3
Radiative corrections

The previous chapter derived some simple consequences of heavy quark symmetry ignoring $1/m_Q$ and radiative corrections. This chapter discusses how radiative corrections can be systematically included in HQET computations. The two main issues are the computation of radiative corrections in the matching between QCD and HQET, and the renormalization of operators in the effective theory. The renormalization of the effective theory is considered first, because it is necessary to understand this before computing corrections to the matching conditions. The $1/m_Q$ corrections will be discussed in the next chapter.

3.1 Renormalization in HQET

The fields and the coupling in the HQET Lagrange density Eq. (2.49) are actually bare quantities,

$$\mathcal{L}_{\text{eff}} = i\bar{Q}_v^{(0)} v^\mu \left[\partial_\mu + ig^{(0)} A_\mu^{(0)}\right] Q_v^{(0)}, \tag{3.1}$$

where the superscript (0) denotes bare quantities. It is convenient to define renormalized fields that have finite Green's functions. The renormalized heavy quark field is related to the bare one by wave-function renormalization,

$$Q_v = \frac{1}{\sqrt{Z_h}}\, Q_v^{(0)}. \tag{3.2}$$

The coupling constant $g^{(0)}$ and the gauge field $A_\mu^{(0)}$ are also related to the renormalized coupling and gauge field by multiplicative renormalization. In the background field gauge, gA_μ is not renormalized, so $g^{(0)} A_\mu^{(0)} = g\mu^{\epsilon/2} A_\mu$, where $n = 4 - \epsilon$ is the dimension of space–time.

In terms of renormalized quantities, the HQET Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= iZ_h\, \bar{Q}_v v^\mu \left(\partial_\mu + ig\mu^{\epsilon/2} A_\mu\right) Q_v \\ &= i\bar{Q}_v v^\mu \left(\partial_\mu + ig\mu^{\epsilon/2} A_\mu\right) Q_v + \text{ counterterms.}\end{aligned} \tag{3.3}$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

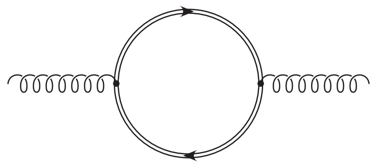

Fig. 3.1. Heavy quark loop graph, which vanishes in the effective theory. Heavy quark propagators are denoted by a double line.

Equation (3.3) has been written in $n = 4-\epsilon$ dimensions, with μ the dimensionful scale parameter of dimensional regularization.

Heavy quarks do not effect the renormalization constants for light quark fields Z_q, the gluon field Z_A, and the strong coupling Z_g, because heavy quark loops vanish in the effective theory. That loops do not occur is evident from the propagator in Eq. (2.41). In the rest frame $v = v_r$ the propagator $i/(k \cdot v + i\varepsilon)$ has one pole below the real axis at $k^0 = -i\varepsilon$. A closed heavy quark loop graph such as in Fig. 3.1 involves an integration over the loop momentum k. The heavy quark propagators in the loop both have poles below the real axis, so the k^0 integral can be closed in the upper half-plane, giving zero for the loop integral. The HQET field Q_v annihilates a heavy quark but does not create the corresponding antiquark.

In the full theory of QCD, the light quark wave-function renormalization Z_q is independent of the quark mass in the $\overline{\text{MS}}$ scheme. A heavy quark with mass m_Q contributes to the QCD β function even for $\mu \ll m_Q$. At first glance, this would imply that heavy particle effects do not decouple at low energies. This nondecoupling is an artifact of the $\overline{\text{MS}}$ scheme. The finite parts of loop graphs have a logarithmic dependence on the quark mass and become large as $\mu \ll m_Q$. One can show that the logarithmic dependence of the finite parts exactly cancels the logarithmic heavy quark contribution to the renormalization group equation, so that the total heavy quark contribution vanishes as $\mu \ll m_Q$. This cancellation can be made manifest in the zero heavy quark sector by constructing an effective theory for $\mu < m_Q$ in which the heavy quark has been integrated out. Such effective theories were considered in Sec. 1.5 of Chapter 1. Similarly, in HQET, one matches at $\mu = m_Q$ to a new theory in which the Dirac propagator for the heavy quark is replaced by the HQET propagator Eq. (2.41). This changes the renormalization scheme for the heavy quarks, so that Z_h for the heavy quark differs from Z_q for the light quarks.

Z_h can be computed by studying the one-loop correction to the heavy quark propagator in Fig. 3.2. In the Feynman gauge, the graph is

$$\begin{aligned}
&\int \frac{d^n q}{(2\pi)^n} \left(-igT^A \mu^{\epsilon/2} \right) v_\lambda \frac{i}{(q+p)\cdot v} \left(-igT^A \mu^{\epsilon/2} \right) v^\lambda \frac{(-i)}{q^2} \\
&\quad = -\left(\frac{4}{3}\right) g^2 \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2\, v\cdot (q+p)},
\end{aligned} \tag{3.4}$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

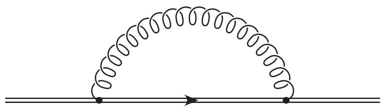

Fig. 3.2. Gluon interaction with a heavy quark.

where p is the external residual momentum, q is the loop momentum, and we have used the identity $T^A T^A = (4/3)\mathbb{1}$ for the **3** of $SU(3)$. The one-loop wave-function renormalization is given by the ultraviolet divergent part of Eq. (3.4). If one expands in $v \cdot p$, Eq. (3.4) is also infrared divergent, and it is convenient to regulate the infrared divergence by giving the gluon a mass m that will be set to zero at the end of the computation. This infrared regulator allows one to isolate the ultraviolet divergence by computing the $1/\epsilon$ term in the integral. The regulated integral that has to be evaluated is

$$-\left(\frac{4}{3}\right) g^2 \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m^2)[v \cdot (q+p)]}, \tag{3.5}$$

where m is the gluon mass. The integral Eq. (3.5) will be computed in detail, since it provides an example of some standard tricks that are useful in computing loop graphs in HQET. The denominators can be combined by using the identity

$$\frac{1}{a^r b^s} = 2^s \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^\infty d\lambda \frac{\lambda^{s-1}}{(a + 2b\lambda)^{r+s}}, \tag{3.6}$$

so that Eq. (3.5) can be rewritten as

$$-\left(\frac{8}{3}\right) g^2 \mu^\epsilon \int_0^\infty d\lambda \int \frac{d^n q}{(2\pi)^n} \frac{1}{[q^2 - m^2 + 2\lambda v \cdot (q+p)]^2}. \tag{3.7}$$

Shifting the loop integration momentum by $q \to q - \lambda v$ gives

$$-\left(\frac{8}{3}\right) g^2 \mu^\epsilon \int_0^\infty d\lambda \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m^2 - \lambda^2 + 2\lambda v \cdot p)^2}. \tag{3.8}$$

Evaluating Eq. (3.8) using the standard dimensional regularization formula in Eq. (1.44) gives

$$-\left(\frac{8}{3}\right) g^2 \mu^\epsilon \int_0^\infty d\lambda \frac{i}{(4\pi)^{2-\epsilon/2}} \Gamma(\epsilon/2)[\lambda^2 - 2\lambda v \cdot p + m^2]^{-\epsilon/2}. \tag{3.9}$$

The λ integral can be evaluated by using the recursion relation,

$$\begin{aligned} I(a,b,c) &\equiv \int_0^\infty d\lambda (\lambda^2 + 2b\lambda + c)^a \\ &= \frac{1}{1+2a}\left[(\lambda^2 + 2b\lambda + c)^a (\lambda + b)\big|_0^\infty + 2a(c - b^2) I(a-1,b,c)\right], \end{aligned} \tag{3.10}$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

to convert it to one that is convergent when $\epsilon = 0$,

$$\begin{aligned}\int_0^\infty & d\lambda[\lambda^2 - 2\lambda v\cdot p + m^2]^{-\epsilon/2} \\ &= \frac{1}{1-\epsilon}\Bigg\{ (\lambda^2 - 2\lambda v\cdot p + m^2)^{-\epsilon/2}(\lambda - v\cdot p)\Big|_0^\infty \\ &\qquad - \epsilon\left[m^2 - (v\cdot p)^2\right]\int_0^\infty d\lambda(\lambda^2 - 2\lambda v\cdot p + m^2)^{-1-\epsilon/2}\Bigg\}. \end{aligned}\tag{3.11}$$

The Γ functions in a one-loop dimensionally regularized integral can have at most a $1/\epsilon$ singularity. Since the last term in Eq. (3.11) is multiplied by ϵ, one can set $\epsilon = 0$ in the integrand. The other terms can be evaluated by noting that in dimensional regularization,

$$\lim_{\lambda\to\infty} \lambda^z = 0, \tag{3.12}$$

as long as z depends on ϵ in a way that allows one to analytically continue z to negative values. This gives for Eq. (3.9)

$$\begin{aligned} &-i\frac{g^2}{6\pi^2}(4\pi\mu^2)^{\epsilon/2}\Gamma(\epsilon/2)\frac{1}{1-\epsilon}\Bigg\{(m^2)^{-\epsilon/2}(v\cdot p) \\ &-\epsilon\left[m^2 - (v\cdot p)^2\right]\int_0^\infty d\lambda(\lambda^2 - 2\lambda v\cdot p + m^2)^{-1}\Bigg\} \\ &\quad = -i\frac{g^2}{3\pi^2\epsilon}v\cdot p + \text{finite}. \end{aligned}\tag{3.13}$$

