Canad. J. Math. Vol. **66** (2), 2014 pp. 373–386
http://dx.doi.org/10.4153/CJM-2013-009-2
© Canadian Mathematical Society 2013

Uniform Convexity and the Bishop–Phelps–Bollobás Property

Sun Kwang Kim and Han Ju Lee

Abstract. A new characterization of the uniform convexity of Banach space is obtained in the sense of the Bishop–Phelps–Bollobás theorem. It is also proved that the couple of Banach spaces (X, Y) has the Bishop–Phelps–Bollobás property for every Banach space Y when X is uniformly convex. As a corollary, we show that the Bishop–Phelps–Bollobás theorem holds for bilinear forms on $\ell_p \times \ell_q$ $(1 < p, q < \infty)$.

1 Introduction

Throughout this paper, X is a Banach space over a real or complex field $\mathbb{K}$ and B_X (resp. S_X) is the closed unit ball (resp. unit sphere) of X. The closed ball with center $x \in X$ and radius $\epsilon > 0$ is denoted by $B(x, \epsilon)$. For Banach spaces X, Y over the same scalar field $\mathbb{K}$, $\mathcal{L}(X, Y)$ is the Banach space of all bounded linear operators from X into Y and $X^* = \mathcal{L}(X, \mathbb{K})$ stands for the dual space of X. We say that an operator $T \in \mathcal{L}(X, Y)$ *attains its norm* if there exists a point $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\| = \sup\{\|T(x)\| : x \in B_X\}$.

In 1961, Bishop and Phelps showed that the set of norm-attaining functionals on a Banach space X is dense in its dual space X^* (the *Bishop-Phelps Theorem* [6]). There has been a great effort to extend this theorem to bounded linear operators between Banach spaces. In general, the set of norm-attaining operators is not dense in $\mathcal{L}(X, Y)$, but there are many positive answers on classical Banach spaces [18, 24, 25, 27]. Moreover, for a reflexive Banach space X, it is true for every Banach space Y [23], and this result is generalized to a Banach space X with the Radon–Nikodým property [8]. Very recently, this study has also been extended to non-linear mappings, such as multi-linear mappings, polynomials and holomorphic mappings [1, 5, 11, 14, 15, 20, 21].

Meanwhile, Bollobás sharpened the Bishop–Phelps Theorem by simultaneously approximating both functional and point. He approximates the norm of the functional with norm-attaining functionals and corresponding points at which they attain their norms.

Received by the editors June 19, 2012; revised March 6, 2013.
Published electronically April 2, 2013.
The first author is the corresponding author. The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science, and Technology (2012R1A1A1006869).
AMS subject classification: **46B20**, 46B22.
Keywords: Bishop–Phelps–Bollobás property, Bishop–Phelps–Bollobás theorem, norm attaining, uniformly convex.

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

Theorem 1.1 ([7]) *For an arbitrary $\epsilon > 0$, if $x^* \in S_{X^*}$ satisfies $|1 - x^*(x)| < \frac{\epsilon^2}{4}$ for $x \in B_X$, then there are both $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \epsilon$ and $\|y^* - x^*\| < \epsilon$.*

Very recently, Acosta et al. [2] began extending this theorem to bounded linear operators between Banach spaces and introduced the *Bishop–Phelps–Bollobás property*.

Definition 1.2 ([2, Definition 1.1]) Let X and Y be Banach spaces over $\mathbb{K}$. We say that the pair (X, Y) has the Bishop–Phelps–Bollobás property for operators (BPBP) if, given $\epsilon > 0$ there exist $\beta(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon \to 0^+} \beta(\epsilon) = 0$ such that if there exist both $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist an operator $S \in S_{\mathcal{L}(X,Y)}$ and $u_0 \in S_X$ such that

$$\|Su_0\| = 1, \quad \|x_0 - u_0\| < \beta(\epsilon) \quad \text{and} \quad \|T - S\| < \epsilon.$$

In the same paper it is shown that the pair (X, Y) has the BPBP for finite dimensional Banach spaces X and Y, and that the pair (ℓ_∞^n, Y) has the BPBP for every n, whenever Y is a uniformly convex space. They also gave a geometric characterization of a Banach space Y, which is called AHSP such that (ℓ_1, Y) has the BPBP, and we know that uniformly convex spaces and lush spaces have this property [13]. It is worth mentioning that some results on $L_1(\mu)$ were obtained in [4, 12].

On the other hand, Aron et al. [3] studied this property for the case $Y = C_0(L)$, where L is a locally compact Hausdorff space, and showed that the pair $(X, C_0(L))$ has the BPBP if X is Asplund. It follows from this result that the pair $(X, C_0(L))$ has the BPBP for every uniformly convex Banach space X.

In this paper, we study the relation between the uniform convexity and BPBP.

In Section 2, we characterize a uniformly convex Banach space from the view point of the Bishop–Phelps–Bollobás Theorem. Notice that the James theorem [19] says that a Banach space is reflexive if and only if every continuous linear functional is norm-attaining. In the sense of Bishop–Phelps–Bollobás theorem, we consider the following property for a Banach space X:

> For all $\epsilon > 0$, there is some $\eta(\epsilon) > 0$ such that for all $f \in S_{X^*}$ and $x \in B_X$ satisfying $|f(x)| > 1 - \eta(\epsilon)$, there exists $x_0 \in S_X$ such that $|f(x_0)| = 1$ and $\|x - x_0\| < \epsilon$.

We show that the above property is equivalent to uniform convexity. As a corollary, if a Banach space X has a uniformly strongly exposed family $\{x_\alpha\}_\alpha \subset S_X$ with respect to $\{f_\alpha\}_\alpha \subset S_{X^*}$ and the convex hull of $\{f_\alpha\}_\alpha$ is weak-*-dense in B_{X^*}, then X is uniformly convex.

In Section 3, we show that (X, Y) has the BPBP for every Banach space Y if X is uniformly convex. As a corollary, the Bishop–Phelps–Bollobás theorem holds for bilinear forms on $\ell_p \times \ell_q$ ($1 < p, q < \infty$). This is an affirmative answer to one of the questions in a forthcoming paper by Cheng and Dai [9]. We also consider the

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

following property for a Banach space X:

> Given $\epsilon > 0$, there exist positive real valued functions $\eta(\epsilon)$ and $\beta(\epsilon)$ (satisfying $\lim_{\epsilon\to 0}\beta(\epsilon) = 0$) such that for every Banach space Y, if there are both $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfying $\|Tx\| > 1-\eta(\epsilon)$, then there exist both $S \in S_{\mathcal{L}(X,Y)}$ and $u \in S_X$ such that
>
> $$\|Su\| = 1, \|x-u\| < \beta(\epsilon) \text{ and } \|T-S\| < \epsilon.$$

If a Banach space X has the above property, then the following hold:

(a) any face of S_X does not contain a relatively open subset of S_X;
(b) if X is isomorphic to a strictly convex Banach space, then the set of all extreme points of B_X is dense in S_X;
(c) if X is isomorphic to a uniformly convex Banach space, then the set of all strongly exposed points of B_X is dense in S_X.

