the present value of an annuity of £a for $n-m$ years. Hence, the portion of capital unpaid at the end of m years is equal to the value at the same period of the remainder of the annuity.

Again, the interest included in the payment of the annuity made at the end of any one year, is that accruing on the amount of capital unpaid at the end of the preceding year. So that, the capital at the end of the $(m-1)$th year being

$$\frac{a(1-v^{n-m+1})}{r},$$

a year's interest upon this, that is, $a(1-v^{n-m+1})$, is the amount of interest included in the mth payment; and the total of that payment being a, it follows that av^{n-m+1} is the amount of capital included therein. Thus, making $m=1, 2, 3, \ldots n$, successively, we see that the portions of capital included in the

1st, 2nd, 3rd, ... nth payments,

are av^n, av^{n-1}, av^{n-2}, ... av, respectively;

the sum of these being, as we know,

$$\frac{a(1-v^n)}{r}$$

which was the amount advanced.

The mth payment is thus shown to consist of

Capital repaid, av^{n-m+1},
and interest, $a(1-v^{n-m+1})$.

But the foregoing results may be obtained in a briefer manner, and by the aid of a less profuse array of symbols. Since $a(1-v^n):r$ is the sum that will be paid off in n years by an annuity of £a, so, changing n into $n-m$, $a(1-v^{n-m}):r$ is the amount that will be paid off by the same annuity in $n-m$ years. Hence, $a(1-v^n):r$ being the sum advanced—the capital unpaid—m years ago, $a(1-v^{n-m}):r$ is the balance of that sum now, after the mth payment of the annuity, remaining unpaid.

Likewise, $a(1-v^{n-m}):r$ being the amount now unpaid, and $a(1-v^{n-m+1}):r$ the amount unpaid a year ago, the difference of these, viz., av^{n-m+1} is the amount of capital included in the mth payment of the annuity; whence also it follows, as before, that $a(1-v^{n-m+1})$ is the interest included in that payment.

Attend for a moment to the two expressions I have marked (A) and (B). They furnish, incidentally, a proof of the fallacy of a notion that is prevalent among the uninstructed, and which also, I believe, pervaded many of the schemes that were proposed for the redemption of the National Debt. The notion to which I refer is, that in consequence of the rapidity with which money increases when improved at compound interest, a debt will be sooner and more easily extinguished if the redemption-money, instead of being applied as it accrues to the gradual reduction of the debt, be retained and accumulated till it suffice for its extinction. The fallacy of this is shown by the expressions I have referred to. The first (A), shows the state of the account, so to speak, at the end of m years, on the hypothesis of the redemption-money being retained and improved by itself; and the second (B), shows the state, on the other hypothesis, of the redemption-money being directly applied as it accrues to the reduction of the debt. The identity of the two expressions shows that, apart from moral con-

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siderations, it is indifferent which arrangement is adopted. Those who entertain the notion animadverted on are unaware, or forget, that a deficiency accumulates with the same rapidity as a surplus.

The practical application of the problem with which we have been occupied is the separation into their component parts of the several payments of an annuity, for the purpose of avoiding the deduction of income tax by the payer, from the portion which consists of capital. I give an example:—

£1,000 is advanced, at 5 per cent., to be repaid by way of annuity in ten years.

On reference to Jones's Table VII.,* p. 98, we find immediately that the annuity is £129·5046; and we have to apportion the successive payments of it into principal and interest. The operation is as follows:—

		Sum advanced, £1000·	Interest.
	a	129·5046 50·0000	
1st year . .	av^{10}	79·5046 3·9752	50·0000
2nd ,, . .	av^{9}	83·4798 4·1740	46·0248
3rd ,, . .	av^{8}	87·6538 4·3827	41·8508
4th ,, . .	av^{7}	92·0365 4·6018	37·4681
5th ,, . .	av^{6}	96·6383 4·8319	32·8663
6th ,, . .	av^{5}	101·4702 5·0735	28·0344
7th ,, . .	av^{4}	106·5437 5·3272	22·9609
8th ,, . .	av^{3}	111·8709 5·5935	17·6337
9th ,, . .	av^{2}	117·4644 5·8732	12·0402
10th ,, . .	av	123·3376 6·1669	6·1670
			295·0462 Sum.
	a	129·5045	

* I do not think this very convenient table is so well known as it deserves to be. I have certainly often seen Table VI. employed, when, by the use of Table VII., a tedious division would have been saved. The only other publications of this table that I know of are in Corbaux's books *On Interest* and *On Population.*

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The initial term of the first series, av^{10}, is formed by subtracting from a a year's interest on the sum advanced, for $a-a(1-v^{10})=av^{10}$. The remaining terms are formed in succession, by addition to each of its product by r (here equal to ·05), which is equivalent to continual multiplication by $1+r$, or v^{-1}. Verification of the work is obtained by forming an additional term, giving a. The terms of this series are the amounts of capital included in the several payments of the annuity.*

The terms of the second series, which are the amounts of interest included in the several payments, are derived from the corresponding terms of the first by subtracting these from a; and verification of this series (as also of the whole work) is had by addition of the terms, which gives the difference between na and the sum advanced. Thus, here,

$$10(129{\cdot}5046)-1000=295{\cdot}046.$$

I can refer to instances in which the method I have sought to elucidate, or some equivalent method, has been used, and for the purpose to which I have referred—viz., to separate into their component parts the successive payments of an annuity, so as to exhibit the portion of each on which alone income-tax is (or ought to be) exigible. Mr. Newmarch, in his examination before Mr. Hubbard's Committee on the Property and Income Tax, in 1861 (pp. 23, 24), hands in two schedules, having reference to actual transactions in which the separation in question is fully carried out. The first is strictly analogous to that exemplified above (including also, however, premiums for life assurance), the rate of interest being $2\frac{1}{2}$ per cent. per half-year; but the second is in a somewhat different form. In it the repayments of principal are annual and uniform, while the interest, payable half-yearly, varies. The Hon. William Napier, Manager of the Lands Improvement Company, likewise hands in two schedules (Appendix, pp. 293, 294), referring to two similar transactions, the payments in the one case being half-yearly and in the other yearly.†

There is something in this matter that I do not quite understand. Mr. Newmarch intimates, distinctly enough, that the Office with which he was connected found no difficulty in realizing the object with a view to which the arrangement described was adopted. The body represented by Mr. Napier, on the other hand, met with great difficulty. They had sought to protect themselves, by a clause in their Special Act of Parliament, against the overcharge, notwithstanding which the Inland Revenue Office claimed tax on the entire annuity payments. And the claim was only got rid of by procuring another Special Act to *interpret* the former protecting clause! (*See* Evidence, pp. 144, 145.)

I am, Sir,

Your most obedient servant,

Camden Town, 1*st June*, 1863. P. GRAY.

* It is worth while to remark, that if, as will frequently happen, the annuity be 10 or any power of 10, *the figures* of the terms of this series will be those of succeeding powers of v.

† The latter affords an illustration of the remark made in the preceding Note.