There is also a tree-level contribution from the counterterm:

$$iv\cdot p(Z_h - 1). \tag{3.14}$$

The sum of Eqs. (3.14) and (3.13) must be finite as $\epsilon \to 0$, so in the $\overline{\text{MS}}$ scheme

$$Z_h = 1 + \frac{g^2}{3\pi^2\epsilon}. \tag{3.15}$$

Note that Z_h is different from the wave-function renormalization of light quark fields given in Eq. (1.86). The anomalous dimension of a heavy quark field is

$$\gamma_h = \frac{1}{2}\frac{\mu}{Z_h}\frac{dZ_h}{d\mu} = -\frac{g^2}{6\pi^2}. \tag{3.16}$$

Composite operators require additional subtractions beyond wave-function renormalization. Consider the heavy-light bare operator

$$O_\Gamma^{(0)} = \bar q^{(0)}\Gamma Q_v^{(0)} = \sqrt{Z_q Z_h}\,\bar q\Gamma Q_v, \tag{3.17}$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

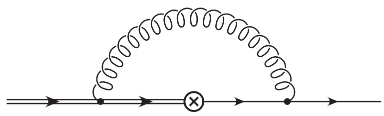

Fig. 3.3. One-loop renormalization of the heavy-light operator $\bar{q}\Gamma Q_v$. The heavy quark is denoted by a double line, the light quark by a single line, and the operator insertion by $\otimes$.

where Γ is any Dirac matrix. The renormalized operator is defined by

$$\begin{aligned} O_\Gamma &= \frac{1}{Z_O} O_\Gamma^{(0)} = \frac{\sqrt{Z_q Z_h}}{Z_O}\, \bar{q}\Gamma Q_v \\ &= \bar{q}\Gamma Q_v + \text{counterterms}, \end{aligned} \tag{3.18}$$

where the additional operator renormalization Z_O can be determined by computing a Green's function with an insertion of O_Γ. For example, Z_O can be determined by considering the one-particle irreducible Green's function of q, $\bar{Q}_v$, and O_Γ. The counterterm in Eq. (3.18) contributes

$$\left(\frac{\sqrt{Z_q Z_h}}{Z_O} - 1 \right) \Gamma \tag{3.19}$$

to this time-ordered product. The one-loop diagram in Fig. 3.3 also gives a divergent contribution to the time-ordered product. Neglecting external momenta (the operator O_Γ contains no derivatives) and using the Feynman gauge, the diagram gives

$$\begin{aligned} &\int \frac{d^n q}{(2\pi)^n} \left(-ig\mu^{\epsilon/2} T^A\right) \gamma^\lambda \frac{i \not{q}}{q^2} \Gamma \frac{i}{v \cdot q} \left(-ig\mu^{\epsilon/2} T^A\right) v_\lambda \frac{(-i)}{q^2} \\ &\quad = -i \frac{4}{3} g^2 \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{\not{v} \not{q} \Gamma}{q^4 v \cdot q}. \end{aligned} \tag{3.20}$$

Combining denominators using Eq. (3.6), introducing a gluon mass m to regulate the infrared divergence, and making the change of variables $q \to q - \lambda v$ gives

$$-i \frac{16}{3} g^2 \mu^\epsilon \int d\lambda \int \frac{d^n q}{(2\pi)^n} \frac{\not{v}(\not{q} - \lambda \not{v})\Gamma}{(q^2 - \lambda^2 - m^2)^3}. \tag{3.21}$$

The term proportional to $\not{q}$ is odd in q, and it vanishes on integration. The identity $\not{v}\not{v} = 1$ reduces the remaining integral to be the same as $i/2$ times the derivative of Eq. (3.8) with respect to $v \cdot p$ at $v \cdot p = 0$. Consequently, Fig. 3.3 yields

$$\frac{g^2 \Gamma}{6\pi^2 \epsilon}, \tag{3.22}$$

up to terms that are not divergent as $\epsilon \to 0$. The sum of Eqs. (3.19) and (3.22)

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

must be finite as $\epsilon \to 0$. Using the expressions for $\sqrt{Z_h}$ and $\sqrt{Z_q}$ in Eqs. (3.15) and (1.86) gives

$$Z_O = 1 + \frac{g^2}{4\pi^2\epsilon}, \tag{3.23}$$

and the anomalous dimension is

$$\gamma_O = -\frac{g^2}{4\pi^2}. \tag{3.24}$$

Note that the renormalization of O_Γ is independent of the gamma matrix Γ in the operator. This is a consequence of heavy quark spin symmetry and light quark chiral symmetry, and it is very different from what occurs in the full theory of QCD. For example, in the full theory the operator $\bar{q}_i q_j$ requires renormalization whereas the operator $\bar{q}_i \gamma_\mu q_j$ does not.

As a final example of operator renormalization, consider a composite operator with two heavy quark fields with velocity v and v',

$$T_\Gamma^{(0)} = \bar{Q}_{v'}^{(0)} \Gamma Q_v^{(0)} = Z_h\, \bar{Q}_{v'} \Gamma Q_v. \tag{3.25}$$

The renormalized operator is related to the bare one by means of

$$\begin{aligned} T_\Gamma &= \frac{1}{Z_T} T_\Gamma^{(0)} \\ &= \frac{Z_h}{Z_T} \bar{Q}_{v'} \Gamma Q_v = \bar{Q}_{v'} \Gamma Q_v + \text{counterterms}. \end{aligned} \tag{3.26}$$

One can always choose a frame where $v = v_r$ or where $v' = v_r$, but it is not possible, in general, to go to a frame where both heavy quarks are at rest. Hence T_Γ depends on $w = v \cdot v'$ and we anticipate that its renormalization will also depend on this variable. Heavy quark spin symmetry implies that the renormalization of T_Γ will be independent of Γ. The operator renormalization factor Z_T can be determined from the time-ordered product of $Q_{v'}$, $\bar{Q}_v$ and T_Γ. The counterterm gives the contribution

$$\left(\frac{Z_h}{Z_T} - 1 \right) \Gamma, \tag{3.27}$$

and the one-loop Feynman diagram in Fig. 3.4 gives (neglecting external

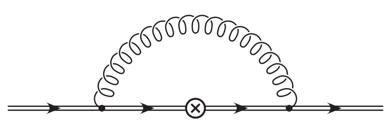

Fig. 3.4. One-loop renormalization of the heavy-heavy operator $\bar{Q}_{v'} \Gamma Q_v$. The heavy quark is denoted by a double line and the operator insertion by $\otimes$.

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

momenta) the contribution

$$\int \frac{d^n q}{(2\pi)^n}\left(-igT^A\mu^{\epsilon/2}\right)v'_\lambda\left(-igT^A\mu^{\epsilon/2}\right)v^\lambda \frac{i}{v'\cdot q}\Gamma\frac{i}{v\cdot q}\frac{(-i)}{q^2}$$
$$= -ig^2\mu^\epsilon\left(\frac{4}{3}\right)w\int\frac{d^n q}{(2\pi)^n}\frac{\Gamma}{q^2(q\cdot v)(q\cdot v')} \tag{3.28}$$

to this three-point function. Using the Feynman trick to first combine the $q \cdot v$ and $q \cdot v'$ terms, and then using Eq. (3.6), gives

$$-ig^2\left(\frac{32}{3}\right)\mu^\epsilon\Gamma w\int_0^\infty d\lambda\int_0^1 dx$$
$$\times\int\frac{d^n q}{(2\pi)^n}\frac{\lambda}{\{q^2+2\lambda[xv+(1-x)v']\cdot q-m^2\}^3}, \tag{3.29}$$

where m has been introduced to regulate the infrared divergence. Performing the q integration by completing the square in the denominator, shifting the q integration and dropping finite terms gives

$$-\frac{g^2}{3\pi^2}\mu^\epsilon w\Gamma\int_0^\infty d\lambda\int_0^1 dx\frac{\lambda}{\{\lambda^2[1+2x(1-x)(w-1)]+m^2\}^{1+\epsilon/2}}, \tag{3.30}$$

where $w = v \cdot v'$. The λ integral can be evaluated explicitly to give

$$-\frac{16}{3}\frac{g^2}{16\pi^2\epsilon}w\Gamma(m^2)^{-\epsilon/2}\int_0^1 dx\frac{1}{[1+2x\,(1-x)\,(w-1)]}. \tag{3.31}$$

Performing the x integral yields for the part proportional to $1/\epsilon$,

$$-\left(\frac{16}{3}\right)\frac{g^2}{16\pi^2\epsilon}w\,r(w)\,\Gamma, \tag{3.32}$$

where

$$r(w) = \frac{1}{\sqrt{w^2-1}}\ln\left(w+\sqrt{w^2-1}\right), \tag{3.33}$$

Demanding that the sum of Eq. (3.27) and Eq. (3.32) be finite as $\epsilon \to 0$ determines the operator renormalization factor Z_T. Using Eq. (3.15) we find that

$$Z_T = 1-\frac{g^2}{3\pi^2\epsilon}[w\,r(w)-1], \tag{3.34}$$

and the operator anomalous dimension is

$$\gamma_T = \frac{g^2}{3\pi^2}[w\,r(w)-1]. \tag{3.35}$$

Note that the renormalization of $T_\Gamma = \bar{Q}_{v'}\Gamma Q_v$ depends on the dot product of four velocities $w = v \cdot v'$. This is reasonable since Q_v is a different field for each

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

value of the four-velocity. At the zero-recoil point $w = 1$ the operator $\bar{Q}_v \gamma_\mu Q_v$ is a conserved current associated with heavy quark flavor symmetry and hence is not renormalized. The anomalous dimension γ_T near $w = 1$ has the expansion

$$\gamma_T = \frac{g^2}{\pi^2}\left[\frac{2}{9}(w-1) - \frac{1}{15}(w-1)^2 + \cdots\right], \tag{3.36}$$

and vanishes at $w = 1$.