From this result, we see that a 2-dimensional real Banach space with the aforementioned property must be uniformly convex.

2 Uniform Convexity and the Bishop–Phelps–Bollobás Theorem

The norm-attaining operators have played key roles to characterize some properties of a Banach space. James showed that a Banach space is reflexive if and only if every bounded linear functional attains its norm. As an another example, a Banach space X has the *Radon–Nikodým property* if and only if X has the *Bishop–Phelps property* [8], that is, for every nonempty closed bounded convex subset C of X and for every Banach space Y, the set of all bounded linear operators T such that $\|T(\cdot)\|$ attains its maximum on C is dense in $L(X,Y)$.

In this section, we provide new criteria for characterizations of the uniform convexity and the uniform smoothness of Banach spaces in the sense of the Bishop–Phelps–Bollobás Theorem.

For $\epsilon \in (0,2]$, the *modulus of convexity* of a Banach space $(X,\|\cdot\|)$ is defined by

$$\delta(\epsilon) = \inf\Big\{ 1 - \Big\|\frac{x+y}{2}\Big\| : x,y \in B_X, \|x-y\| \geq \epsilon \Big\},$$

and for $\tau > 0$, the *modulus of smoothness* of $(X,\|\cdot\|)$ is defined by

$$\rho(\tau) = \sup\Big\{ \frac{\|x+\tau h\| + \|x-\tau h\| - 2}{2} : \|x\| = \|h\| = 1 \Big\}.$$

A Banach space $(X,\|\cdot\|)$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon \in (0,2]$, and uniformly smooth if $\lim_{\tau\to 0^+}\frac{\rho(\tau)}{\tau} = 0$. For the geometric meaning and basic properties of these moduli, see [17].

It is convenient to use the sequential characterization of uniform convexity. A Banach space X is uniformly convex if and only if, whenever $x_n, y_n \in X$ $(n \in \mathbb{N})$, $\lim_{n\to\infty}(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n+y_n\|^2) = 0$, and $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

In the proof of the theorem, we will use a well-known property of Banach spaces having the Radon–Nikodým property. A functional $f \in X^*$ is said to strongly expose B_X at x if f attains its norm at x and whenever there is a sequence $\{x_n\}$ in B_X such that $\lim_{n\to\infty} \operatorname{Re} f(x_n) = \|f\|$, $\{x_n\}$ converges to x in norm. It is well known [8, 26] that if X has the Radon-Nikodým property, then the set of all functionals that strongly expose B_X is dense in X^*.

Theorem 2.1 *A Banach space X is uniformly convex if and only if for every $\epsilon > 0$ there is $0 < \eta(\epsilon) < 1$ such that for all $f \in S_{X^*}$ and all $x \in B_X$ satisfying $|f(x)| > 1 - \eta(\epsilon)$, there exists $x_0 \in S_X$ satisfying $|f(x_0)| = 1$ and $\|x - x_0\| < \epsilon$.*

Proof First, assume that X is a uniformly convex Banach space. Let $\delta(\epsilon)$ be the modulus of uniform convexity and put $\eta(\epsilon) = \min(\delta(\frac{\epsilon}{2}), \frac{\epsilon}{2})$. If there exist both $x^* \in S_{X^*}$ and $x \in B_X$ satisfying $|x^*(x)| > 1 - \eta(\epsilon)$, then $\|x - \frac{x}{\|x\|}\| < \frac{\epsilon}{2}$. Because X is reflexive, there is a $x_0 \in S_X$ satisfying $x^*(x_0) = 1$. Choosing $c \in S_{\mathbb{K}}$ with $x^*(c\frac{x}{\|x\|}) > 1 - \eta(\epsilon)$, then

$$\delta\Big(\frac{\epsilon}{2}\Big) > 1 - x^*\Big(\frac{x_0 + c\frac{x}{\|x\|}}{2}\Big) \geq 1 - \Big\|\frac{x_0 + c\frac{x}{\|x\|}}{2}\Big\|.$$

Hence, $\|x_0 - c\frac{x}{\|x\|}\| < \frac{\epsilon}{2}$ and we get $\|\bar{c}x_0 - x\| < \epsilon$.

To prove the converse, let $0 < \epsilon < 1$ and suppose $x, y \in S_X$ satisfy $\|x - y\| \geq \epsilon$. Let Γ be the set of all bounded linear functionals in S_{X^*} that strongly expose B_X.

We claim that each $x^* \in \Gamma$ satisfies either

$$\operatorname{Re} x^*(x) \leq 1 - \min\Big\{\eta\Big(\frac{\epsilon^2}{64}\Big), \frac{\epsilon^2}{64}\Big\}$$

or

$$\operatorname{Re} x^*(y) \leq 1 - \min\Big\{\eta\Big(\frac{\epsilon^2}{64}\Big), \frac{\epsilon^2}{64}\Big\}.$$

Otherwise, there is $z^* \in S_{X^*}$ that strongly exposes B_X at z, and satisfies both

$$\operatorname{Re} z^*(x) > 1 - \min\Big\{\eta\Big(\frac{\epsilon^2}{64}\Big), \frac{\epsilon^2}{64}\Big\}$$

and

$$\operatorname{Re} z^*(y) > 1 - \min\Big\{\eta\Big(\frac{\epsilon^2}{64}\Big), \frac{\epsilon^2}{64}\Big\}.$$

By assumption, we have that for some α_1 and α_2 in $S_{\mathbb{C}}$,

$$\|x - \alpha_1 z\| < \frac{\epsilon^2}{64} \text{ and } \|y - \alpha_2 z\| < \frac{\epsilon^2}{64}.$$

Hence, we get $|z^*(x) - \alpha_1| < \frac{\epsilon^2}{64}$ and $|z^*(y) - \alpha_2| < \frac{\epsilon^2}{64}$. This implies that

$$\operatorname{Re} \alpha_1 > \operatorname{Re} z^*(x) - \frac{\epsilon^2}{64} > 1 - \frac{\epsilon^2}{32}$$

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

and

$$\operatorname{Re}\alpha_2 > \operatorname{Re} z^*(y) - \frac{\epsilon^2}{64} > 1 - \frac{\epsilon^2}{32}.$$

Finally, we have

$$\begin{aligned}|\alpha_1 - \alpha_2| &\le \sqrt{(\operatorname{Re}\alpha_1 - \operatorname{Re}\alpha_2)^2 + (\operatorname{Im}\alpha_1 - \operatorname{Im}\alpha_2)^2}\\ &< \sqrt{(\epsilon^2/32)^2 + 4(1-(1-\epsilon^2/32)^2)} < \frac{\epsilon}{2},\end{aligned}$$

and

$$\begin{aligned}\|x - y\| &\le \|x - \alpha_1 z\| + \|\alpha_1 z - \alpha_2 z\| + \|\alpha_2 z - y\|\\ &< \frac{\epsilon^2}{64} + \frac{\epsilon}{2} + \frac{\epsilon^2}{64} < \epsilon,\end{aligned}$$

which is a contradiction.

It follows from the hypothesis and James theorem that X is reflexive, hence Γ is dense in S_{X^*} ([26]). Therefore, by the above claim, we get

$$\begin{aligned}\Big\|\frac{x+y}{2}\Big\| &= \sup\Big\{\operatorname{Re}\frac{x^*(x)+x^*(y)}{2} : x^* \in \Gamma\Big\}\\ &\le \frac{2 - \min\big\{\eta(\frac{\epsilon^2}{64}), \frac{\epsilon^2}{64}\big\}}{2} = 1 - \min\Big\{\frac{1}{2}\eta\Big(\frac{\epsilon^2}{64}\Big), \frac{\epsilon^2}{128}\Big\},\end{aligned}$$

which completes the proof. ∎

By Theorem 2.1 and Smulian's theorem, that is, the dual space of a uniformly convex Banach space is uniformly smooth, we can get the following characterization of a uniformly smooth Banach space.

Corollary 2.2 *A reflexive Banach space X is uniformly smooth if and only if for every $\epsilon > 0$ there is $0 < \eta(\epsilon) < 1$ such that, for all $f \in B_{X^*}$ and all $x \in S_X$ satisfying $|f(x)| > 1 - \eta(\epsilon)$, there exists $f_0 \in S_{X^*}$ satisfying $|f_0(x)| = 1$ and $\|f - f_0\| < \epsilon$.*

In the next corollary, if the set of strongly exposing functionals is weak-*-dense in S_{X^*}, then we can get the same result without any difficulty. We omit the details of the proof.

Corollary 2.3 *Suppose that the set Γ of every elements $x^* \in S_{X^*}$ that strongly expose B_X is weak-*-dense in S_{X^*} and that for each $\epsilon > 0$, there is $\eta(\epsilon) > 0$ such that if there exist both $x \in S_X$ and $x^* \in S_{X_*}$ such that x^* strongly exposes B_X at x_0 and $|x^*(x)| > 1 - \eta(\epsilon)$, then $\|x - \alpha x_0\| < \epsilon$ holds for some $\alpha \in S_{\mathbb{C}}$. Then X is a uniformly convex Banach space with modulus of convexity*

$$\delta(\epsilon) \ge \min\Big\{\frac{1}{2}\eta\Big(\frac{\epsilon^2}{64}\Big), \frac{\epsilon^2}{128}\Big\}.$$

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

Lindenstrauss [23] introduced the notion of a uniformly strongly exposed family to study the denseness of norm-attaining operators. We say that a family $\{x_\alpha\}_\alpha \subset S_X$ is uniformly strongly exposed with respect to $\{f_\alpha\}_\alpha \subset S_{X^*}$ if for every $\epsilon > 0$ there is a function $\delta(\epsilon) > 0$ such that for every α, $f_\alpha(x_\alpha) = 1$, and for any $x \in B_X$, $f_\alpha(x) \geq 1 - \delta(\epsilon)$ implies $\|x - x_\alpha\| \leq \epsilon$. A slight modification of the proof of Theorem 2.1 shows the following corollary.

Corollary 2.4 *Let X be a Banach space with a uniformly strongly exposed family $\{x_\alpha\}_\alpha \subset S_X$ with respect to $\{f_\alpha\}_\alpha \subset S_{X^*}$. The convex hull of $\{f_\alpha\}_\alpha$ is weak-*-dense in B_{X^*} if and only if X is uniformly convex.*

We note that Theorem 2.1 cannot be extended to vector-valued mappings. Indeed, let $\mathbb{R}^2$ be the 2-dimensional Euclidean space. For every $k \in \mathbb{N}$ define $T_k\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_k(x, y) = \left(x, \left(1 - \frac{1}{k}\right)^{1/2} y\right).$$

It is easily computed that

$$\|T_k\| = \sup\left\{\alpha^2 + (1 - \alpha^2)\left(1 - \frac{1}{k}\right) : 0 \leq \alpha \leq 1\right\},$$

which implies that $\|T_k\| = 1$ and only α with $\alpha^2 = 1$ gives the norm of T_k. Hence each T_k attains its norm only at $(\pm 1, 0)$. However $\|T_k(0,1)\| > (1 - \frac{1}{k})^{1/2}$ for every $k \in \mathbb{N}$ and $\|(\pm 1, 0) - (0, 1)\| = \sqrt{2}$.

3 The Bishop–Phelps–Bollobás Property for a Uniformly Convex Space

Even though the Radon-Nikodým property is equivalent to the Bishop–Phelps property [8], there exists a Banach space Y such that (ℓ_1, Y) fails to have the BPBP [2]. On the other hand, the *uniform convexity* of X implies the BPBP of (X, Y) for every Banach space Y.

Theorem 3.1 *Let $0 < \epsilon < 1$ and $\delta(\epsilon) > 0$ be the modulus of convexity of a uniformly convex Banach space X. Then (X, Y) has the BPBP for every Banach space Y. More precisely, if $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy*

$$\|Tx\| > 1 - \frac{\epsilon}{2^5}\delta\left(\frac{\epsilon}{2}\right),$$

then there exist $S \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ such that $\|Sx_0\| = 1$, $\|S - T\| < \epsilon$ and $\|x - x_0\| < \epsilon$.

Proof Suppose that $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|Tx\| > 1 - \frac{\epsilon}{2^5}\delta\left(\frac{\epsilon}{2}\right).$$

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

Choose $f \in S_{Y^*}$ satisfying

$$\operatorname{Re} f(Tx) > 1 - \frac{\epsilon}{2^5}\delta\Big(\frac{\epsilon}{2}\Big).$$

Define a sequence $(x_i, f_i, T_i)_{i=1}^{\infty} \subset S_X \times S_{Y^*} \times S_{\mathcal{L}(X,Y)}$ inductively.