3.2 Matching between QCD and HQET

The computation of physical quantities in QCD using HQET requires relating QCD operators to HQET operators, which is referred to as "matching." Consider the QCD vector current operator,

$$V_\nu = \bar{q}\gamma_\nu Q, \tag{3.37}$$

involving a heavy quark field Q and a light quark field q. Matrix elements of this operator are important for semileptonic decays such as $\bar{B} \to \pi e \bar{\nu}_e$ and $D \to \pi \bar{e} \nu_e$. In QCD this operator is not renormalized, since it is conserved in the limit that the (heavy and light) quark masses vanish. Quark mass terms are dimension-three operators, and therefore do not affect anomalous dimensions. Matrix elements of the full QCD vector current between physical states contain large logarithms of the quark mass m_Q divided by a typical hadronic momentum, which is of the order of $\Lambda_{\rm QCD}$. These logarithms can be resummed using HQET. In HQET, matrix elements of operators renormalized at μ can only contain logarithms of $\Lambda_{\rm QCD}/\mu$. There are no logarithms of m_Q/μ, since HQET makes no reference to the large-momentum scale m_Q. The logarithms of m_Q/μ are obtained by scaling the HQET operators between m_Q and μ, using the anomalous dimensions computed in the previous section.

The first step in computing matrix elements of V_ν is to relate the QCD operator to HQET operators. One can do this by computing matrix elements of the QCD operator between quarks at a scale μ, and comparing this with matrix elements of HQET operators renormalized at the same scale. Both calculations are done in perturbation theory, and are in general infrared divergent. However, the matching conditions depend on the difference between the computations in QCD and HQET. Since HQET is constructed to reproduce the low-momentum dynamics of QCD, the infrared divergences cancel in the matching conditions. One can therefore compute the matching conditions by using any convenient infrared regulator. It is crucial that the matching conditions do not depend on infrared effects; otherwise they would depend on the nonperturbative scale $\Lambda_{\rm QCD}$, and they would not be computable by using perturbation theory. Two common ways to regulate infrared divergences are to use a gluon mass and to use dimensional regularization. In this chapter, we will use dimensional regularization. If the scale

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

μ is chosen to be of the order of the heavy quark mass m_Q, the computation of the matching between the full and effective theories will be an expansion in $\alpha_s(\mu)$, with no large logarithms. For the specific example of the heavy $\to$ light vector current, this expansion takes the form

$$V^\lambda = C_1^{(V)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right]\bar{q}\gamma^\lambda Q_v + C_2^{(V)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right]\bar{q}v^\lambda Q_v. \tag{3.38}$$

The right-hand side of Eq. (3.38) includes all dimension-three operators with the same quantum numbers as the vector current V^λ. Higher dimension operators are suppressed by powers of $1/m_Q$. They can also be computed in a systematic expansion to determine the $1/m_Q$ corrections, as will be discussed in Chapter 4. Other dimension-three operators can be rewritten in terms of the two operators given above. For example, $\bar{q}i\sigma^{\mu\nu}v_\nu Q_v = -(1/2)\bar{q}(\gamma^\mu \not{v} - \not{v}\gamma^\mu)Q_v = -\bar{q}\gamma^\mu Q_v + v^\mu \bar{q}Q_v$, and so is not a linearly independent operator.

The matching calculation between QCD and HQET at the scale m_Q determines $C_i^{(V)}[1, \alpha_s(m_Q)]$. At lowest order in α_s (tree level), the matching condition is trivial,

$$\begin{aligned} C_1^{(V)}[1, \alpha_s(m_Q)] &= 1 + \mathcal{O}[\alpha_s(m_Q)], \\ C_2^{(V)}[1, \alpha_s(m_Q)] &= \mathcal{O}[\alpha_s(m_Q)], \end{aligned} \tag{3.39}$$

since at tree level, the field Q can be replaced by Q_v up to corrections of the order of $1/m_Q$. The one-loop corrections to $C_i^{(V)}$ will be computed in Sec. 3.3.

In the general case, one has a QCD operator $O_{\rm QCD}$ renormalized at the scale m_Q, which can be expressed as a linear combination of HQET operators O_i renormalized at the scale μ,

$$O_{\rm QCD}(m_Q) = \sum_i C_i\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right] O_i(\mu), \tag{3.40}$$

where the coefficients $C_i[1, \alpha_s(\mu)]$ are computed by doing a perturbative matching condition calculation at the scale $\mu = m_Q$. One can then obtain the coefficients $C_i[m_Q/\mu, \alpha_s(\mu)]$ at some lower scale $\mu < m_Q$ by renormalization group scaling in the effective theory, using the same procedure as that used for the weak Hamiltonian in Sec. 1.6. The operators O_i satisfy the renormalization group equation in Eq. (1.129). Since the left-hand side of Eq. (3.40) is μ independent, this implies that the coefficients satisfy the renormalization group equation shown in Eq. (1.133), with the solution given by Eq. (1.134).

The renormalization group equation solution in Eq. (1.134) can be written out explicitly in the case in which a single operator is multiplicatively renormalized, so that γ is a number rather than a matrix. The anomalous dimension, β function,

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

and matching coefficient have the perturbative expansions

$$\gamma(g) = \gamma_0 \frac{g^2}{4\pi} + \gamma_1 \left(\frac{g^2}{4\pi}\right)^2 + \cdots$$
$$\beta(g) = -\beta_0 \frac{g^3}{4\pi} - \beta_1 \frac{g^5}{(4\pi)^2} + \cdots, \tag{3.41}$$
$$C[1, \alpha_s(m_Q)] = C_0 + C_1 \alpha_s(m_Q) + \cdots .$$

Integrating Eq. (1.134) gives

$$\begin{aligned} C\left[\frac{m_Q}{\mu}, \alpha_s(m_Q)\right] &= [C_0 + C_1\alpha_s(m_Q) + \cdots] \\ &\quad \times \left\{\exp \int_{g(\mu)}^{g(m_Q)} \frac{dg}{g}\left[\frac{\gamma_0}{\beta_0} + \left(\frac{\gamma_1}{\beta_0} - \frac{\gamma_0\beta_1}{\beta_0^2}\right)\frac{g^2}{4\pi} + \cdots\right]\right\} \\ &= \left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right]^{-(\gamma_0/2\beta_0)} \\ &\quad \times \left\{C_0 + C_0\left(\frac{\gamma_1}{2\beta_0} - \frac{\gamma_0\beta_1}{2\beta_0^2}\right)[\alpha_s(m_Q) - \alpha_s(\mu)] + C_1\alpha_s(m_Q) + \cdots\right\}. \end{aligned} \tag{3.42}$$

The terms explicitly displayed in this equation sum all subleading logarithms of the form $\alpha_s^{n+1} \ln^n(m_Q/\mu)$. To evaluate the subleading logarithms requires knowing the two-loop anomalous dimension and β function, and the one-loop matching coefficient C_1. The two-loop β function is scheme independent, but C_1 and γ_1 are both scheme dependent in general. Retaining only the one-loop anomalous dimension γ_0 and the one-loop β function β_0 sums all the leading logarithms $\alpha_s^n \ln^n(m_Q/\mu)$.

The leading logarithms can be summed in the case of operator mixing by diagonalizing the anomalous dimension matrix γ_0, and then using Eq. (3.42). The two-loop equations with operator mixing cannot be simplified in the same way, because in general, γ_0 and γ_1 cannot be simultaneously diagonalized, and the equation has to be integrated numerically.

It should now be clear how to interpret the predictions for heavy meson decay constants and form factors obtained in Secs. 2.8–2.11. For the decay constants, the coefficient a is subtraction-point dependent, and Eq. (2.62) holds up to perturbative matching corrections when a is evaluated at $\mu = m_Q$. The μ dependence of a is determined by the anomalous dimension in Eq. (3.24). The situation is similar for the Isgur-Wise functions that occur in $\bar{B} \to D^{(*)} e \bar{\nu}_e$ and $\Lambda_b \to \Lambda_c e \bar{\nu}_e$ decays. The Isgur-Wise functions are matrix elements of HQET operators and also depend on the subtraction point μ due to the anomalous dimension in Eq. (3.35). The expression for the form factors in terms of the Isgur-Wise functions are valid up to perturbative matching corrections provided the Isgur-Wise functions are evaluated at a subtraction point around $m_{c,b}$, e.g.,

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

$\mu = \sqrt{m_c m_b}$. Note, however, that the anomalous dimension γ_T vanishes at $w = 1$, and therefore the normalization conditions $\xi(1) = 1$ and $\zeta(1) = 1$ in Eqs. (2.93) and (2.109) are μ independent.

3.3 Heavy-light currents

The tree-level matching conditions for heavy $\to$ light currents are given in Eq. (3.39). The one-loop corrections to this result can be determined by computing at order α_s a matrix element of the left-hand side of Eq. (3.38) in the full theory of QCD and equating it with the corresponding matrix element of the right-hand side of Eq. (3.38) calculated in HQET. A convenient matrix element is that between an on-shell heavy quark with four-momentum $p = m_Q v$ as the initial state and an on-shell massless quark state with four-momentum zero as the final state. These are not physical states since the strong interactions confine. However, Eq. (3.38) holds at the operator level and so these unphysical states can be used to determine the matching coefficients, $C_1^{(V)}$ and $C_2^{(V)}$.

The order α_s matrix element in QCD contains the one-loop vertex correction, as well as the one-loop correction to the propagator for the heavy and light quark fields. The quark propagators have the form [analytic $+\, iR^{(Q)}/(\not p - m_Q)$] and [analytic $+\, iR^{(q)}/\not p$] near the poles $p^2 = m_Q^2$ and $p^2 = 0$, respectively. The residues $R^{(Q)}$ and $R^{(q)}$ have perturbative expansions

$$R^{(Q)} = 1 + R_1^{(Q)} \alpha_s(\mu) + \cdots \tag{3.43}$$

and

$$R^{(q)} = 1 + R_1^{(q)} \alpha_s(\mu) + \cdots . \tag{3.44}$$

The desired matrix element in full QCD is obtained from the LSZ reduction formula,

$$\langle q(0, s')| V^\lambda | Q(p, s)\rangle = \left[R^{(Q)} R^{(q)}\right]^{1/2} \bar u(0, s')\left[\gamma^\lambda + V_1^\lambda \alpha_s(\mu)\right] u(p, s), \tag{3.45}$$

where γ^λ is the tree-level vertex, and $\alpha_s V_1^\lambda$ is the one-loop correction to the vertex from Fig. 1.4. The one-loop correction to the vertex has the expansion $(p = m_Q v)$

$$V_1^\lambda = V_1^{(1)} \gamma^\lambda + V_1^{(2)} v^\lambda, \tag{3.46}$$

as will be shown in Eq. (3.65).

The expression for the analogous matrix element in HQET is

$$\langle q(0, s')| \bar q \Gamma Q_v | Q(v, s)\rangle = \left[R^{(h)} R^{(q)}\right]^{1/2} \bar u(0, s')\left[1 + V_1^{\rm eff} \alpha_s(\mu)\right] \Gamma u(0, s), \tag{3.47}$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

where $R^{(h)}$ is the the residue of the heavy quark propagator near its pole, $iR^{(h)}/p \cdot v +$ analytic, and $\alpha_s V_1^{\rm eff} \Gamma$ is the one-loop vertex correction in Fig. 3.3, which is independent of the Γ matrix structure of the operator $\bar{q}\Gamma Q_v$.

Comparing Eqs. (3.45)–(3.47) and (3.38) gives

$$\begin{aligned} C_1^{(V)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right] &= 1 + \left\{\frac{1}{2}\left[R_1^{(Q)} - R_1^{(h)}\right] + V_1^{(1)} - V_1^{\rm eff}\right\}\alpha_s(\mu) + \cdots, \\ C_2^{(V)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right] &= V_1^{(2)}\alpha_s(\mu) + \cdots, \end{aligned} \tag{3.48}$$

where the ellipses denote terms higher order in $\alpha_s(\mu)$. $R_1^{(q)}$ does not occur in Eqs. (3.48) because it is common to both the HQET and full QCD calculations of the matrix element. The quantities R_1 and V_1 are ultraviolet finite as $\epsilon \to 0$ but they have infrared divergences, which must be regulated before computing these quantities. The coefficients $C_1^{(V)}$ and $C_2^{(V)}$ are not infrared divergent, so the infrared divergence cancels in the matching condition, which involves differences $R_1^{(Q)} - R_1^{(h)}$ and $V_1^{(1)} - V_1^{\rm eff}$ in the full and effective theories. It is important to use the same infrared regulator in both theories when computing matching conditions.

In this section, dimensional regularization will be used to regulate both the infrared and ultraviolet divergences. All graphs are computed in $4 - \epsilon$ dimensions, and the limit $\epsilon \to 0$ is taken at the end of the computation. Graphs will have $1/\epsilon$ poles, which arise from ultraviolet and infrared divergences. Only the $1/\epsilon$ ultraviolet divergences are canceled by counterterms. As a simple example, consider the integral

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^4} = 0. \tag{3.49}$$

The integral is ultraviolet and infrared divergent, but it is zero when evaluated in dimensional regularization. The infrared divergence can be regulated by introducing a mass to give

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m^2)^2} = \frac{i}{8\pi^2\epsilon} + \text{finite}. \tag{3.50}$$

Thus the original integral can be written as

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^4} = \frac{i}{8\pi^2\epsilon} - \frac{i}{8\pi^2\epsilon}, \tag{3.51}$$

where the first term is the ultraviolet divergence, and the second term is the infrared divergence. The counterterm contribution to the integral is $-i/8\pi^2\epsilon$, which cancels the ultraviolet divergence and leaves

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^4} + \text{counterterm} = -\frac{i}{8\pi^2\epsilon}, \tag{3.52}$$

where the right-hand side now only has an infrared divergence.

3.3.1 The QCD computation

The two-point function of renormalized heavy quark fields in the full QCD theory gets two contributions at order α_s. One is the one-loop diagram in Fig. 1.2 denoted by the subscript fd, and the other is the tree-level matrix element of the counterterm that cancels the $1/\epsilon$ ultraviolet divergence, denoted by the subscript ct. In the Feynman gauge, the one-loop contribution in Fig. 1.2 gives the quark self-energy Σ_{fd},

$$
\begin{aligned}
-i\Sigma_{\text{fd}} &= \int \frac{d^n q}{(2\pi)^n}\left(-igT^A\mu^{\epsilon/2}\right)\gamma^\alpha \frac{i(\not{p}+\not{q}+m_Q)}{\left[(p+q)^2-m_Q^2\right]}\left(-igT^A\mu^{\epsilon/2}\right)\gamma_\alpha \frac{(-i)}{q^2} \\
&= -g^2\left(\frac{4}{3}\right)\mu^\epsilon \int \frac{d^n q}{(2\pi)^n}\frac{\gamma^\alpha(\not{q}+\not{p})\gamma_\alpha + n m_Q}{q^2\left[(q+p)^2-m_Q^2\right]}. \qquad (3.53)
\end{aligned}
$$

Using the identity $\gamma^\alpha\gamma_\mu\gamma_\alpha = 2\gamma_\mu - \gamma^\alpha\gamma_\alpha\gamma_\mu = (2-n)\gamma_\mu$ and combining denominators gives

$$
\begin{aligned}
-i\Sigma_{\text{fd}} &= -g^2\left(\frac{4}{3}\right)\mu^\epsilon \int_0^1 dx \int \frac{d^n q}{(2\pi)^n}\frac{(2-n)(\not{q}+\not{p})+n m_Q}{\left[q^2+2q\cdot px - m_Q^2 x + p^2 x\right]^2} \\
&= -g^2\left(\frac{4}{3}\right)\mu^\epsilon \int_0^1 dx \int \frac{d^n q}{(2\pi)^n}\frac{(2-n)(1-x)\not{p}+n m_Q}{\left[q^2+p^2x(1-x)-m_Q^2 x\right]^2}. \qquad (3.54)
\end{aligned}
$$

The self-energy has the form

$$
\Sigma(p) = A(p^2)m_Q + B(p^2)\not{p}. \qquad (3.55)
$$

Since the full propagator is $i/[\not{p} - m_Q - \Sigma(p)]$, it is straightforward to see that the residue at the pole is

$$
R_1^{(Q)}\alpha_s(\mu) = B\left(m_Q^2\right) + 2m_Q^2 \left.\frac{\mathrm{d}(A+B)}{\mathrm{d}p^2}\right|_{p^2=m_Q^2}. \qquad (3.56)
$$

Performing the $d^n q$ integration in Eq. (3.54) yields the following expressions for A and B:

$$
\begin{aligned}
A_{\text{fd}}(p^2) &= \frac{g^2}{12\pi^2}(4\pi\mu^2)^{\epsilon/2}\Gamma(\epsilon/2)(4-\epsilon)\int_0^1 dx\left[m_Q^2 x - p^2 x(1-x)\right]^{-\epsilon/2}, \\
B_{\text{fd}}(p^2) &= -\frac{g^2}{12\pi^2}(4\pi\mu^2)^{\epsilon/2}\Gamma(\epsilon/2)(2-\epsilon) \qquad (3.57) \\
&\quad\times \int_0^1 dx(1-x)\left[m_Q^2 x - p^2 x(1-x)\right]^{-\epsilon/2}.
\end{aligned}
$$

The on-shell renormalization factor R_1 of Eq. (3.56) can be obtained by

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

substituting for A and B and integrating over x, using the identity

$$\int_0^1 x^a(1-x)^b = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}. \tag{3.58}$$

Expanding around $\epsilon = 0$ gives

$$R_{1,\text{fd}}\alpha_s = -\frac{g^2}{12\pi^2}\left(\frac{6}{\epsilon} + 4 - 3\gamma + 3\ln\frac{4\pi\mu^2}{m_Q^2}\right). \tag{3.59}$$

The $1/\epsilon$ terms include both the infrared and ultraviolet divergences. The counterterm contribution is $-i\Sigma_{\text{ct}} = i(Z_q - 1)\not{p} - i(Z_m - 1)m$, i.e., $A_{\text{ct}} = (Z_m - 1)$ and $B_{\text{ct}} = -(Z_q - 1)$, which gives the counterterm contribution to $R_{1,\text{ct}}\alpha_s$ of $-(Z_q - 1)$. Adding this [from Eq. (1.86)] to $R_{1,\text{fd}}\alpha_s$ and rescaling $4\pi\mu^2 \to \mu^2 e^\gamma$ to convert to the $\overline{\text{MS}}$ scheme gives the final result,

$$R_1^{(Q)}\alpha_s = -\frac{g^2}{12\pi^2}\left(\frac{4}{\epsilon} + 4 + 3\ln\frac{\mu^2}{m_Q^2}\right), \tag{3.60}$$

where the $1/\epsilon$ divergence in Eq. (3.60) is only an infrared divergence.