First, set $(x_1, f_1, T_1) = (x, f, T)$. When the k-th sequence is constructed, set

$$\widetilde{T}_{k+1}x = T_k x + \frac{\epsilon}{2^{k+2}} f_k(T_k x) T_k x_k \quad T_{k+1} = \frac{\widetilde{T}_{k+1}}{\|\widetilde{T}_{k+1}\|}$$

and choose $x_{k+1} \in S_X$ and $f_{k+1} \in S_{Y^*}$ satisfying

$$\operatorname{Re} f_{k+1}(\widetilde{T}_{k+1}x_{k+1}) > \|\widetilde{T}_{k+1}\| - \frac{\epsilon}{2^{k+5}}\delta\Big(\frac{\epsilon}{2^{k+1}}\Big),$$
$$\operatorname{Re} f_k(\widetilde{T}_k x_{k+1}) = \big| f_k(\widetilde{T}_k x_{k+1})\big|.$$

It follows that

$$\operatorname{Re} f_{k+1}(T_{k+1}x_{k+1}) > \|T_{k+1}\| - \frac{\epsilon}{2^{k+4}}\delta\Big(\frac{\epsilon}{2^{k+1}}\Big).$$

Hence, $\|T_k - T_{k+1}\| \le \|T_k - \widetilde{T}_{k+1}\| + \|\widetilde{T}_{k+1} - T_{k+1}\| < \frac{\epsilon}{2^{k+1}}$ and $\{T_k\}_k$ is a Cauchy sequence. This implies that $(T_k)_{k=1}^{\infty}$ converges to some $T_\infty \in S_{\mathcal{L}(X,Y)}$ and $\|T - T_\infty\| < \epsilon$.

To show that the sequence $(x_i)_{i=1}^{\infty}$ is a Cauchy sequence we need to check the following:

$$\begin{aligned}
\|\widetilde{T}_k\| - \frac{\epsilon}{2^{k+4}}\delta\Big(\frac{\epsilon}{2^k}\Big) &< \big| f_k(\widetilde{T}_k x_k)\big| \\
&= \Big| f_k(T_{k-1}x_k) + \frac{\epsilon}{2^{k+1}} f_{k-1}(T_{k-1}x_k)\cdot f_k(T_{k-1}x_{k-1})\Big| \\
&\le \Big| f_k(T_{k-1}x_k)| + \frac{\epsilon}{2^{k+1}}|f_{k-1}(T_{k-1}x_k)\Big| \cdot \big| f_k(T_{k-1}x_{k-1})\big| \\
&\le \|T_{k-1}\| + \frac{\epsilon}{2^{k+1}} \operatorname{Re} f_{k-1}(T_{k-1}x_k),
\end{aligned}$$

and

$$\begin{aligned}
\|\widetilde{T}_k\| &\ge \big| f_{k-1}(\widetilde{T}_k x_{k-1})\big| \\
&= \Big| f_{k-1}(T_{k-1}x_{k-1}) + \frac{\epsilon}{2^{k+1}} f_{k-1}(T_{k-1}x_{k-1})\cdot f_{k-1}(T_{k-1}x_{k-1})\Big| \\
&\ge \Big| \Big(1 + \frac{\epsilon}{2^{k+1}} f_{k-1}(T_{k-1}x_{k-1})\Big)\cdot f_{k-1}(T_{k-1}x_{k-1})\Big| \\
&\ge \Big(1 + \frac{\epsilon}{2^{k+1}} \operatorname{Re} f_{k-1}(T_{k-1}x_{k-1})\Big)\cdot \operatorname{Re} f_{k-1}(T_{k-1}x_{k-1}) \\
&\ge \|T_{k-1}\| - \frac{\epsilon}{2^{k+2}}\delta\Big(\frac{\epsilon}{2^{k-1}}\Big) + \frac{\epsilon}{2^{k+1}}\Big(\|T_{k-1}\| - \frac{\epsilon}{2^{k+2}}\delta\Big(\frac{\epsilon}{2^{k-1}}\Big)\Big)^2.
\end{aligned}$$

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

Therefore, from the monotonicity of the modulus of convexity (*cf.* [17]), we get

$$\begin{aligned}\operatorname{Re} f_{k-1}(T_{k-1}x_k) &> \Big(\|T_{k-1}\| - \frac{\epsilon}{2^{k+2}}\delta\Big(\frac{\epsilon}{2^{k-1}}\Big)\Big)^2 - \frac{1}{2}\delta\Big(\frac{\epsilon}{2^{k-1}}\Big) - \frac{1}{2^3}\delta\Big(\frac{\epsilon}{2^k}\Big)\\ &\geq 1 - \frac{\epsilon}{2^{k+1}}\delta\Big(\frac{\epsilon}{2^{k-1}}\Big) - \frac{1}{2}\delta\Big(\frac{\epsilon}{2^{k-1}}\Big) - \frac{1}{2^3}\delta\Big(\frac{\epsilon}{2^k}\Big)\\ &\geq 1 - \delta\Big(\frac{\epsilon}{2^{k-1}}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\Big\|\frac{x_{k-1}+x_k}{2}\Big\| &\geq \operatorname{Re} f_{k-1}\Big(T_{k-1}\frac{x_{k-1}+x_k}{2}\Big)\\ &> 1 - \frac{\epsilon}{2^{k+3}}\delta\Big(\frac{\epsilon}{2^{k-1}}\Big) - \frac{1}{2}\delta\Big(\frac{\epsilon}{2^{k-1}}\Big) \geq 1 - \delta\Big(\frac{\epsilon}{2^{k-1}}\Big).\end{aligned}$$

This implies that $\|x_{k-1} - x_k\| < \frac{\epsilon}{2^{k-1}}$. Thus, $(x_k)_k$ converges to some $x_\infty \in S_X$ and $\|x - x_\infty\| < \epsilon$.

From the fact that $\lim_{k\to\infty} \|T_k x_k\| = 1$ and that both T_k and x_k converge in norm, it follows that $\|T_\infty x_\infty\| = 1$. ■

Theorem 2.2 in [2] implies that for Banach spaces X and Y such that Y has the *property* β of Lindenstrauss with $0 \leq \rho < 1$, for given $\epsilon > 0$, if $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy $\|T(x)\| > 1 - \epsilon^2/4$, then for each real number η such that $\eta > \rho/(1-\rho)(\epsilon + \epsilon^2/4)$, there are $S \in \mathcal{L}(X,Y)$, $z \in S_X$ such that

$$\|Sz\| = \|S\|, \quad \|z - x\| < \epsilon, \quad \|S - T\| < \eta + \epsilon + \frac{\epsilon^2}{4}.$$

This means that (X,Y) has the BPBP when Y has the property β. Moreover, the real valued functions $\eta(\epsilon)$ and $\beta(\epsilon)$ in the definition of the BPBP do not depend on the Banach space X. In Theorem 3.1, we can see similarly that the real valued functions $\eta(\epsilon)$ and $\beta(\epsilon)$ do not depend on the target space Y. A natural question arises as to whether or not this implies the uniform convexity of X. We could get a necessary condition and an affirmative answer for a real 2-dimensional Banach space.