Next, consider the order α_s contribution to the one-particle irreducible vertex in full QCD shown in Fig. 1.4. In the Feynman gauge the graph gives

$$\int \frac{d^n q}{(2\pi)^n}(-ig\mu^{\epsilon/2}T^A)\gamma_\alpha \frac{i\not{q}}{q^2}\gamma^\lambda i\frac{(\not{p} + \not{q} + m_Q)}{\left[(p+q)^2 - m_Q^2\right]}(-ig\mu^{\epsilon/2}T^A)\gamma^\alpha \frac{(-i)}{q^2}. \tag{3.61}$$

Combining denominators, shifting the integration variable $q \to q - px$, and using $p^2 = m_Q^2$ gives

$$-ig^2\mu^\epsilon\left(\frac{8}{3}\right)\int_0^1 dx(1-x)\int\frac{d^n q}{(2\pi)^n}\frac{1}{\left(q^2 - m_Q^2 x^2\right)^3}$$
$$\times\{\gamma_\alpha(\not{q} - \not{p}x)\gamma^\lambda[\not{q} + \not{p}(1-x)]\gamma^\alpha + m_Q\gamma_\alpha(\not{q} - \not{p}x)\gamma^\lambda\gamma^\alpha\}. \tag{3.62}$$

The numerator can be simplified using the relations $\gamma_\alpha \not{a}\not{b}\not{c}\gamma^\alpha = -2\not{c}\not{b}\not{a} - (n-4)\not{a}\not{b}\not{c}$, and $\gamma_\alpha\not{a}\not{b}\gamma^\alpha = 4a\cdot b + (n-4)\not{a}\not{b}$. Terms odd in q vanish on integration. Terms involving $\not{p}$ can be simplified by anticommuting $\not{p}$ through any γ matrices until it is at the right, where it can be eliminated using $\not{p} = m_Q$ when acting on the heavy quark spinor. The final expression is

$$-ig^2\mu^\epsilon\left(\frac{8}{3}\right)\int_0^1 dx(1-x)\int\frac{d^n q}{(2\pi)^n}\frac{1}{\left(q^2 - m_Q^2 x^2\right)^3}$$
$$\times\left\{\frac{q^2}{n}(2-n)^2\gamma^\lambda - 2m_Q p^\lambda(n-2)x^2 + m_Q^2\gamma^\lambda x[x(n-2) - 2]\right\}. \tag{3.63}$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

Evaluating the q integrals and using $p = m_Q v$ gives

$$\frac{g^2}{12\pi^2}(4\pi\mu^2)^{\epsilon/2}\int_0^1 dx(1-x)\left(m_Q^2 x^2\right)^{-\epsilon/2}\left\{\frac{1}{2}\Gamma(\epsilon/2)(2-\epsilon)^2\gamma^\lambda\right.$$
$$\left. + 2\Gamma(1+\epsilon/2)v^\lambda(2-\epsilon) - \Gamma(1+\epsilon/2)\gamma^\lambda\frac{1}{x}\left[x(2-\epsilon)-2\right]\right\}. \tag{3.64}$$

Evaluating the x integral and expanding in ϵ gives

$$\frac{g^2}{12\pi^2}\{-2\gamma^\lambda + 2v^\lambda\}. \tag{3.65}$$

The counterterm contribution is determined by the renormalization of the current $\bar{q}\gamma^\lambda Q$ in QCD. Since this is a partially conserved current (i.e., is conserved in the limit that the masses vanish), it is not renormalized. The only remaining counterterm contribution is the QCD wave-function renormalization $Z_q - 1 = -2\alpha_s/3\pi\epsilon$ to $V_1^{(1)}\alpha_s$, from Eq. (1.86). Adding this to Eq. (3.65) gives

$$\begin{aligned} V_1^{(1)}\alpha_s &= -\frac{2\alpha_s}{3\pi}\left(\frac{1}{\epsilon}+1\right), \\ V_1^{(2)}\alpha_s &= \frac{2\alpha_s}{3\pi}. \end{aligned} \tag{3.66}$$

3.3.2 The HQET computation

We have now calculated all the quantities in full QCD that occur in Eq. (3.48) for $C_1^{(V)}$ and $C_2^{(V)}$. It remains to calculate the HQET quantities. In the Feynman gauge the HQET heavy quark self-energy obtained from the Feynman diagram in Fig. 3.2 is

$$-i\Sigma_{\text{fd}}(p) = -\left(\frac{4}{3}\right)g^2\mu^\varepsilon\int\frac{d^n q}{(2\pi)^n}\frac{1}{q^2\, v\cdot(p+q)}, \tag{3.67}$$

The residue at the pole is

$$R_1^{(h)}\alpha_s = v^\alpha\left.\frac{\partial\Sigma}{\partial p^\alpha}\right|_{p\cdot v=0}. \tag{3.68}$$

Evaluating Eq. (3.67) by combining denominators, the q integral gives

$$\begin{aligned} -i\Sigma_{\text{fd}} &= -i\frac{g^2}{6\pi^2}(4\pi\mu^2)^{\epsilon/2}\Gamma(\epsilon/2)\int_0^\infty d\lambda(\lambda^2 - 2\lambda p\cdot v)^{-\epsilon/2} \\ &= -i\frac{g^2}{6\pi^2}(4\pi\mu^2)^{\epsilon/2}(-p\cdot v)^{1-\epsilon}\frac{\Gamma(\epsilon/2)\Gamma(1-\epsilon/2)\Gamma(-1/2+\epsilon/2)}{2\sqrt{\pi}}. \end{aligned} \tag{3.69}$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

This yields $R_{1,\text{fd}}^{(h)} = 0$, since $\lim_{p\to 0}(-p \cdot v)^{-\epsilon} = 0$. The only contribution to $R_1^{(h)}$ is $-(Z_h - 1)$ from the counterterm, Eq. (3.14),

$$R_1^{(h)}\alpha_s = R_{1,\text{ct}}^{(h)}\alpha_s = -\frac{4\alpha_s}{3\pi\epsilon}. \tag{3.70}$$

The vertex calculation is also much simpler in HQET than in full QCD. The Feynman diagram in Fig. 3.3 gives

$$-ig^2\mu^\epsilon\left(\frac{4}{3}\right)\int\frac{d^nq}{(2\pi)^n}\frac{\not{v}\not{q}\Gamma}{(q^2)^2 v\cdot q}. \tag{3.71}$$

Combining denominators and evaluating the q integral gives

$$\frac{g^2}{6\pi^2}\Gamma(4\pi\mu^2)^{\epsilon/2}\Gamma(1+\epsilon/2)\int_0^\infty d\lambda\,\lambda^{-1-\epsilon}, \tag{3.72}$$

which is zero in dimensional regularization. The only contribution is from the counterterm, the negative of Eq. (3.22), which implies that

$$V_1^{\text{eff}}\alpha_s = -\frac{2\alpha_s}{3\pi\epsilon}. \tag{3.73}$$

Putting the pieces Eqs. (3.48), (3.60), (3.66), (3.70), and (3.73) of the matching calculation together yields

$$\begin{aligned} C_1^{(V)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right] &= 1 + \frac{\alpha_s(\mu)}{\pi}\left[\ln(m_Q/\mu) - \frac{4}{3}\right], \\ C_2^{(V)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right] &= \frac{2}{3}\frac{\alpha_s(\mu)}{\pi}. \end{aligned} \tag{3.74}$$

All the $1/\epsilon$ infrared divergences have canceled in the matching conditions. Note that in $C_1^{(V)}$ there is a logarithm of (m_Q/μ). That is why in our initial condition for the $C^{(V)}$'s we took $\mu = m_Q$. If μ was chosen very different from m_Q, large logarithms would prevent a perturbative evaluation of the initial values for the $C^{(V)}$'s. Of course, we do not have to pick $\mu = m_Q$ precisely. One may just as well use $\mu = m_Q/2$ or $\mu = 2m_Q$, for example. The μ dependence of the coefficients $C_i^{(V)}$ is connected with the anomalous dimension of the HQET operator $\bar{q}\gamma^\lambda Q_v$. Here $\mu[\mathrm{d}C_1^{(V)}/\mathrm{d}\mu]$ is the anomalous dimension γ_O given in Eq. (3.24). The absence of a logarithm in $C_2^{(V)}$ shows explicitly that $\bar{q}\gamma^\lambda Q_v$ does not mix with $\bar{q}v^\lambda Q_v$, which is consistent with our expectations based on spin and chiral symmetries.

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

A similar matching condition holds for the axial current, $A^\mu = \bar{q}\gamma^\mu\gamma_5 Q$.

$$A^\mu = C_1^{(A)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right]\bar{q}\gamma^\mu\gamma_5 Q_v + C_2^{(A)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right]\bar{q}v^\mu\gamma_5 Q_v. \tag{3.75}$$

It is simple to deduce the $C_j^{(A)}$, given our calculation of the $C_j^{(V)}$'s. Rewrite the axial current as $A^\mu = -\bar{q}\gamma_5\gamma^\mu Q$. γ_5 acting on the massless quark q gives $\pm$ depending on the chirality of the quark. Chirality is conserved by the gluon vertices, so the calculation of matching conditions proceeds just as in the vector current case, except that $\bar{q}$ should be replaced everywhere by $\bar{q}\gamma_5$. At the end of the calculation, the γ_5 is moved back next to Q_v, producing a compensating minus sign for $\gamma^\mu\gamma_5$, but not for $v^\mu\gamma_5$. Thus

$$C_1^{(A)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right] = C_1^{(V)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right], \tag{3.76}$$

$$C_2^{(A)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right] = -C_2^{(V)}\left[\frac{m_Q}{\mu}, \alpha_s(\mu)\right]. \tag{3.77}$$

The results of this section can be used to compute the α_s corrections to the pseudoscalar and vector meson decay constant relations given in Sec. 2.8. The QCD vector and axial current operators match the linear combination of HQET operators given in Eqs. (3.38) and (3.75). Computing the matrix elements of the HQET operators $\bar{q}\Gamma^\mu Q_v$ (renormalized at μ) as in Eq. (2.63) gives

$$a(\mu) \times \begin{cases} -iv^\mu P_v^{(Q)} & \text{if } \Gamma^\mu = \gamma^\mu\gamma_5, \\ iv^\mu P_v^{(Q)} & \text{if } \Gamma^\mu = v^\mu\gamma_5, \\ P_v^{*(Q)\mu} & \text{if } \Gamma^\mu = \gamma^\mu, \\ 0 & \text{if } \Gamma^\mu = v^\mu. \end{cases} \tag{3.78}$$

Combining this with the matching conditions gives

$$\begin{aligned} f_{P^*} &= \sqrt{m_{P*}}\, a(\mu) C_1^{(V)}(\mu), \\ f_P &= \frac{1}{\sqrt{m_P}}\, a(\mu)\left[C_1^{(A)}(\mu) - C_2^{(A)}(\mu)\right]. \end{aligned} \tag{3.79}$$

The μ dependence of the matrix element $a(\mu)$ is given by the anomalous dimension of the heavy-light operators, Eq. (3.24),

$$\mu\frac{\mathrm{d}a}{\mathrm{d}\mu} = -\gamma_O a = \frac{\alpha_s}{\pi}a. \tag{3.80}$$

This μ dependence is canceled by the μ dependence in the coefficients $C_i^{(V,A)}$, so that the complete answer for the measurable quantity $f_{P,P*}$ is μ independent.

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

For example,

$$\begin{aligned}\sqrt{m_P}\,\mu\frac{\mathrm{d}f_P}{\mathrm{d}\mu} &= \mu\frac{\mathrm{d}a}{\mathrm{d}\mu}\left[C_1^{(A)} - C_2^{(A)}\right] + a\,\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left[C_1^{(A)} - C_2^{(A)}\right] \\ &= \frac{\alpha_s}{\pi}\,a\left[C_1^{(A)} - C_2^{(A)}\right] + a\left(-\frac{\alpha_s}{\pi} + 0\right) \\ &= 0 + \mathcal{O}\left(\alpha_s^2\right).\end{aligned} \tag{3.81}$$

Equation (3.79) gives the α_s correction to the ratio of the pseudoscalar and vector meson decay constants,

$$\frac{f_{P^*}}{f_P} = \sqrt{m_{P^*}m_P}\left[\frac{C_1^{(V)}}{C_1^{(A)} - C_2^{(A)}}\right] = \sqrt{m_{P^*}m_P}\left[1 - \frac{2}{3}\frac{\alpha_s(m_Q)}{\pi}\right]. \tag{3.82}$$

The α_s correction to the ratio of pseudoscalar meson decay constants for the D and B mesons can also be determined. Heavy quark flavor symmetry implies that $a\,(\mu)$, the matrix element in the effective theory, is independent of the quark mass. The matching from QCD to the effective theory is done at the scale $m_Q = m_b$ for the $\bar{B}$ meson system, and $m_Q = m_c$ for the D meson system. This determines

$$\begin{aligned}\frac{f_B\sqrt{m_B}}{f_D\sqrt{m_D}} &= \left[\frac{a\,(m_b)}{a(m_c)}\right]\frac{C_1^{(A)}[1,\alpha_s(m_b)] - C_2^{(A)}[1,\alpha_s(m_b)]}{C_1^{(A)}[1,\alpha_s(m_c)] - C_2^{(A)}[1,\alpha_s(m_c)]} \\ &= \left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right]^{-6/25} \\ &\quad\times\left\{1 + [\alpha_s(m_b) - \alpha_s(m_c)]\left[-\frac{2}{3\pi} + \left(\frac{\gamma_{1O}}{2\beta_0} - \frac{\gamma_{0O}\beta_1}{2\beta_0^2}\right)\right]\right\}.\end{aligned} \tag{3.83}$$

To complete the prediction for the ratio of B and D meson decay constants, the two-loop correction to the anomalous dimension of O_Γ, γ_{1O}, and the two-loop contribution to the β function, β_1, are needed. These can be found in the literature. The leading logarithmic prediction for the ratio of B and D meson decay constants is

$$\frac{f_B\sqrt{m_B}}{f_D\sqrt{m_D}} = \left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right]^{-6/25}. \tag{3.84}$$

The matching conditions in this section have been computed keeping the $1/\epsilon$ infrared divergent quantities, to show explicitly that the divergences cancel in the matching coefficients. This cancellation provides a useful check on the calculation. The matching conditions can be computed more simply if one is willing to forego this check. One can simply compute only the finite parts of the dimensionally regulated graphs in the full and effective theory to compute the matching conditions. The $1/\epsilon$ ultraviolet divergences are canceled by counterterms, and the $1/\epsilon$ infrared divergences will cancel in the matching conditions, and so need

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

not be retained. One also need not compute any diagrams in the effective theory, since all on-shell graphs in the effective theory vanish on dimensional regularization. We saw this explicitly in Eqs. (3.69) and (3.72). The reason is that graphs that contain no dimensionful parameter vanish in dimensional regularization.

Since m_b/m_c is not very large, there is no reason to sum the leading logarithms of m_b/m_c. If one matches onto HQET simultaneously for the b and c quarks at a scale μ, then Eqs. (3.74), (3.76), and (3.77) imply that

$$\frac{f_B\sqrt{m_B}}{f_D\sqrt{m_D}} = 1 + \frac{\alpha_s(\mu)}{\pi}\ln\left(\frac{m_b}{m_c}\right). \tag{3.85}$$

Eq. (3.85) can also be derived by expanding Eq. (3.84) to order α_s.

3.4 Heavy-heavy currents

$\bar{B} \to D^{(*)}e\bar{\nu}_e$ and $\Lambda_b \to \Lambda_c e\bar{\nu}_e$ decay rates are determined by matrix elements of the vector current, $\bar{c}\gamma_\mu b$, and the axial vector current $\bar{c}\gamma_\mu\gamma_5 b$. The matching of these currents in full QCD onto operators in HQET has the form

$$\begin{aligned}\bar{c}\gamma_\mu b = \; & C_1^{(V)}\left[\frac{m_b}{\mu}, \frac{m_c}{\mu}, \alpha_s(\mu), w\right]\bar{c}_{v'}\gamma_\mu b_v \\ & + C_2^{(V)}\left[\frac{m_b}{\mu}, \frac{m_c}{\mu}, \alpha_s(\mu), w\right]\bar{c}_{v'}v_\mu b_v \\ & + C_3^{(V)}\left[\frac{m_b}{\mu}, \frac{m_c}{\mu}, \alpha_s(\mu), w\right]\bar{c}_{v'}v'_\mu b_v\end{aligned} \tag{3.86}$$

and

$$\begin{aligned}\bar{c}\gamma_\mu\gamma_5 b = \; & C_1^{(A)}\left[\frac{m_b}{\mu}, \frac{m_c}{\mu}, \alpha_s(\mu), w\right]\bar{c}_{v'}\gamma_\mu\gamma_5 b_v \\ & + C_2^{(A)}\left[\frac{m_b}{\mu}, \frac{m_c}{\mu}, \alpha_s(\mu), w\right]\bar{c}_{v'}v_\mu\gamma_5 b_v \\ & + C_3^{(A)}\left[\frac{m_b}{\mu}, \frac{m_c}{\mu}, \alpha_s(\mu), w\right]\bar{c}_{v'}v'_\mu\gamma_5 b_v.\end{aligned} \tag{3.87}$$