In [23], Lindenstrauss showed the following for a Banach space X such that the set of norm attaining operators is dense in $\mathcal{L}(X,Y)$ for any Banach space Y:

(a) If X is isomorphic to a strictly convex space, then S_X is the closed convex hull of its extreme points.

(b) If X is isomorphic to a locally uniformly convex space, then S_X is the closed convex hull of its strongly exposed points.

In the next theorem, we get stronger results for X when (X,Y) has the BPBP with the positive real valued functions $\eta(\epsilon)$ and $\beta(\epsilon)$ that are independent of the target space Y.

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

Theorem 3.2 *Let X be a Banach space. Suppose that given $\epsilon > 0$ there exist positive real valued functions $\eta(\epsilon)$ and $\beta(\epsilon)$ that go to 0 as ϵ goes to 0 and satisfying the following.*

- *For every Banach space Y if $\|Tx\| > 1-\eta(\epsilon)$ for $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$, there exist $u \in S_X$ and $S \in S_{\mathcal{L}(X,Y)}$ such that $\|Su\| = 1$, $\|x-u\| < \beta(\epsilon)$ and $\|T-S\| < \epsilon$.*

Then

(i) *if X is a real Banach space, then there is no face of S_X that contains a nonempty relatively open subset of S_X;*

(ii) *if X is isomorphic to a strictly convex Banach space, then the set of all extreme points of B_X is dense in S_X;*

(iii) *if X is isomorphic to a uniformly convex Banach space, then the set of all strongly exposed points of B_X is dense in S_X.*

Proof For the proof of (i), assume that there is $x^* \in S_{X^*}$ such that the face $F(x^*) = \{x \in S_X \mid x^*(x) = 1\}$ contains a nonempty relatively open subset U of S_X.

Choose a positive number $0 < \epsilon' < 1$ and points $x_0, y_0 \in U$ such that $B_X(y_0, \epsilon') \cap S_X \subset U$, $\|x_0 - y_0\| < \epsilon'$, and $x_0 \neq y_0$. Let $p = x_0 - y_0$. Choose $y^* \in S_{X^*}$ such that $y^*(p) = \|p\|$, and set

$$y_n^* = \frac{x^* + \frac{1}{n} y^*}{\|x^* + \frac{1}{n} y^*\|}.$$

Then $(y_n^*)_n$ converges to x^*.

For each $n \in \mathbb{N}$, define an equivalent norm $|||\cdot|||_n$ of X by

$$|||x|||_n^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}|y_n^*(x)|^2$$

and let $X_n = (X, |||\cdot|||_n)$. We can see that for each $x \in S_X$ there exists unique $t_x > 0$ such that $t_x x \in S_{X_n}$.

Set

$$U_n' = \{t_x x \in S_{X_n} : x \in U,\ t_x > 0\}.$$

It is easy to see that the map $x \mapsto t_x x$ is a homeomorphism, and so U_n' is relatively open in S_{X_n}.

Claim There is no nonempty relatively open convex subset $\widetilde{U}$ in S_{X_n} that is contained in U_n'.

If not, there exists a nonempty relatively open convex set $\widetilde{U}$ in S_{X_n} that is contained in U_n'. Choose $x \in U$ and $t > 0$ such that $x + tp \in U$ and $t_x x,\ t_{x+tp}(x+tp) \in \widetilde{U}$. Then, by the assumption

$$\Big|\Big|\Big| \frac{t_x x + t_{x+tp}(x+tp)}{2} \Big|\Big|\Big|_n = 1.$$

Since

$$|||t_x x|||_n^2 = \frac{1}{2}\|t_x x\|^2 + \frac{1}{2}|y_n^*(t_x x)|^2 = 1,$$

$$|||t_{x+tp}(x+tp)|||_n^2 = \frac{1}{2}\|t_{x+tp}(x+tp)\|^2 + \frac{1}{2}|y_n^*(t_{x+tp}(x+tp))|^2 = 1,$$

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

we get

$$\begin{aligned}
1 &= \left|\!\left|\!\left| \frac{t_x x + t_{x+tp}(x+tp)}{2} \right|\!\right|\!\right|_n^2 \\
&= \frac{1}{8}\big\| t_x x + t_{x+tp}(x+tp)\big\|^2 + \frac{1}{8}\big| y_n^*\big(t_x x + t_{x+tp}(x+tp)\big)\big|^2 \\
&\le \frac{1}{8}\big(\|t_x x\| + \|t_{x+tp}(x+tp)\|\big)^2 + \frac{1}{8}\big| y_n^*(t_x x) + y_n^*\big(t_{x+tp}(x+tp)\big)\big|^2 \\
&\le \frac{1}{8}\Big(\|t_x x\|^2 + \|t_{x+tp}(x+tp)\|^2 + 2\|t_x x\|\|t_{x+tp}(x+tp)\| \\
&\qquad + |y_n^*(t_x x)|^2 + |y_n^*\big(t_{x+tp}(x+tp)\big)|^2 + 2|y_n^*(t_x x)|\cdot|y_n^*\big(t_{x+tp}(x+tp)\big)|\Big) \\
&\le \frac{1}{8}\Big(4 + 2\big(\|t_x x\|\|t_{x+tp}(x+tp)\| + |y_n^*(t_x x)|\cdot|y_n^*\big(t_{x+tp}(x+tp)\big)|\big)\Big) \\
&\le \frac{1}{8}\Big(4 + 2\big(\|t_x x\|^2 + |y_n^*(t_x x)|^2\big)^{1/2}\big(\|t_{x+tp}(x+tp)\|^2 + |y_n^*\big(t_{x+tp}(x+tp)\big)|^2\big)^{1/2}\Big) \\
&\le 1.
\end{aligned}$$

The equality holds only when $\|t_x x\| = \|t_{x+tp}(x+tp)\|$ and $y_n^*(t_x x) = y_n^*(t_{x+tp}(x+tp))$. It follows from $\|t_x x\| = \|t_{x+tp}(x+tp)\|$ that $t_x = t_{x+tp}$. Hence, $y_n^*(x) = y_n^*(x+tp)$. This is a contradiction to the fact that $y_n^*(p) > 0$.