The right-hand side contains all dimension three operators with the same quantum numbers as the left-hand side. Higher dimension operators give effects suppressed by powers of $(\Lambda_{\rm QCD}/m_{c,b})$ and will be considered in the next chapter. In the matching condition of Eqs. (3.86) and (3.87) the transition to HQET is made simultaneously for both quarks. Usually one chooses a subtraction point, $\mu = \bar{m} = \sqrt{m_b m_c}$, which is between the bottom and charm quark masses for the initial value for the C_j's and then runs down to a lower value of μ by using the HQET renormalization group equation. At order α_s, the matching condition

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

contains terms of the order of $\alpha_s(\bar{m})\ln(m_b/m_c)$, but since this logarithm is not very large there is no need to sum all terms of the order of $\alpha_s(\bar{m})^n \ln^n(m_c/m_b)$. Tree-level matching at $\bar{m}$ gives

$$
\begin{aligned}
C_1^{(V,A)}\left[\frac{m_b}{\bar{m}}, \frac{m_c}{\bar{m}}, \alpha_s(\bar{m}), w\right] &= 1 + \mathcal{O}\left[\alpha_s(\bar{m})\right], \\
C_2^{(V,A)}\left[\frac{m_b}{\bar{m}}, \frac{m_c}{\bar{m}}, \alpha_s(\bar{m}), w\right] &= 0 + \mathcal{O}\left[\alpha_s(\bar{m})\right], \\
C_3^{(V,A)}\left[\frac{m_b}{\bar{m}}, \frac{m_c}{\bar{m}}, \alpha_s(\bar{m}), w\right] &= 0 + \mathcal{O}\left[\alpha_s(\bar{m})\right].
\end{aligned}
\tag{3.88}
$$

The additional operators $\bar{c}_{v'} v^\mu b_v$ and $\bar{c}_{v'} v'^\mu b_v$ induced at one loop do not cause a loss of predictive power in computing decay rates. In HQET the $\bar{B} \to D^{(*)}$ matrix elements of any operator of the form $\bar{c}_{v'} \Gamma b_v$ (where Γ is a 4×4 matrix in spinor space) can be expressed in terms of the Isgur-Wise function, so the matrix elements of the new operators are related to the matrix elements of the old operators. This was also the case for heavy-light matrix elements in Eq. (3.78).

The calculation of the $C_j^{(V,A)}$ at order α_s is straightforward but somewhat tedious, since these coefficients depend not only on the bottom and charm quark masses but also on the dot product of four velocities $w = v \cdot v'$. In this chapter we shall explicitly calculate the matching condition at the zero-recoil kinematic point, $w = 1$. Here the matching condition simplifies because $\bar{c}_v \gamma_5 b_v = 0$ and $\bar{c}_v \gamma_\mu b_v = \bar{c}_v v_\mu b_v$. Consequently we can write the matching relation as

$$
\begin{aligned}
\bar{c}\gamma_\mu b &= \eta_V \; \bar{c}_v \gamma_\mu b_v, \\
\bar{c}\gamma_\mu \gamma_5 b &= \eta_A \; \bar{c}_v \gamma_\mu \gamma_5 b_v.
\end{aligned}
\tag{3.89}
$$

As in the case of heavy-light currents, the coefficients η_V and η_A are determined by equating a full QCD matrix element of these currents with the corresponding one in HQET. The matrix element we choose is between an on-shell b-quark state with four-momentum $p_b = m_b v$ and an on-shell c-quark state with four-momentum $p_c = m_c v$. Since $\bar{c}_v \gamma_\mu b_v$ is the conserved current associated with heavy quark flavor symmetry, and $\bar{c}_v \gamma_\mu \gamma_5 b_v$ is related to it by heavy quark spin symmetry, we know the matrix elements of these currents. To all orders in the strong coupling,

$$
\left\langle c(v, s')\right| \bar{c}_v \Gamma b_v \left|b(v, s)\right\rangle = \bar{u}(v, s')\, \Gamma\, u(v, s),
\tag{3.90}
$$

where Γ is any matrix in spinor space (including γ_μ or $\gamma_\mu \gamma_5$), and the right-hand side is absolutely normalized by heavy quark symmetry. This relation is subtraction-point independent and so $\eta_{(V,A)}$ must be μ independent:

$$
\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \eta_{(V,A)} = 0.
\tag{3.91}
$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

The matching condition will be computed by using the procedure outlined at the end of the previous section, so only the finite parts of dimensionally regulated graphs will be computed. The vector current matrix element in QCD is

$$\langle c(p_c, s')|\bar{c}\gamma^\lambda b|b(p_b, s)\rangle = \bar{u}(p_c, s')\left\{1 + \frac{1}{2}\left[R_1^{(c)} + R_1^{(b)}\right]\alpha_s(\mu) + V_1\alpha_s(\mu)\right\}\gamma^\lambda u(p_b, s) + \cdots, \tag{3.92}$$

where $p_c = m_c v$, $p_b = m_b v$, and the ellipsis denotes terms higher order in α_s. Here $R_1^{(Q)}$ has already been computed, so it only remains to compute the one-particle irreducible vertex at the order of α_s. It is given by the Feynman diagram in Fig. (1.4). In the Feynman gauge Fig. 1.4 yields

$$-ig^2\mu^\varepsilon\left(\frac{4}{3}\right)\int\frac{d^nq}{(2\pi)^n}\frac{\gamma_\alpha(\not{q} + \not{p}_c + m_c)\gamma^\lambda(\not{q} + \not{p}_b + m_b)\gamma^\alpha}{(q^2 + 2p_c\cdot q)(q^2 + 2p_b\cdot q)q^2}. \tag{3.93}$$

The charm and bottom quarks have the same four velocity and so a factor of $\not{p}_{c,b}$ on the far left or right can be replaced by $m_{c,b}$. Hence Eq. (3.93) can be written as

$$\begin{aligned}&-ig^2\mu^\varepsilon\left(\frac{4}{3}\right)\int\frac{d^nq}{(2\pi)^n}\frac{(2m_cv_\alpha + \gamma_\alpha\not{q})\gamma^\lambda(2m_bv^\alpha + \not{q}\gamma^\alpha)}{(q^2 + 2q\cdot p_c)(q^2 + 2q\cdot p_b)q^2}\\ &\quad = -ig^2\mu^\varepsilon\left(\frac{4}{3}\right)\int\frac{d^nq}{(2\pi)^n}\\ &\qquad\times\left[\frac{4m_cm_b\gamma^\lambda + 2m_c\gamma^\lambda\not{q} + 2m_b\not{q}\gamma^\lambda + (2-n)\not{q}\gamma^\lambda\not{q}}{(q^2 + 2q\cdot p_c)(q^2 + 2q\cdot p_b)q^2}\right].\end{aligned} \tag{3.94}$$

It is convenient to first combine the two quark propagator denominators using the Feynman parameter x, and then combine the result with the gluon propagator using y. Shifting the q integration variable, $q \to q - y[m_cx + m_b(1-x)]v$ and performing the d^nq integration gives

$$\begin{aligned}&\frac{g^2}{12\pi^2}\gamma^\lambda(4\pi\mu^2)^{\epsilon/2}\int_0^1 dx\int_0^1 y\,dy\left(m_x^2y^2\right)^{-\epsilon/2}\left\{\frac{1}{2}(2-\epsilon)^2\,\Gamma(\epsilon/2)\right.\\ &\quad\left.-\Gamma(1+\epsilon/2)\left[\frac{4m_cm_b}{m_x^2y^2} - 2\frac{m_c + m_b}{m_xy} - (2-\epsilon)\right]\right\}\end{aligned} \tag{3.95}$$

where

$$m_x = m_cx + m_b(1-x).$$

Evaluating the y integral, expanding in ϵ, and rescaling μ to the $\overline{\text{MS}}$ scheme yields

$$\frac{g^2}{6\pi^2}\gamma^\lambda\int_0^1 dx\left[\left(1 + \frac{2m_bm_c}{m_x^2}\right)\frac{1}{\epsilon} + \frac{m_b + m_c}{m_x} - \left(1 + \frac{2m_bm_c}{m_x^2}\right)\ln\left(\frac{m_x}{\mu}\right)\right]. \tag{3.96}$$

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

Integrating with respect to x and keeping the finite part gives

$$V_1\alpha_s = -\frac{g^2}{6\pi^2}\left[1 + 3\frac{m_b\ln(m_c/\mu) - m_c\ln(m_b/\mu)}{m_b - m_c}\right]. \tag{3.97}$$

Equations (3.90) and (3.92) imply that the matching coefficient is

$$\eta_V = 1 + \alpha_s(\mu)\left[\frac{R_1^{(b)}}{2} + \frac{R_1^{(c)}}{2} + V_1\right] + \cdots, \tag{3.98}$$

where the ellipsis denotes terms of the order of α_s^2 and higher. Using Eq. (3.97) and the finite part of Eq. (3.60), we find that at order α_s,

$$\eta_V = 1 + \frac{\alpha_s(\mu)}{\pi}\left[-2 + \left(\frac{m_b + m_c}{m_b - m_c}\right)\ln\left(\frac{m_b}{m_c}\right)\right]. \tag{3.99}$$

Note that the coefficient of $\alpha_s(\mu)$ is independent of μ. This is a consequence of Eq. (3.91), which states that η_V is independent of the subtraction point μ. Terms higher order in α_s compensate for the dependence of α_s on μ in Eq. (3.99). Usually for numerical evaluation of $\eta_{(V,A)}$ one uses $\mu = \sqrt{m_b m_c} = \bar{m}$.

In the case $m_b = m_c$, the vector current $\bar{c}\gamma^\lambda b$ is a conserved current in QCD and its on-shell matrix element is $\langle c(p_c, s')|\bar{c}\gamma^\lambda b|b(p_b, s)\rangle = \bar{u}(p_c, s')\gamma^\lambda\, u(p_b, s')$, to all orders in α_s. Consequently the coefficient of α_s in Eq. (3.99) vanishes in the limit $m_b = m_c$.

The axial current matching condition is almost the same as in the vector case. In the calculation of the one-particle irreducible vertex, Eq. (3.94) is replaced by

$$\begin{aligned} &-ig^2\mu^\varepsilon\left(\frac{4}{3}\right)\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q^2 + 2q\cdot p_c)(q^2 + 2q\cdot p_b)q^2} \\ &\quad\times\left[4m_c m_b\gamma^\lambda\gamma_5 + 2m_c\gamma^\lambda\gamma_5\not{q} + 2m_b\not{q}\gamma^\lambda\gamma_5 + (2-n)\not{q}\gamma^\lambda\not{q}\gamma_5\right]. \end{aligned} \tag{3.100}$$

One can then combine denominators and change the integration variable as for the computation of η_V. The only difference between η_V and η_A is that for η_V, $(2-n)\not{q}\gamma^\lambda\not{q}$ generates the term $(2-n)m_x^2y^2\gamma^\lambda$ on shifting the integration variable, whereas for η_A, $(2-n)\not{q}\gamma^\lambda\not{q}\gamma_5$ generates $-(2-n)m_x^2y^2\gamma^\lambda\gamma_5$. Thus

$$\begin{aligned} \eta_A &= \eta_V + ig^2\left(\frac{4}{3}\right)2(2-n)\int_0^1 dx\int_0^1 2y\,dy\int\frac{d^nq}{(2\pi)^n}\frac{m_x^2y^2}{\left(q^2 - m_x^2y^2\right)^3} \\ &= \eta_V - \frac{2}{3\pi}\alpha_s(\mu) \\ &= 1 + \frac{\alpha_s(\mu)}{\pi}\left[-\frac{8}{3} + \frac{(m_b + m_c)}{(m_b - m_c)}\ln\left(\frac{m_b}{m_c}\right)\right]. \end{aligned} \tag{3.101}$$

Here $\eta_{(V,A)}$ are important for the $B \to D^{(*)}e\bar{\nu}_e$ differential decay rates near

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

$w = v \cdot v' = 1$, i.e., $\mathcal{F}_{D^*}(1) = \eta_A$ and $\mathcal{F}_D(1) = \eta_V$ up to corrections suppressed by powers of m_Q.

3.5 Problems

1. The effective Hamiltonian for $B^0 - \bar{B}^0$ mixing is proportional to the operator

$$(\bar{d}\gamma_\mu P_L b)(\bar{d}\gamma^\mu P_L b).$$

After the transition to HQET, it becomes

$$O^{\Delta S=2} = (\bar{d}\gamma_\mu P_L b_v)(\bar{d}\gamma^\mu P_L b_v).$$

Calculate the anomalous dimension of $O^{\Delta S=2}$ at one loop.

2. Analytic expressions for the matching coefficients $C_j^{(V)}$ and $C_j^{(A)}$ can be found in an expansion about $w = 1$.

(a) Show that if the c and b quarks are matched onto the HQET fields $c_{v'}$ and b_v at the common scale $\mu = \bar{m} = \sqrt{m_c m_b}$, then $C_j^{V,A}(w) = 1 + (\alpha_s(\bar{m})/\pi)\delta C_j^{V,A}(w)$, where

$$\delta C_1^{(V)}(1) = -\frac{4}{3} - \frac{1+z}{1-z},$$
$$\delta C_2^{(V)}(1) = -\frac{2(1 - z + z\ln z)}{3(1-z)^2},$$
$$\delta C_3^{(V)}(1) = \frac{2z(1 - z + \ln z)}{3(1-z)^2},$$
$$\delta C_1^{(A)}(1) = -\frac{8}{3} - \frac{1+z}{1-z}\ln z,$$
$$\delta C_2^{(A)}(1) = -\frac{2[3 - 2z - z^2 + (5-z)z\ln z]}{3(1-z)^3},$$
$$\delta C_3^{(A)}(1) = \frac{2z[1 + 2z - 3z^2 + (5z-1)\ln z]}{3(1-z)^3},$$

where $z = m_c/m_b$.

(b) Show that

$$\delta C_1'^{(V)}(1) = -\frac{2[13 - 9z + 9z^2 - 13z^3 + 3(2 + 3z + 3z^2 + 2z^3)\ln z]}{27(1-z)^3},$$
$$\delta C_2'^{(V)}(1) = \frac{2(2 + 3z - 6z^2 + z^3 + 6z\ln z)}{9(1-z)^4},$$
$$\delta C_3'^{(V)}(1) = \frac{2z(1 - 6z + 3z^2 + 2z^3 - 6z^2\ln z)}{9(1-z)^4},$$
$$\delta C_1'^{(A)}(1) = -\frac{2[7 + 9z - 9z^2 - 7z^3 + 3(2 + 3z + 3z^2 + 2z^3)\ln z]}{27(1-z)^3},$$
$$\delta C_2'^{(A)}(1) = \frac{2[2 - 33z + 9z^2 + 25z^3 - 3z^4 - 6z(1+7z)\ln z]}{9(1-z)^5},$$
$$\delta C_3'^{(A)}(1) = -\frac{2z[3 - 25z - 9z^2 + 33z^3 - 2z^4 - 6z^2(7+z)\ln z]}{9(1-z)^5},$$

where $'$ denotes differentiation with respect to w.

https://doi.org/10.1017/9781009402125.004 Published online by Cambridge University Press

(c) Using $m_c = 1.4$ GeV and $m_b = 4.8$ GeV, calculate the perturbative QCD corrections to the ratios of form factors $R_1(1)$ and $R_2(1)$ defined in Chapter 2.

3. Prove the identity in Eq. (3.6).
4. Calculate the renormalization of the operators

$$\begin{aligned}
O_1 &= \bar{c}_{v'}\,\Gamma i D_\mu\, b_v \\
O_2 &= \bar{c}_{v'}\,\Gamma i \overleftarrow{D}_\mu\, b_v \\
O_3 &= \bar{c}_{v'}\,\Gamma i (v' \cdot D)\, b_v\, v_\mu \\
O_4 &= \bar{c}_{v'}\,\Gamma i (v' \cdot D)\, b_v\, v'_\mu \\
O_5 &= \bar{c}_{v'}\,\Gamma i (v \cdot \overleftarrow{D})\, b_v\, v_\mu \\
O_6 &= \bar{c}_{v'}\,\Gamma i (v \cdot \overleftarrow{D})\, b_v\, v'_\mu
\end{aligned}$$

and use it to compute the anomalous dimension matrix for $O_1 - O_6$.

5. Consider the ratio $r_f(w) = f_2(w)/f_1(w)$ of the form factors for $\Lambda_b \to \Lambda_c e \bar{\nu}_e$ decay. Show that in the $m_b \to \infty$ limit, the perturbative α_s correction gives

$$r_f(w) = -\frac{2\alpha_s(m_c)}{3\pi} r(w),$$

where $r(w)$ is defined in Eq. (3.33).

3.6 References

The renormalization Z factors of QCD in the $m_Q \to \infty$ limit, and their implications for heavy quark physics, were discussed in:

M.A. Shifman and M.B. Voloshin, Sov. J. Nucl. Phys. 45 (1987) 292 [Yad. Fiz. 45 (1987) 463]

see also:

H.D. Politzer and M.B. Wise, Phys. Lett. B206 (1988) 681, B208 (1988) 504

Matching of QCD onto HQET was developed in:

A.F. Falk, H. Georgi, B. Grinstein, and M.B. Wise, Nucl. Phys. B343 (1990) 1

Matching of heavy-heavy currents was discussed in:

A.F. Falk and B. Grinstein, Phys. Lett. B247 (1990) 406
M. Neubert, Phys. Rev. D46 (1992) 2212

For some work on two-loop matching and anomalous dimensions, see:

X. Ji and M.J. Musolf, Phys. Lett. B257 (1991) 409
D.J. Broadhurst and A.G. Grozin, Phys. Lett. B267 (1991) 105, Phys. Rev. D52 (1995) 4082
W. Kilian, P. Manakos, and T. Mannel, Phys. Rev. D48 (1993) 1321
M. Neubert, Phys. Lett. B341 (1995) 367
A. Czarnecki, Phys. Rev. Lett. 76 (1996) 4124
G. Amoros, M. Beneke, and M. Neubert, Phys. Lett. B401 (1997) 81

see also:

G.P. Korchemsky and A.V. Radyushkin, Nucl. Phys. B283 (1987) 343, Phys. Lett. B279 (1992) 359

Perturbative matching for heavy-light currents at order $1/m_Q$ is considered in:

A.F. Falk and B. Grinstein, Phys. Lett. B247 (1990) 406
M. Neubert, Phys. Rev. D46 (1992) 1076

For an excellent review on perturbative corrections, see:

M. Neubert, Phys. Rep. 245 (1994) 259