Now, we are ready to prove that there are no positive real valued functions $\eta(\epsilon)$ and $\beta(\epsilon)$ satisfying the assumption in Theorem 3.2.

Otherwise, choose ρ so that $0 < \rho < \frac{\epsilon'}{8}$ and $\beta(\rho) < \frac{\epsilon'}{4}$, and $N \in \mathbb{N}$ such that

$$\sqrt{\frac{1 + |y_N^*(y_0)|^2}{2}} > 1 - \eta(\rho).$$

Considering the identity operator $I\colon X \to X_N$, we can see easily that $\|I\| = 1$. Hence,

$$|||Iy_0|||_N = |||y_0|||_N = \sqrt{\frac{1}{2}\|y_0\|^2 + \frac{1}{2}|y_N^*(y_0)|^2} > 1 - \eta(\rho).$$

There exist $V \in S_{\mathcal{L}(X,Y)}$ and $y_1 \in B_X(y_0, \frac{\epsilon'}{4}) \cap S_X \subset U$ such that $\|V - I\| < \rho < \frac{\epsilon'}{8}$ and $|||Vy_1|||_N = 1$. Clealy, V is an isomorphism.

We will show that U_N' contains a nonempty relatively open convex subset in S_{X_N}, which contradicts the *claim*.

Now Vy_1 is $t_u u$ for some $u \in U$ and $t_u > 0$. Indeed, we can write Vy_1 in S_{X_N} uniquely in the form $t_u u$ for some $u \in S_X$ and $t_u > 0$. From the fact that $\|x\| \le 2\, |||\, x|||_N \le 2\|x\|$, it follows that $1 \le t_u < 1 + \epsilon'/4$. More precisely, the fact that

$$\begin{aligned}
1 \le t_u = \|t_u u\| = \|Vy_1\| &\le \|y_1\| + \|Vy_1 - y_1\| \\
&< \|y_1\| + 2\, |||\, Vy_1 - y_1|||_N < 1 + 2\rho < 1 + \frac{\epsilon'}{4}
\end{aligned}$$

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

implies that $\|u - y_1\| \le \|u - t_u u\| + \|t_u u - y_1\| < \frac{\epsilon'}{2}$. Thus $\|u - y_0\| < \epsilon'$ and $Vy_1 = t_u u \in U'_N$.

Choose $0 < \delta < \frac{\epsilon'}{4}$ so that $B_{X_N}(Vy_1, \delta) \cap S_{X_N} \subset U'_N$. We can see that

$$V\big(B_X(y_1, \delta) \cap S_X\big) = V\big(B_X(y_1, \delta) \cap F(x^*)\big) \subset B_{X_N}(Vy_1, \delta)$$

and $V(B_X(y_1, \delta) \cap F(x^*)) \subset S_{X_N}$, because $|||Vy_1|||_N = 1$. Hence $V(B_X(y_1, \delta) \cap S_X)$ is a convex subset of U'_N. Further, from the following we can see that $V(B_X(y_1, \frac{\delta}{2}) \cap S_X) = V(B_X(y_1, \frac{\delta}{2})) \cap S_{X_N}$, which implies that S_{X_N} contains a relatively open convex subset contained in U'_n. Indeed, we have checked that $|||Vx|||_N = 1$ for any $x \in B_X(y_1, \frac{\delta}{2}) \cap S_X$ in the above, and it is easy to see that $|||Vx|||_N < 1$ for any $x \in B_X(y_1, \frac{\delta}{2})$ with $\|x\| < 1$. For any $x' \in B_X(y_1, \frac{\delta}{2})$ with $\|x'\| > 1$ we can write $x' = \alpha x$ for some $\alpha > 1$ and $x \in B_X(y_1, \delta) \cap S_X$. Hence, from $|||Vx|||_N = 1$ we can get that $|||Vx'|||_N > 1$.

For the proof of (ii), suppose that there are both $x_0 \in S_X$ and $\epsilon_0 > 0$ such that the subset $B_X(x_0, \epsilon_0) \cap S_X$ does not contain any extreme point of B_X. Let $|||\cdot|||$ be a norm on X with which the Banach space $(X, |||\cdot|||)$ is strictly convex and we may assume that $|||x||| \le \|x\|$ for all $x \in X$. Then for each $n \in \mathbb{N}$, define the equivalent norm $\|x\|_n = (\|x\|^2 + \frac{1}{n}\,|||x|||^2)^{1/2}$ on X. Then $(X, \|\cdot\|_n)$ is strictly convex. Choose $0 < \rho$ satisfying $\beta(\rho) < \epsilon_0/2$ and $m \in \mathbb{N}$ satisfying

$$\frac{\sqrt{m}}{\sqrt{1+m}} > 1 - \eta(\rho).$$

Let $I\colon (X, \|\cdot\|) \to (X, \|\cdot\|_m)$ be the identity operator on X and let $T = I/\|I\|$. Because $1 \le \|I\| \le (1 + 1/m)^{1/2}$,

$$\|Tx_0\|_m = \frac{\|x_0\|_m}{\|I\|} \ge \frac{\|x_0\|\sqrt{m}}{\sqrt{m+1}} > 1 - \eta(\rho).$$

Hence, there exist both an operator $S\colon (X, \|\cdot\|) \to (X, \|\cdot\|_m)$ and $x_1 \in S_X$ such that $\|x_0 - x_1\| < \beta(\rho) < \epsilon_0/2$, $\|Sx_1\|_m = 1$ and $\|S - T\| \le \rho < 1/4$. This implies that $x_1 \in B_X(x_0, \epsilon_0) \cap S_X$, and it is not an extreme point of B_X. Choose a nonzero $p \in X$ and $t_0 > 0$ such that $x_1 + tp \in B_X$ for all t satisfying $|t| < t_0$. Let $y^* \in S_{(X, \|\cdot\|_m)^*}$ such that $y^*(Sx_1) = 1$. Then for all t satisfying $|t| < t_0$,

$$1 = \operatorname{Re} y^* S(x_1) = \frac{\operatorname{Re} y^* S(x_1 + tp) + \operatorname{Re} y^* S(x_1 - tp)}{2} \le 1.$$

Hence it is clear that $\|S(x + tp)\|_m = 1$ for all t satisfying $|t| < t_0$. Because $(X, \|\cdot\|_m)$ is strictly convex, $Sp = 0$. Finally we show that S is invertible, and this is a contradiction to $p \ne 0$. Indeed,

$$\begin{aligned}\|T^{-1}S - I\| &= \|T^{-1}(S - T)\| \le \|T^{-1}\| \cdot \|T - S\| \\ &= \|I^{-1}\| \cdot \|I\| \cdot \|T - S\| < \frac{1}{4}(1 + \frac{1}{m})^{1/2} < 1,\end{aligned}$$

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

which implies that S is invertible.

The proof of (iii) is almost the same as that of (ii), but for the sake of completeness we give it here.

Suppose that there are both $x_0 \in S_X$ and $\epsilon_0 > 0$ such that the subset $B_X(x_0, \epsilon_0) \cap S_X$ does not contain any strongly exposed point of B_X. Let $|||\cdot|||$ be a norm on X with which the Banach space $(X, |||\cdot|||)$ is uniformly convex, and we may assume that $|||x||| \le \|x\|$ for all $x \in X$. Then for each $n \in \mathbb{N}$, define the equivalent norm $\|x\|_n = (\|x\|^2 + \frac{1}{n} |||x|||^2)^{1/2}$ on X. Then $(X, \|\cdot\|_n)$ is uniformly convex. Choose $0 < \rho$ satisfying $\beta(\rho) < \epsilon_0/2$ and $m \in \mathbb{N}$ satisfying

$$\frac{\sqrt{m}}{\sqrt{1+m}} > 1 - \eta(\rho).$$

Let $I\colon (X, \|\cdot\|) \to (X, \|\cdot\|_m)$ be the identity operator on X and let $T = I/\|I\|$. Because $1 \le \|I\| \le (1 + 1/m)^{1/2}$,

$$\|Tx_0\|_m = \frac{\|x_0\|_m}{\|I\|} \ge \frac{\|x_0\|\sqrt{m}}{\sqrt{m+1}} > 1 - \eta(\rho).$$

Hence, there exist both an operator $S\colon (X, \|\cdot\|) \to (X, \|\cdot\|_m)$ and $x_1 \in S_X$ such that $\|x_0 - x_1\| < \beta(\rho) < \epsilon_0/2$, $\|Sx_1\|_m = 1$ and $\|S - T\| \le \rho < 1/4$. This implies that $x_1 \in B_X(x_0, \epsilon_0) \cap S_X$, and it is not a strongly exposed point of B_X. Let $y^* \in S_{(X, \|\cdot\|_m)^*}$ such that $y^*(Sx_1) = 1$. By the uniform convexity of the X, Sx_1 is a strongly exposed point of $B_{(X, \|\cdot\|_m)}$, and this implies that for every sequence $(u_i)_{i=1}^\infty \subset B_X$ if $y^*(Su_i)$ converges to 1 then Su_i converges to Sx_1. Thus S is invertible, and u_i converges to x_1. This is a contradiction. ■

In a 2-dimensional Banach space X, a nontrivial line segment in S_X is a face of S_X with a nonempty relatively open subset in S_X. Hence Theorem 3.2 implies the following corollary from the fact that a finite dimensional Banach space is strictly convex if and only if it is uniformly convex. (*cf.* see [17]).

Corollary 3.3 *If X is a real 2-dimensional Banach space, the assumption in Theorem 3.2 is equivalent to uniform convexity.*

Indeed, for given $\epsilon > 0$, there exist positive real valued functions $\eta(\epsilon)$ and $\beta(\epsilon)$ that go to 0 as ϵ goes to 0 and satisfying that for every Banach space Y if $\|Tx\| > 1 - \eta(\epsilon)$ for $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$, then there exist $u \in S_X$ and $S \in S_{\mathcal{L}(X,Y)}$ such that $\|Su\| = 1$, $\|x - u\| < \beta(\epsilon)$ and $\|T - S\| < \epsilon$, if and only if X is uniformly convex.

We say that (X, Y) has the *Bishop–Phelps–Bollobás property for bilinear forms* if given $\epsilon > 0$, there exist $\eta(\epsilon)$ and $\beta(\epsilon) > 0$ with $\lim_{t\to 0} \beta(t) = 0$ such that for all $\phi \in S_{\mathcal{L}^2(X\times Y)}$, if $x \in S_X$, $y \in S_Y$ satisfy $|\phi(x, y)| > 1 - \eta(\epsilon)$, then there exist points $x_\epsilon \in S_X$, $y_\epsilon \in S_Y$ and a bilinear form $\phi_\epsilon \in S_{\mathcal{L}^2(X\times Y)}$ that satisfy

$$|\phi_\epsilon(x_\epsilon, y_\epsilon)| = 1, \quad \|x - x_\epsilon\| < \beta(\epsilon), \quad \|y - y_\epsilon\| < \beta(\epsilon), \quad \|\phi - \phi_\epsilon\| < \epsilon.$$

It is known that (ℓ_1, ℓ_∞) has the BPBP [2], but (ℓ_1, ℓ_1) does not have the Bishop–Phelps–Bollobás property for bilinear forms [10, 16]. However, Cheng and Dai [9]

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

could show that (ℓ_1, ℓ_p) has the Bishop-Phelps-Bollobás property for bilinear forms for $(1 < p < \infty)$ by obtaining Theorem 3.4, and they asked whether (ℓ_p, ℓ_q) has the Bishop–Phelps–Bollobás property for bilinear forms for $(1 < p, q < \infty)$. Using Theorems 3.2 and 3.4, we get an affirmative answer to this question.

Theorem 3.4 ([9]) *Assume that a Banach space Y is uniformly convex. Then (X, Y) has the Bishop-Phelps-Bollobás property for bilinear forms if and only if the pair (X, Y^*) has the BPBP.*

Corollary 3.5 *(X, Y) has the Bishop–Phelps–Bollobás property for bilinear forms for any uniformly convex Banach spaces X and Y.*

We want to finish this paper with some open questions.

(a) Very recently, it is shown that (c_0, Y) has the BPBP when Y is uniformly convex [22]. It would be interesting to find conditions of a Banach space Y that guarantee that (c_0, Y) has the BPBP.

(b) In [22], it is also shown that (c_0, ℓ_p) $(1 < p < \infty))$ has the BPBP for bilinear forms. However, we still do not know whether (c_0, c_0) has the Bishop–Phelps–Bollobás property for bilinear forms.

Acknowledgments The authors wish to express their thanks to Prof. Yun Sung Choi for his valuable suggestions and fruitful conversations about the subject of this paper. The authors also wish to express their thanks to the anonymous referee whose careful reading and suggestions have improved the final form of the paper.

References

[1] M. D. Acosta, J. Alaminos, D. García, and M. Maestre, *On holomorphic functions attaining their norms*. J. Math. Anal. Appl. **297**(2004), no. 2, 625–644. http://dx.doi.org/10.1016/j.jmaa.2004.04.010

[2] M. D. Acosta, R. M. Aron, D. García, and M. Maestre, *The Bishop-Phelps-Bollobás theorem for operators*. J. Funct. Anal. **254**(2008), no. 11, 2780–2799. http://dx.doi.org/10.1016/j.jfa.2008.02.014

[3] R. M. Aron, B. Cascales, and O. Kozhushkina, *The Bishop-Phelps-Bollobás theorem and Asplund operators.* Proc. Amer. Math. Soc. **139**(2011), no. 10, 3553–3560. http://dx.doi.org/10.1090/S0002-9939-2011-10755-X

[4] R. M. Aron, Y. S. Choi, D. García, and M. Maestre, *The Bishop-Phelps-Bollobás theorem for $L(L_1(\mu), L_\infty[0,1])$*. Adv. Math. **228**(2011), no. 1, 617–628. http://dx.doi.org/10.1016/j.aim.2011.05.023

[5] R. M. Aron, C. Finet, and E. Werner, *Some remarks on norm attaining N-linear forms.* In: Functions spaces (Edwardsville, IL, 1994), Lecture Notes in Pure and Appl. Math., 172, Dekker, New York, 1995, pp. 19–28.

[6] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive.* Bull. Amer. Math. Soc. **67**(1961), 97–98. http://dx.doi.org/10.1090/S0002-9904-1961-10514-4

[7] B. Bollobás, *An extension to the theorem of Bishop and Phelps*. Bull. London. Math. Soc. **2**(1970), 181–182. http://dx.doi.org/10.1112/blms/2.2.181

[8] J. Bourgain, *Dentability and the Bishop-Phelps property.* Israel J. Math. **28**(1977), no. 4, 265–271. http://dx.doi.org/10.1007/BF02760634

[9] L. Cheng and D. Dai, *The Bishop-Phelps-Bollobás Theorem for bilinear forms.* Preprint.

[10] Y. S. Choi, *Norm attaining bilinear forms on $L^1[0,1]$*. J. Math. Anal. Appl. **211**(1997), no. 1, 295–300. http://dx.doi.org/10.1006/jmaa.1997.5461

[11] Y. S. Choi and S. G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials.* J. London Math. Soc. **54**(1996), no. 1, 135–147. http://dx.doi.org/10.1112/jlms/54.1.135

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press

[12] Y. S. Choi and S. K. Kim, *The Bishop-Phelps-Bollobás theorem for operators from $L_1(\mu)$ to Banach spaces with the Radon-Nikodým property.* J. Funct. Anal. **261**(2011), no. 6, 1446–1456. http://dx.doi.org/10.1016/j.jfa.2011.05.007

[13] ———, *The Bishop-Phelps-Bollobás property and lush spaces*. J. Math. Anal. Appl. **390**(2012), no. 2, 549–555. http://dx.doi.org/10.1016/j.jmaa.2012.01.053

[14] Y. S. Choi, H. J. Lee, and H. G. Song, *Denseness of norm-attaining mappings on Banach spaces.* Publ. Res. Inst. Math. Sci. **46**(2010), no. 1, 171–182. http://dx.doi.org/10.2977/PRIMS/4

[15] ———, *Bishop's theorem and differentiability of a subspace of $C_b(K)$*. Israel J. Math. **180**(2010), 93–118. http://dx.doi.org/10.1007/s11856-010-0095-9

[16] Y. S. Choi and H. G. Song, *The Bishop-Phelps-Bollobás theorem fails for bilinear forms on $l_1 \times l_1$*. J. Math. Anal. Appl. **360**(2009), no. 2, 752–753. http://dx.doi.org/10.1016/j.jmaa.2009.07.008

[17] M. Fabian, P. Habala, P. Hàjek, V. Montesinos, and V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis.* CMS Books in Mathematics, Springer, New York, 2011.

[18] C. Finet and R. Payá, *Norm attaining operators from L_1 into L_∞*. Israel J. Math. **108**(1998), 139–143. http://dx.doi.org/10.1007/BF02783045

[19] R. C. James, *Weak compactness and reflexivity*. Israel J. Math. **2**(1964), 101–119. http://dx.doi.org/10.1007/BF02759950

[20] J. Kim and H. J. Lee, *Strong peak points and strongly norm attaining points with applications to denseness and polynomial numerical indices*. J. Funct. Anal. **257**(2009), no. 4, 931–947. http://dx.doi.org/10.1016/j.jfa.2008.11.024

[21] S. G. Kim and H. J. Lee, *Numerical peak holomorphic functions on Banach spaces.* J. Math. Anal. Appl. **364**(2010), no. 2, 437–452. http://dx.doi.org/10.1016/j.jmaa.2009.10.046

[22] S. K. Kim, *The Bishop-Phelps-Bollobás theorem for operators from c_0 to uniformly convex spaces.* Israel. J. Math., to appear. http://dx.doi.org/10.1007/s11856-012-0186-x

[23] J. Lindenstrauss, *On operators which attain their norm*. Israel J. Math. **1**(1963), 139–148. http://dx.doi.org/10.1007/BF02759700

[24] R. Payá and Y. Saleh, *Norm attaining operators from $L_1(\mu)$ into $L_\infty(\nu)$*. Arch. Math. **75**(2000), no. 5, 380–388. http://dx.doi.org/10.1007/s000130050519

[25] W. Schachermayer, *Norm attaining operators on some classical Banach spaces*. Pacific J. Math. **105**(1983), no. 2, 427–438. http://dx.doi.org/10.2140/pjm.1983.105.427

[26] C. Stegall, *Optimization and differentiation in Banach spaces*. Proceedings of the symposium on operator theory (Athens, 1985), Linear Algebra Appl. **84**(1986), 191–211. http://dx.doi.org/10.1016/0024-3795(86)90314-9

[27] J. J. Uhl, Jr, *Norm attaining operators on $L_1[0,1]$ and the Radon-Nykodým property.* Pacific J. Math. **63**(1976), no. 1, 293–300. http://dx.doi.org/10.2140/pjm.1976.63.293

School of Mathematics, Korea Institute for Advanced Study (KIAS), 85 Hoegiro, Dongdaemun-gu, Seoul 130-722, Republic of Korea
e-mail: lineksk@gmail.com

Department of Mathematics Education, Dongguk University–Seoul, 100-715 Seoul, Republic of Korea
e-mail: hanjulee@dongguk.edu

https://doi.org/10.4153/CJM-2013-009-2 Published online by Cambridge University Press